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A new approximate functional equation for Hurwitz zeta function for rational parameter

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Abstract. For Hurwitz zeta function $\zeta(s,(a/k))$ with $a=1,2,3,\ldots,k$, we obtain a new simple approximate functional equation (uniform in k and t) in critical strip. Our method should prove to be an alternative approach to Atkinson's method in dealing with $\Sigma_{\chi(\mathrm{mod}\,q)}\int_0^T|L(s,\chi)|^2\mathrm{d}t$, where $L(s,\chi)$ is Dirichlet L-series modulo q and $s=\sigma+it$.

Keywords. Hurwitz zeta function; Dirichlet L-series.

1. Introduction

For integer $q\geqslant 1$, let $\chi(\mathrm{mod}\,q)$ be a Dirichlet character modulo q and let $L(s,\chi)$ be the corresponding Dirichlet L-series, where $s=\sigma+it$ with σ,t real. In what follows, we shall have $t>0$. For $0<\alpha\leqslant 1$. Let $\zeta(s,\alpha)$ be Hurwitz zeta function defined by $\zeta(s,\alpha)=\Sigma_{n\geqslant 0}(n+\alpha)^{-s}$ for Re $s>1$; and its analytic continuation. Let $\zeta_1(s,\alpha)=\zeta(s,\alpha)-\alpha^{-s}$. In what follows $\ll$ and 0-constants are absolute. Also $[u]$ for real u, shall denote integral part of u. In the light of the fact

$$\Sigma_{\chi(\mathrm{mod}\,q)}|L(\sigma+it,\chi)|^2=\frac{\phi(q)}{q^{2\sigma}}\cdot\sum_{k/q}\mu(q/k)\left(\sum_{a=1}^{k}\left|\zeta\left(\sigma+it,\frac{a}{k}\right)\right|^2\right),$$

where ϕ and μ are Euler and Moebius functions respectively, one needs an estimate for $\Sigma_{a=1}^{k}|\zeta(s,(a/k))|^2$ for a divisor k of q. The object of this paper is to prove the following theorem, which gives a new form of the approximate functional equation for $\zeta(s,(a/k))$ in critical strip (for $a=1,2,\ldots,k$), with a mean value estimate for the error term.

Theorem. *Let $s=\sigma+it$ with $0<\sigma<1$ and $x\geqslant 1$ an integer. Let $2\pi xy=t$ with $y>0$. Let $K(s)=(2\pi/i)^{s-1}.\,\Gamma(1-s)$. Then, we have*

1) *for $y\geqslant k$,*

$$\zeta\left(s,\frac{a}{k}\right)=\sum_{0\leqslant m\leqslant x-1}\left(m+\frac{a}{k}\right)^{-s}+K(s)\sum_{n\leqslant y}\mathrm{e}^{-(2\pi ina)/k}.n^{s-1}$$
$$+\sum_{1\leqslant n\leqslant y+k}b_n(s)\mathrm{e}^{-(2\pi ina)/k}+R\left(s,x,\frac{a}{k}\right),$$

where

$$b_n(s)=-\int_0^x\frac{\mathrm{e}^{2\pi inu}}{u^s}\mathrm{d}u\qquad \textit{for}\qquad n\leqslant y$$
$$=\int_x^\infty\frac{\mathrm{e}^{2\pi inu}}{u^s}\mathrm{d}u\qquad \textit{for}\qquad y<n\leqslant y+k;$$

with

$$\sum_{a=1}^{k}\left|R\left(s,x,\frac{a}{k}\right)\right|^2 \ll x^{-2\sigma}\log^2 kt.$$

