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On asymptotic distribution on the a-adic integers

R NAIR

Department of Pure Mathematics, University of Liverpool, P.O. Box 147, Liverpool L69 3BX, U.K.

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Abstract. We show that the values of a polynomial with **a**-adic coefficients at integer and rational prime arguments are asymptotically distributed on the **a**-adic integers and that the integer parts of certain sequences known to be uniformly distributed modulo one, are uniformly distributed on the **a**-adic integers.

Keywords. **a**-adic integers; asymptotic distribution.

1. Introduction

By construction the natural numbers are dense in the p-adic integers. In fact, they can by shown to be uniformly distributed there. On the other hand the squares of natural numbers are dense but not uniformly distributed. Motivated by this observation in this paper we study the asymptotic distribution of certain sequences on the **a**-adic integers. To fix ideas we begin with some basic information about the **a**-adic integers. Let $\mathbf{a} = (a_n)_{n=0}^{\infty}$ be a sequence of rational integers greater than one. Using the notation of [HR, § 10] we define the **a**-adic integers $\Delta_{\mathbf{a}}$ to be the set of infinite sequences $(x_n)_{n=0}^{\infty}$ in $\prod_{n=1}^{\infty}\{0,1,\ldots,a_n-1\}$.

For $\mathbf{x} = (x_n)_{n=0}^{\infty}$ and $\mathbf{y} = (y_n)_{n=0}^{\infty}$ let $\mathbf{z} = (z_n)_{n=0}^{\infty}$ be defined as follows. Write $x_0 + y_0 = t_0 a_0 + z_0$, where $z_0 \in \{0,1,\ldots,a_0-1\}$ and t_0 is a rational integer. Suppose $z_0,\ldots,z_k$ and $t_0,\ldots,t_k$ have been defined. Then write $x_{k+1} + y_{k+1} + t_k = t_{k+1}a_{k+1} + z_{k+1}$, where $z_{k+1} \in \{0,1,\ldots,a_{k+1}-1\}$ and t_{k+1} is a rational integer. We have thus inductively defined the sequence $\mathbf{z} = (z_n)_{n=0}^{\infty}$, which we deem to be $\mathbf{x} + \mathbf{y}$. The binary operation $+$ which we call addition makes $\Delta_{\mathbf{a}}$ an abelian group.

For each non-negative integer k, let

$$\Lambda_k = \{x \in \Delta_{\mathbf{a}} : x_n = 0 \quad \text{if} \quad n < k\}.$$

These sets form a basis at $\mathbf{0} = (0,0,\ldots)$ for a topology on $\Delta_{\mathbf{a}}$. With respect to this topology $\Delta_{\mathbf{a}}$ is compact and the group operations are continuous, making $\Delta_{\mathbf{a}}$ a compact Abelian topological group. A second binary operation called multiplication, denoted by $\times$ and compatible with addition is defined as follows. Let $\mathbf{u} = (1,0,0,\ldots)$. Note that $(n\mathbf{u})_{n=0}^{\infty}$ is dense in $\Delta_{\mathbf{a}}$. First on $(n\mathbf{u})_{n=0}^{\infty}$ define $k_1\mathbf{u} \times k_2\mathbf{u}$ to be $k_1k_2\mathbf{u}$. Deeming multiplication to be continuous on $\Delta_{\mathbf{a}}$ defines it off $(n\mathbf{u})_{n=0}^{\infty}$. The binary operations addition and multiplication makes $\Delta_{\mathbf{a}}$ a topological ring.

For each non-negative integer n, let $\lambda_n(A)$ denote the measure on $\{0,1,\ldots,a_n-1\}$ given by $\lambda_n(A) = \text{card}(A)/a_n$. Haar measure is the corresponding product measure on $\Delta_{\mathbf{a}}$.

The dual group of $\Delta_{\mathbf{a}}$, which we denote $\mathbf{Z}(\mathbf{a}^{\infty})$ consists of all rationals $t = l/A_r$ where $A_r = a_0\ldots a_r$ and $0 \leqslant l \leqslant A_r$ for some non-negative integer r. To evaluate a character χ_t at $\mathbf{x}$ in $\Delta_{\mathbf{a}}$ we write

$$\chi_t(\mathbf{x}) = e\left(\frac{l}{A_r}(x_0 + a_0x_1 + \cdots + a_0\cdots a_{r-1}x_r)\right),$$

where as usual, for a real number x, $e(x)$ denotes $e^{2\pi i x}$.

