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A weak Brun–Titchmarsh theorem for multiplicative functions

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Abstract. An important theorem of Shiu gives a (precise) bound for the average of values of multiplicative functions, of a certain class, over 'short' intervals. Here we obtain, by simple means, the above result of same qualitative order.

Keywords. Multiplicative functions; short intervals.

1. Introduction

In the sequel, $f(n)$ denotes a multiplicative function satisfying (i) $0 \leqslant f(n) = O_\varepsilon(n^\varepsilon)$, for every $\varepsilon > 0$; (ii) $f(p^l) \leqslant A_1^l$ with a positive constant A_1. Here and throughout, the letter p denotes primes. Under these conditions the following result was proved in [1]:

Theorem 1. *Let* $0 < \alpha < \frac{1}{2}$, $0 < \beta < \frac{1}{2}$, *and let* a, k *be integers satisfying* $0 < a < k$, $(a,k) = 1$. *Then, as* $x \to \infty$, *under* (i) *and* (ii)

$$\sum_{\substack{x-y<n\leqslant x\\ n\equiv a(\mathrm{mod}\,k)}} f(n) \ll \frac{y}{\varphi(k)}\frac{1}{\log x}\exp\left(\sum_{\substack{p\leqslant x\\ p\nmid k}}\frac{f(p)}{p}\right), \tag{1}$$

uniformly in a, k *and* y *provided that*

$$1 \leqslant k < y^{1-\alpha}, \quad x^\beta \leqslant y \leqslant x. \tag{2}$$

Here we present a weaker version of the above result noted while attempting for a simpler proof of it. Actually, we obtain mainly the following

Theorem 1′. *Let* $0 < \alpha (< 1)$, $0 < \beta (< 1)$. *Then, as* $x \to \infty$, *under* (i) *and* (ii),

$$\sum_{\substack{x-y<n\leqslant x\\ n\equiv a(\mathrm{mod}\,k)}} f(n) \ll (a,k)\frac{y}{k}\exp\left(\sum_{\substack{p\leqslant y/k\\ (p,k)|(a,k)}}\frac{(f(p))^R}{p}\right) \tag{1′}$$

with a certain constant R *depending only on* α, β, f, *uniformly in* y *and integers* a, k *provided that*

$$1 \leqslant k < y^{1-\alpha}, \quad x^\beta \leqslant y \leqslant x. \tag{2′}$$

Also we note in course that Theorem 1′ and its proof, contain (qualitative) consequences similar to Theorems 2 and 3 of [1], apart from implying Theorem 4 there.

2. Proof of Theorem 1′

Let $n \in P_Y$ mean that there exist p and b such that $p \leqslant Y < p^b$, $p^b \| n$. Set $Y = (y/k)^\delta$, where δ denotes a small positive constant, to be fixed shortly. We start by observing that

$$|\{n: M-y<n\leqslant M, n\equiv a(\bmod k); n\in P_Y\}| = O\left(Y+(a,k)\left(\frac{y}{k}\right)^{1-\frac{1}{2}\delta}\right), \quad (3)$$

holds, with an absolute constant, uniformly in M. For this, obviously we have that the quantity on the left side is

$$\leqslant \sum_{p\leqslant Y}\left\{1+\sum_{\substack{b\\ Y<p^b\leqslant (y/k)(p^b,k)}}\left(1+\frac{y}{[p^b,k]}\right)\right\} = O\left(Y+\frac{(a,k)y}{k}\sum_{p\leqslant Y<p^b}\frac{1}{p^b}\right),$$

on noting that if p^b gives a non-zero contribution then $(p^b, k)|(a,k)$. This implies (3). Therefore, for a small enough $\delta>0$,

$$\sum_{n,n\in P_Y}{}' f(n) = O\left((a,k)\frac{y}{k}c_{\alpha,\beta,f,\delta}\right) = O_{\alpha,\beta,f,\delta}\left((a,k)\frac{y}{k}\right), \quad (4)$$

where $'$ denotes (here and hereafter) the condition $x-y<n\leqslant x$, $n\equiv a(\bmod k)$. Now we shall assume that δ is fixed to yield (4).