2) *for* $y<k$,

$$\zeta\left(s,\frac{a}{k}\right)=\sum_{0\leqslant m\leqslant x-1}\left(m+\frac{a}{k}\right)^{-s}+K(s)\sum_{n\leqslant y}\mathrm{e}^{-(2\pi ina)/k}\cdot n^{s-1}$$
$$+\frac{x^{1-s}}{s-1}+\sum_{1\leqslant n\leqslant y+k}b_n(s)\mathrm{e}^{-(2\pi ina)/k}+\sum_{1\leqslant n\leqslant k}c_n(s)\,\mathrm{e}^{(2\pi ina)/k}+R'\left(s,x,\frac{a}{k}\right),$$

where

$$c_n(s)=-\int_0^x\frac{\mathrm{e}^{-2\pi inu}}{u^s}\mathrm{d}u \qquad for \qquad n\leqslant y$$
$$=\int_x^{\infty}\frac{\mathrm{e}^{-2\pi inu}}{u^s} \qquad for \qquad y<n\leqslant k$$

with

$$\sum_{a=1}^{k}\left|R'\left(s,x,\frac{a}{k}\right)\right|^2 \ll x^{-2\sigma}\log^2 kt$$

and

$$\sum_{a=1}^{k}\left|\sum_{n\leqslant k}c_n(s)\cdot\mathrm{e}^{(2\pi ina)/k}\right|^2 \ll kx^{-2a}\cdot\min(1,1/y)$$

Our approximate functional equation can be used to obtain asymptotic formula for $\Sigma_{\chi(\mathrm{mod}\,q)}|L(s,\chi)|^2$ and $\Sigma_{\chi(\mathrm{mod}\,q)}\int_0^T|L(s,\chi)|^2\mathrm{d}t$ (uniform in q and t (or T)) on critical line. In fact, our approach may serve as an alternative to Atkinson's method, while dealing with $\int_0^T\mathrm{d}t\Sigma_{\chi(\mathrm{mod}\,q)}|L(s,\chi)|^2$ on critical line. Our approach also gives an interesting expression for $\int_0^1|\zeta_1(s,\alpha)|^2\mathrm{d}\alpha$ in critical strip and an expression for $h/k-[h/k]-1/2$ for integers h and k with $k\geqslant 1$ (see Remarks below).

Our Theorem may be viewed as the continuation of [1]. In the context of our Theorem, it is pertinent to compare our Theorem to Riemann–Siegel type formula for $\zeta((1/2)+it,\alpha)$ of [2]. Also see [3] and [4].

In what follows, we shall be using the following facts. For integer $n\geqslant 1$ and sufficiently large integer N and with $y=t/2\pi x$, we have

a) for $n>y$, $\displaystyle\int_x^N\frac{\mathrm{e}^{2\pi inu}}{u^{s+1}}\mathrm{d}u\ll\frac{x^{-\sigma-1}}{n-y}$; and $\displaystyle\int_x^N\frac{\mathrm{e}^{2\pi inu}}{u^s}\mathrm{d}u\ll\frac{x^{-\sigma}}{n-y}$,

b) $\displaystyle\int_x^N\frac{\mathrm{e}^{-2\pi inu}}{u^{s+1}}\mathrm{d}u\ll\frac{x^{-\sigma-1}}{n+y}$ and $\displaystyle\int_x^N\frac{\mathrm{e}^{-2\pi inu}}{u^s}\mathrm{d}u\ll\frac{x^{-\sigma}}{n}$ for $n\geqslant 1$,

c) $\displaystyle\int_0^x\frac{\mathrm{e}^{2\pi inu}}{u^s}\mathrm{d}u\ll\frac{x^{-\sigma}}{y-n}$ for $n<y$; and $\ll t^{(1/2)-\sigma}\cdot y^{\sigma-1}$ for $n\geqslant 1$,

d) $\displaystyle\int_0^x\frac{\mathrm{e}^{-2\pi inu}}{u^s}\mathrm{d}u\ll\frac{x^{-\sigma}}{y}$ for $n\geqslant 1$,

e) For $n \geqslant 1$, we have $\int_0^{\infty} \frac{e^{2\pi i n u}}{u^s}\, du = K(s)\cdot n^{s-1}$ and