For a sequence $(x_n)_{n=0}^{\infty}$ in Δ_a, let $A(E;N)$ denote the number of elements of the set $\{x_0,\ldots,x_{n-1}\}$ that belong to E. If for every set E belonging to the algebra generated by the sets $\Lambda_k(k=0,1,\ldots)$ and their translates, the limit

$$\mu(E)=\lim_{N\to\infty}\frac{A(E;N)}{N}$$

exists, we say $(x_n)_{n=0}^{\infty}$ is asymptotically distributed on Δ_a with distribution μ. If μ coincides with Haar measure λ, we say $(x_n)_{n=0}^{\infty}$ is uniformly distributed.

Let $\alpha_0,\ldots,\alpha_k$ be an arbitrary but fixed set of elements of Δ_a and let

$$\rho(x)=\alpha_k x^k+\cdots+\alpha_1 x+\alpha_0.$$

In § 2 we prove the following theorem.

Theorem 1. *Suppose $(p_n)_{n=0}^{\infty}$ denotes the sequence of rational primes. The sequences $(\rho(n))_{n=0}^{\infty}$ and $(\rho(p_n))_{n=0}^{\infty}$ are asymptotically distributed on Δ_a. Moreover their distributions are continuous with respect to Haar measure.*

It is known [Me] that except in special cases the sequences considered in Theorem 1 are not uniformly distributed.

Suppose the sequence of integers $(k_n)_{n=1}^{\infty}$ is described by any of the following constructions:

(i) Suppose $S=(n_k)_{n=1}^{\infty}\subseteq \mathbf{N}$ is a strictly increasing sequence of natural numbers. By identifying S with its characteristic function I_S we may view it as a point in $\Lambda=\{0,1\}^{\mathbf{N}}$ the set of maps from $\mathbf{N}$ to $\{0,1\}$. We may endow Λ with a probability measure by viewing it as a Cartesian product $\Lambda=\prod_{n=1}^{\infty}X_n$ where for each natural number n we have $X_n=\{0,1\}$ and specify the probability π_n on X_n by $\pi_n(\{1\})=q_n$ with $0\leqslant q_n\leqslant 1$ and $\pi_n(\{0\})=1-q_n$ such that $\lim_{n\to\infty}q_n n=\infty$. The desired probability measure on Λ is the corresponding product measure $\pi=\prod_{n=1}^{\infty}\pi_n$. The underlying σ-algebra β is that generated by the 'cylinders'

$$\{\lambda=(\lambda_n)_{n=1}^{\infty}\in\Lambda:\lambda_{i_1}=\alpha_{i_1},\ldots,\lambda_{i_r}=\alpha_{i_r}\}$$

for all possible choices of $i_1,\ldots,i_r$ and $\alpha_{i_1},\ldots,\alpha_{i_r}$. Let $(k_n)_{n=1}^{\infty}$ be almost any point in Λ with respect to the measure π.

For a real number x let $[x]$ denote the greatest rational integer not exceeding x.

(ii) $k_n=[P(n)](n=1,2,\ldots)$ where

$$P(z)=\alpha_k x^k+\alpha_{k-1}x^{k-1}+\cdots+\alpha_0$$

for real numbers $\alpha_0,\ldots,\alpha_k$ such that the numbers $\alpha_1,\ldots,\alpha_k$ are not all rational multiples of a single real number;

(iii) $k_n=[P(p_n)]$ $(n=1,2,\ldots)$ where $(p_n)_{n=1}^{\infty}$ denotes the sequence of rational primes and $P(z)$ is as in (ii);

(iv) $k_n=[f(n)]$ $(n=1,2,\ldots)$ where $f(z)$ denotes a non-polynomial entire function which is real on the real numbers and such that $|f(z)|\ll e^{(\log z)^{\alpha}}$ with $\alpha<\frac{4}{3}$;

(v) $k_n=[f(p_n)]$ $(n=1,2,\ldots)$ where $f(z)$ is as in (iv) and p_n denotes the nth rational prime;

(vi) $k_n=[n^{\alpha}]$ $(n=1,2,\ldots)$ where α is any real number greater than one.