Next, notice that $n\notin P_Y$ implies that n has factorization of the form $n=m_1\cdots m_S p_1^{b_1}\cdots p_r^{b_r}$, such that (a) $m_1^\delta<\cdots<m_S^\delta\leqslant Y<p_1<\cdots<p_r$, (b) $m_j>(y/k)^{1-\delta}$, $2\leqslant j\leqslant S$, and (c) m_j's are composed of distinct prime powers not exceeding Y, and are mutually relatively prime. Therefore, by using (ii),

$$f(n)\ll_{\alpha,\beta,f}\left(\max_{1\leqslant j\leqslant S} f(m_j)\right)^S; \quad S\leqslant R(\alpha,\beta,f)=:R.$$

Now, any $m\leqslant y/k$ can occur as an $m_j|n$ for at most $(1+(y/[k,m]))$ values of $n\notin P_Y$, coming under the condition $'$. Thus, using $(m_j, k)|(a,k)$, we obtain

$$\sum_{n,n\notin P_Y}{}' f(n)\ll_{\alpha,\beta,f}\sum_{\substack{m\leqslant y/k\\ (m,k)|(a,k)}}(f(m))^R\frac{y}{[k,m]}\ll\frac{(a,k)y}{k}\sum_{\substack{m\leqslant y/k\\ (m,k)|(a,k)}}\frac{(f(m))^R}{m}. \quad (5)$$

Here, on using (i), we observe that

$$\sum_{\substack{m\leqslant Z\\ (m,k)|(a,k)}}\frac{(f(m))^R}{m}\leqslant \exp\left(\sum_{\substack{p\leqslant Z\\ (p,k)|(a,k)}}\frac{(f(p))^R}{p}+c_f\right). \quad (6)$$

By collecting (4), (5) and (6), we obtain (1′) thereby completing the proof of Theorem 1′.

3. Concluding Remarks

From Theorem 1′, by using $f(p)\leqslant A_1$, we immediately deduce the following Corollary.

COROLLARY 1

Under conditions of Theorem 1′ *we have*

$$\sum_{\substack{x-y<n\leqslant x\\ n\equiv a(\bmod k)}} f(n)\leqslant D_1\frac{(a,k)y}{k}(\log x)^{D_2}, \quad (1'')$$

where the constants D_1, D_2 depend at most on α,β,f and ε.

This corollary contains qualitative versions of Theorems 2 and 3 of [1]. Here, we note that (1″) is false without the factor (a,k) in the upper bound. To see this we may take $f(n)=d(n)$, the number of positive divisors of n, $a=0$ and assume that k is such that $d(k)>\exp((\log\log k)^2)$, say. Then we would obtain the lower bound $d(k)(y/k)$ for the sum in (1″), by using $d(k)\leqslant d(n)$ whenever k divides n, thereby leading to a contradiction if the factor (a,k) were absent on the right side in (1″). Further, observe that from (3) and (5) we obtain the following Corollary 2, which incidentally implies Theorem 4 of [1].

COROLLARY 2

Suppose, further that, f has support over a sequence A with $|\{n:n\leqslant x, n\in A\}|=O(x^{1-\eta})$, for some $\eta>0$, then

$$\sum_{\substack{x-y<n\leqslant x\\ n\equiv a(\mathrm{mod}\, k)}} f(n)\leqslant D\frac{(a,k)y}{k},\qquad D=D(\alpha,\beta,f). \tag{7}$$

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Reference

[1] Shiu P, A Brun–Titchmarsh theorem for multiplicative functions, *J. Reine Angew. Math.* **311** (1980) 161–170

Note added in proof.

The reader would find in a forthcoming paper, [2], significant advances on the result of [1].

[2] Mohan Nair and Gérald Tenenbaum, Short sums of certain arithmetic functions (to appear in *Acta Mathematica*)