$$\int_0^{\infty} \frac{e^{-2\pi i n u}}{u^s}\, du = \overline{K(\bar{s})}\cdot n^{s-1} \ll n^{\sigma-1}\cdot e^{-\pi t}\cdot t^{(1/2)-\sigma},$$

f) $\sum_{n \leqslant y+k} |b_n(s)| \ll x^{-\sigma}\cdot \log k(y+2) + t^{(1/2)-\sigma}\cdot y^{\sigma-1}$ and $\sum_{n\leqslant k} |c_n(s)| \ll x^{-\sigma}\cdot \log kt$.

Remarks on Theorem. 1) From the proof of Theorem, we shall find that for $0 < \operatorname{Re} s < 1$ with fixed s, $\zeta_1(s,\alpha)$ as a function of α has a Fourier series $\zeta_1(s,\alpha) = \Sigma_{n=-\infty}^{\infty} a_n(s) e^{2\pi i n\alpha}$, where $a_0(s) = 1/(s-1)$; and for $n \neq 0$, $a_n(s) = \int_1^{\infty}(e^{-2\pi i n u}/u^s)\, du$. As a function of α for $0 < \operatorname{Re} s < 1$, $\zeta_1(s,\alpha)$ is a function of bounded variation and hence it satisfies Parseval's result

$$\int_0^1 |\zeta_1(s,\alpha)|^2 d\alpha = \sum_{n=-\infty}^{\infty} |a_n(s)|^2.$$

2) In the light of the proof of our Theorem it is interesting to find an expression for $h/k - [h/k] - 1/2$, where h, k are integers with $k \geqslant 1$.

$$\frac{h}{k} - \left[\frac{h}{k}\right] - \frac{1}{2} = -\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{e^{2\pi i n h/k}}{2\pi i n} = \frac{1}{2\pi i}\sum_{r=1}^{k-1} a(r,k) e^{2\pi i h r/k},$$

where

$$a(r,k) = -\sum_{\substack{n=-\infty\\ n\neq 0\\ n\equiv r (\operatorname{mod} k)}}^{\infty} \frac{1}{n} = \sum_{m=1}^{\infty} \frac{2r}{m^2k^2 - r^2} - \frac{1}{r}.$$

This expression should be useful in problems related to Dedekind sums.

2. *Proof of Theorem.* We shall write

$$\zeta_x(s,\alpha) = \zeta(s,\alpha) - \sum_{0\leqslant m \leqslant x-\alpha} (m+\alpha)^{-s}.$$

Thus for $\operatorname{Re} s = \sigma > 0$, as in [1] for large N and integer x, we have

$$\begin{aligned}\zeta_x(s,\alpha) &= -s\cdot \int_x^N \frac{(u-\alpha-[u-\alpha]-(1/2))}{u^{s+1}}\, du \\ &\quad + x^{-s}\cdot\left(x-\alpha-[x-\alpha]-\frac{1}{2}\right) + \frac{x^{1-s}}{s-1} + O(tN^{-\sigma}) \\ &= \sum_{n=-\infty}^{\infty}{}' \frac{e^{-2\pi i n\alpha}}{2\pi i n}\left(\int_x^N \frac{s e^{2\pi i n u}}{u^{s+1}}\, du - x^{-s}\right) + \frac{x^{1-s}}{s-1} + O(tN^{-\sigma}),\end{aligned}$$

where dash over the summation sign $\Sigma_{n=-\infty}^{\infty\prime}$ indicates exclusion of $n = 0$. On integration by parts, we get

$$\begin{aligned}\zeta_x(s,\alpha) &= \sum_{n=-\infty}^{\infty}{}' \frac{e^{-2\pi i n\alpha}}{2\pi i n}\left(-\frac{e^{2\pi i n N}}{N^s} + 2\pi i n\int_x^N \frac{e^{2\pi i n u}}{u^s}\, du\right) + \frac{x^{1-s}}{s-1} + O(tN^{-\sigma}) \\ &= -N^{-s}\cdot \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi i n(N-\alpha)}}{2\pi i n} + \sum_{n=-\infty}^{\infty}{}' e^{-2\pi i n\alpha}\cdot \int_x^N \frac{e^{2\pi i n u}}{u^s}\, du\end{aligned}$$

$$+\frac{x^{1-s}}{s-1}+O(tN^{-\sigma})=N^{-s}\cdot\left(N-\alpha-[N-\alpha]-\frac{1}{2}\right)$$

$$+\sum_{n=-\infty}^{\infty}{}' \mathrm{e}^{-2\pi i n\alpha}\cdot\int_x^N \frac{\mathrm{e}^{2\pi i n u}}{u^s}\mathrm{d}u+\frac{x^{1-s}}{s-1}+O(t\cdot N^{-\sigma})$$

$$=\sum_{n=-\infty}^{\infty}{}' \mathrm{e}^{-2\pi i n\alpha}\cdot\int_x^N \frac{\mathrm{e}^{2\pi i n u}}{u^s}\mathrm{d}u+\frac{x^{1-s}}{s-1}+O(t\cdot N^{-\sigma}).$$

Taking $x=1$ and letting $N\to\infty$, we get

$$\zeta_1(s,\alpha)=\sum_{n=-\infty}^{\infty} a_n(s)\cdot \mathrm{e}^{2\pi i n\alpha},$$

with $a_0(s)=1/(s-1)$ and $a_n(s)=\int_1^\infty \mathrm{e}^{-2\pi i n u}/u^s\mathrm{d}u$ for $n\neq 0$.
Now, we continue with the proof of the Theorem. For $\sigma>0$, for large N, we have

$$\zeta_x\left(s,\frac{a}{k}\right)=-s\int_x^N\frac{(u-(a/k)-[u-(a/k)]-(1/2))}{u^{s+1}}\mathrm{d}u$$

$$+\left(x-\frac{a}{k}-\left[x-\frac{a}{k}\right]-\frac{1}{2}\right)\cdot x^{-s}+\frac{x^{1-s}}{s-1}+O(tN^{-\sigma}).$$

$$=\frac{x^{1-s}}{s-1}+\left(x-\frac{a}{k}-\left[x-\frac{a}{k}\right]-\frac{1}{2}\right)x^{-s}$$

$$+\sum_{n=-\infty}^{\infty}{}'\frac{\mathrm{e}^{-2\pi i n a/k}}{2\pi i n}\cdot\int_x^N\frac{s\mathrm{e}^{2\pi i n u}}{u^{s+1}}\mathrm{d}u+O(tN^{-\sigma}).$$

Next,

$$\sum_{n\geqslant 1}\frac{\mathrm{e}^{-2\pi i n a/k}}{2\pi i n}\cdot\int_x^N\frac{s\mathrm{e}^{2\pi i n u}}{u^{s+1}}\mathrm{d}u=\sum_{n\leqslant y+k}+\sum_{n>y+k}, \text{ say.}$$

Now

$$\sum_{a=1}^{k}\left|\sum_{n>y+k}\frac{\mathrm{e}^{-2\pi i n a/k}}{2\pi i n}\cdot\int_x^N\frac{s\mathrm{e}^{2\pi i n u}}{u^{s+1}}\mathrm{d}u\right|^2$$

$$=k\sum_{a=1}^{k}\left|\sum_{\substack{n>y+k\\ n\equiv a(\mathrm{mod}\, k)}}\frac{1}{2\pi i n}\cdot\int_x^N\frac{s\mathrm{e}^{2\pi i n u}}{u^{s+1}}\mathrm{d}u\right|^2$$

$$\ll k\sum_{a=1}^{k}\left|x^{-\sigma}\cdot\sum_{\substack{n>y+k\\ n\equiv a(\mathrm{mod}\, k)}}\frac{y}{n(n-y)}\right|^2$$

$$=kx^{-2\sigma}\cdot\sum_{a=1}^{k}\left|\sum_{m>1+((y-a)/k)}\frac{y}{(mk+a)(mk+a-y)}\right|^2$$

$$\ll\frac{x^{-2\sigma}}{k}\cdot\sum_{a=1}^{k}\left|\sum_{m>1+((y-a)/k)}\frac{y/k}{(m+(a/k))(m-(y-a)/k)}\right|^2\ll x^{-2\sigma}\cdot\log^2 kt.$$

Next

$$\sum_{a=1}^{k}\left|\sum_{n>k}\frac{e^{2\pi ina/k}}{2\pi in}\cdot\int_x^N\frac{se^{-2\pi inu}}{u^{s+1}}du\right|^2$$

$$=k\cdot\sum_{a=1}^{k}\left|\sum_{\substack{n>k\\ n\equiv a(\mathrm{mod}\,k)}}\frac{1}{2\pi in}\cdot\int_x^N\frac{se^{-2\pi inu}}{u^{s+1}}du\right|^2$$

$$\ll kx^{-2\sigma}\cdot\sum_{a=1}^{k}\left|\sum_{\substack{n>k\\ n\equiv a(\mathrm{mod}\,k)}}\frac{y}{n(n+y)}\right|^2$$

$$\ll kx^{-2\sigma}\cdot\sum_{a=1}^{k}\left|\sum_{m\geqslant 1}\frac{y}{(mk+a)(mk+a+y)}\right|^2$$

$$\ll\frac{x^{-2\sigma}}{k}\cdot\sum_{a=1}^{k}\left|\sum_{m\geqslant 1}\frac{y/k}{(m+(a/k))(m+(a/k)+(y/k))}\right|^2\ll x^{-2\sigma}\log^2 kt.$$

Thus, we have

$$\zeta_x\left(s,\frac{a}{k}\right)-\frac{x^{1-s}}{s-1}=\left(x-\frac{a}{k}-\left[x-\frac{a}{k}\right]-\frac{1}{2}\right)x^{-s}$$

$$+\frac{1}{2\pi i}\left(\sum_{n\leqslant y+k}\frac{e^{-2\pi ina/k}}{n}\cdot\int_x^N\frac{se^{2\pi inu}}{u^{s+1}}du\right.$$

$$\left.-\sum_{n\leqslant k}\frac{e^{2\pi ina/k}}{n}\cdot\int_x^N\frac{se^{-2\pi inu}}{u^{s+1}}du\right)+O(tN^{-\sigma})+R_1\left(s,x,\frac{a}{k}\right),$$

where

$$\sum_{a=1}^{k}\left|R_1\left(s,x,\frac{a}{k}\right)\right|^2\ll x^{-2\sigma}\cdot\log^2 kt.$$

Next,

$$\sum_{n\leqslant y+k}\frac{e^{-2\pi ina/k}}{2\pi in}\cdot\int_x^N\frac{se^{2\pi inu}}{u^{s+1}}du$$

$$=x^{-s}\sum_{n\leqslant y+k}\frac{e^{2\pi in(x-(a/k))}}{2\pi in}+\sum_{n\leqslant y+k}e^{-2\pi ina/k}\cdot\int_x^N\frac{e^{2\pi inu}}{u^s}du+O(N^{-\sigma}\cdot\log kt).$$

Similarly,

$$\sum_{n\leqslant k}\frac{e^{2\pi ina/k}}{2\pi in}\cdot\int_x^N\frac{-s\cdot e^{-2\pi inu}}{u^{s+1}}du$$

$$=-x^{-s}\cdot\sum_{n\leqslant k}\frac{e^{-2\pi in(x-(a/k))}}{2\pi in}+\sum_{n\leqslant k}e^{2\pi ina/k}\cdot\int_x^N\frac{e^{-2\pi inu}}{u^s}du+O(N^{-\sigma}\cdot\log kt).$$

Next,

$$\sum_{a=1}^{k}\left|\sum_{k<n\leqslant y+k}\frac{e^{2\pi in(x-(a/k))}}{2\pi in}\right|^2$$

$$= k \sum_{a=1}^{k} \left| \sum_{\substack{k < n \leqslant y+k \\ n \equiv a (\mathrm{mod}\, k)}} \frac{e^{2\pi i n x}}{2\pi i n} \right|^2$$

$$= k \sum_{a=1}^{k} \left| \sum_{\substack{k < n \leqslant y+k \\ n \equiv a (\mathrm{mod}\, k)}} \frac{1}{2\pi i n} \right|^2 \quad \text{as } x \text{ is an integer and thus} \ll \log^2 kt.$$

As x is an integer,

$$x^{-s} \cdot \left(x - \frac{a}{k} - \left[x - \frac{a}{k} \right] - \frac{1}{2} \right) = x^{-s} \cdot \left(\frac{1}{2} - \frac{a}{k} \right).$$

Thus,

$$\left(\frac{1}{2} - \frac{a}{k} \right) + \sum_{n \leqslant k} \frac{e^{-2\pi i n a/k}}{2\pi i n} - \sum_{n \leqslant k} \frac{e^{2\pi i n a/k}}{2\pi i n}$$

$$= - \sum_{|n| > k} \frac{e^{-2\pi i n a/k}}{2\pi i n}.$$

Hence,

$$\zeta_x\left(s, \frac{a}{k}\right) = \frac{x^{1-s}}{s-1} - x^{-s} \cdot \sum_{|n|>k} \frac{e^{-2\pi i n a/k}}{2\pi i n}$$

$$+ \sum_{n \leqslant y+k} e^{-2\pi i n a/k} \cdot \int_x^N \frac{e^{2\pi i n u}}{u^s} du$$

$$+ \sum_{n \leqslant k} e^{2\pi i n a/k} \cdot \int_x^N \frac{e^{-2\pi i n u}}{u^s} du + O(tN^{-\sigma})$$

$$+ R_2\left(s, x, \frac{a}{k}\right),$$

where $\Sigma_{a=1}^{k} |R_2(s, x, (a/k))|^2 \ll x^{-2\sigma} \log^2 kt$.

Next, we show

$$\sum_{a=1}^{k} \left| \sum_{|n|>k} \frac{e^{-2\pi i n a/k}}{2\pi i n} \right|^2 = \sum_{a=1}^{k-1} \left| \sum_{|n|>k} \frac{e^{-2\pi i n a/k}}{2\pi i n} \right|^2 \ll 1.$$

We know for real x, a and b,

$$\sum_{a < n \leqslant b} e^{\pm 2\pi i n x} \ll \frac{1}{\|x\|}.$$

Here $\|x\| = \min_n |x - n|$, where n varies over integers. This gives, by partial summation for large M,

$$\sum_{n=k+1}^{M} \frac{e^{\pm 2\pi i n x}}{2\pi i n} \ll \frac{1}{k\|x\|}.$$

Thus

$$\sum_{a=1}^{k-1}\left|\sum_{|n|>k}\frac{e^{-2\pi ina/k}}{2\pi in}\right|^2$$
$$\ll \sum_{a=1}^{k-1}\left\|\frac{a}{k}\right\|^{-2}\cdot k^{-2} \ll \sum_{\substack{a\neq 0\\ -k/2<a\leqslant k/2}} k^{-2}\cdot\frac{k^2}{a^2} \ll 1.$$

Thus

$$\zeta_x\left(s,\frac{a}{k}\right)-\frac{x^{1-s}}{s-1}=\sum_{n\leqslant y+k} e^{-2\pi ina/k}\cdot\int_x^N \frac{e^{2\pi inu}}{u^s}du$$
$$+\sum_{n\leqslant k} e^{2\pi ina/k}\cdot\int_x^N \frac{e^{-2\pi inu}}{u^s}du+O(t\cdot N^{-\sigma})+R_3\left(s,x,\frac{a}{k}\right),$$

where $\Sigma_{a=1}^{k}|R_3(s,x,(a/k))|^2 \ll x^{-2\sigma}\log^2 kt$.
Thus,

$$\zeta_x\left(s,\frac{a}{k}\right)-\frac{x^{1-s}}{s-1}=\sum_{n\leqslant y} e^{-2\pi ina/k}\cdot\left(\int_0^\infty-\int_0^x-\int_N^\infty\right)\frac{e^{2\pi inu}}{u^s}du$$
$$+\sum_{y<n\leqslant y+k} e^{-2\pi ina/k}\cdot\int_x^N \frac{e^{2\pi inu}}{u^s}du$$
$$+\sum_{n\leqslant k} e^{2\pi ina/k}\cdot\int_x^N \frac{e^{-2\pi inu}}{u^s}du+O(tN^{-\sigma})+R_3\left(s,x,\frac{a}{k}\right).$$

If $k\leqslant y$, we write

$$\sum_{n\leqslant k} e^{2\pi ina/k}\cdot\int_x^N \frac{e^{-2\pi inu}}{u^s}du$$
$$=\sum_{n\leqslant k} e^{2\pi ina/k}\cdot\left(\int_0^\infty-\int_0^x-\int_N^\infty\right)\frac{e^{-2\pi inu}}{u^s}du.$$

Noting

$$\int_0^\infty \frac{e^{2\pi inu}}{u^s}du=K(s)\cdot n^{s-1};\quad \int_N^\infty \frac{e^{2\pi inu}}{u^s}du\ll N^{-\sigma},$$
$$\int_0^\infty \frac{e^{-2\pi inu}}{u^s}du=\overline{K(\bar{s})}\cdot n^{s-1};\quad \int_N^\infty \frac{e^{-2\pi inu}}{u^s}du\ll N^{-\sigma},$$

so that $|K(\bar{s})\cdot\Sigma_{n\leqslant k} e^{2\pi ina/k}\cdot n^{s-1}|$ is small; and noting for $y\geqslant k$,

$$\sum_{a=1}^{k}\left|\sum_{n\leqslant k} e^{2\pi ina/k}\int_0^x \frac{e^{-2\pi inu}}{u^s}du\right|^2$$
$$=k\sum_{n\leqslant k}\left|\int_0^x \frac{e^{-2\pi inu}}{u^s}du\right|^2 \ll k\sum_{n\leqslant k}\frac{x^{-2\sigma}}{y^2}\ll x^{-2\sigma}$$

and $\Sigma_{a=1}^{k} |x^{1-s}/(s-1)|^2 \ll x^{-2\sigma}$ and on letting $N \to \infty$, we find that for $y \geqslant k$,

$$\zeta\left(s, \frac{a}{k}\right) = \sum_{m \leqslant x-1} \left(m + \frac{a}{k}\right)^{-s} + K(s) \cdot \sum_{n \leqslant y} e^{-2\pi i n a/k} \cdot n^{s-1}$$

$$- \sum_{n \leqslant y} e^{-2\pi i n a/k} \cdot \int_0^x \frac{e^{2\pi i n u}}{u^s} du + \sum_{y < n \leqslant y+k} e^{-2\pi i n a/k} \cdot \int_x^\infty \frac{e^{2\pi i n u}}{u^s} du$$

$$+ R_4\left(s, x, \frac{a}{k}\right),$$

where $\Sigma_{a=1}^{k} |R_4(s, x, (a/k))|^2 \ll x^{-2\sigma} \log^2 kt$. Next let $y < k$. In that case

$$\sum_{n \leqslant k} e^{2\pi i n a/k} \cdot \int_x^N \frac{e^{-2\pi i n u}}{u^s} du = \left(\sum_{n \leqslant y} + \sum_{y < n \leqslant k}\right) e^{2\pi i n a/k} \cdot \int_x^N \frac{e^{-2\pi i n u}}{u^s} du.$$

Next

$$\sum_{a=1}^{k} \left| \sum_{y < n \leqslant k} e^{2\pi i n a/k} \cdot \int_x^N \frac{e^{-2\pi i n u}}{u^s} du \right|^2$$

$$= k \sum_{y < n \leqslant k} \left| \int_x^N \frac{e^{-2\pi i n u}}{u^s} du \right|^2 \ll k \sum_{y < n \leqslant k} \frac{x^{-2\sigma}}{n^2} \ll k x^{-2\sigma} \cdot \min(1, 1/y).$$

As before, we find

$$\sum_{n \leqslant y} e^{2\pi i n a/k} \cdot \int_x^N \frac{e^{-2\pi i n u}}{u^s} du$$

$$= \overline{K(\bar{s})} \cdot \sum_{n \leqslant y} e^{2\pi i n a/k} \cdot n^{s-1} - \sum_{n \leqslant y} e^{2\pi i n a/k} \cdot \int_0^x \frac{e^{-2\pi i n u}}{u^s} du + O(N^{-\sigma} \log kt).$$

Noting $|\overline{K(\bar{s})} \cdot \Sigma_{n \leqslant y} e^{2\pi i n a/k} \cdot n^{s-1}|$ is very small and

$$\sum_{a=1}^{k} \left| \sum_{n \leqslant y} e^{2\pi i n a/k} \cdot \int_0^x \frac{e^{-2\pi i n u}}{u^s} du \right|^2$$

$$= k \sum_{n \leqslant y} \left| \int_0^x \frac{e^{-2\pi i n u}}{u^s} du \right|^2 \ll k \sum_{n \leqslant y} \frac{x^{-2\sigma}}{y^2} \ll \frac{k}{y} \cdot x^{-2\sigma};$$

and

$$\sum_{a=1}^{k} \left| \frac{x^{1-s}}{s-1} \right|^2 \ll \frac{k}{y^2} \cdot x^{-2\sigma},$$

we find that for $y < k$,

$$\zeta\left(s, \frac{a}{k}\right) = \sum_{m \leqslant x-1} \left(m + \frac{a}{k}\right)^{-s} + K(s) \cdot \sum_{n \leqslant y} e^{-2\pi i n a/k} \cdot n^{s-1}$$

$$- \sum_{n \leqslant y} e^{-2\pi i n a/k} \cdot \int_0^x \frac{e^{2\pi i n u}}{u^s} du + \sum_{y < n \leqslant y+k} e^{-2\pi i n a/k} \cdot \int_x^\infty \frac{e^{2\pi i n u}}{u^s} du$$

$$+ \left(\frac{x^{1-s}}{s-1} - \sum_{n \leqslant y} e^{2\pi i n a/k} \cdot \int_0^x \frac{e^{-2\pi i n u}}{u^s} du + \sum_{y < n \leqslant k} e^{-2\pi i n u/k} \cdot \int_x^\infty \frac{e^{-2\pi i n u}}{u^s} du\right)$$

$$+ R_4\left(s, x, \frac{a}{k}\right),$$

where

$$\sum_{a=1}^{k}\left|R_4\left(s,x,\frac{a}{k}\right)\right|^2 \ll x^{-2\sigma}\log^2 kt,$$

and

$$\sum_{a=1}^{k}\left(\left|\sum_{n\leqslant y} e^{2\pi ina/k}\cdot\int_0^x \frac{e^{-2\pi inu}}{u^s}du\right|^2 + \left|\sum_{y<n\leqslant k} e^{2\pi ina/k}\cdot\int_x^\infty \frac{e^{-2\pi inu}}{u^s}du\right|^2\right) \ll kx^{-2\sigma}\cdot\min(1, 1/y).$$

This completes the proof of our theorem.

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