(vii) $k_n=[a_n\cos(a_n x)]$ $(n=1,2,\ldots)$ for a strictly increasing sequence of integers $(a_n)_{n=1}^{\infty}$ and almost all x with respect to Lebesgue measure;

(viii) $k_n = [a_n \cos(a_n x)]$ $(n = 1, 2, \ldots)$ for a strictly increasing sequence of integers $(a_n)_{n=1}^{\infty}$ such that $a_n = O(n^p)$ and $p > 1$ and all x outside a set of Hausdorff dimension not greater than $1 - (1/(4p + \frac{1}{2}))$;
(ix) $k_n = [g_n(x)]$ $(n = 1, 2, \ldots)$ for almost all x with respect to Lebesgue measure in $[a, b]$ where $(g_n(x))_{n=1}^{\infty}$ is a sequence of continuously differentiable functions defined on $[a, b]$ satisfying the following hypothesis. For each pair of distinct natural number m and n we have

(a) $g_n'(x) - g_m'(x)$ is monotone on $[a, b]$ and
(b) there is an absolute constant λ such that

$$|g_n'(x) - g'_m(x)| \geqslant \lambda > 0.$$

(x) $k_n = [g_n(x)]$ $(n = 1, 2, \ldots)$ for all x lying outside a set of Hausdorff dimension at most $1 - (1/p)$ in $[a, b]$ where $(g_n(x))_{n=1}^{\infty}$ is a sequence of continuously differentiable functions defined on $[a, b]$ satisfying the hypothesis (a), (b) of (ix) and in addition
(c) for all x in $[a, b]$ we have

$$\sup_{x \in [a, b]} |g_n'(x)| = O(n^p)$$

for some $p > 1$ and with an implied constant independent of x and
(d) for each pair of distinct positive integers m and n the function

$$\frac{g_n'(x) g_m'(x)}{g_m'(x) - g_n'(x)}$$

is monotonic on $[a, b]$.

The proof of the following theorem is spread over §§ 3 and 4.

Theorem 2. *In each of the instances* (i)–(x) *the sequence of integers* $(k_n)_{n=1}^{\infty}$ *is uniformly distributed on* $\Delta_{\mathbf{a}}$.

2. Proof of Theorem 1

To prove Theorem 1 we need some lemmas. The next two lemmas generalize to the **a**-adic integer lemmas already known in the case where for each positive integer i, we have $a_i = p$ for each for a fixed prime p[Ch].

Lemma 3. *The sequence* $(x_n)_{n=0}^{\infty}$ *in* $\Delta_{\mathbf{a}}$ *is asymptotically distributed if and only if for each* χ_t *in* $\mathbf{Z}(\mathbf{a}^{\infty})$

$$c_t = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} \chi_t(x_n)$$

exists. Moreover if

$$\mu_r(\alpha) = \lim_{N \to \infty} \frac{A(\alpha + \Lambda_r;\ N)}{N},$$

then

$$c_t = \frac{1}{A_r} \sum_{j=0}^{A_r - 1} \mu_r(j) e(tj)$$

for each $r > 1$ *and if* $Z_r = \{(l/A_s) \in \mathbf{Z}(\mathbf{a}^\infty) : 0 \leqslant s \leqslant r\}$,

$$\mu_r(\alpha) = \frac{1}{A_r} \sum_{t \in Z_r} c_t e(-t\alpha).$$

Proof. For each rational integer j an element y of $\Delta_{\mathbf{a}}$ being in $j + \Lambda_k$ implies that $e(ty) = e(tj)$. So for any sequence $(y_n)_{n=0}^{\infty}$ we have

$$\frac{1}{N} \sum_{n=0}^{N-1} e(ty_n) = \sum_{j=0}^{A_r - 1} e(jt) N^{-1} |\{y_n \in j + \Lambda_r : 1 \leqslant n \leqslant N\}|.$$

Therefore if $(y_n)_{n=0}^{\infty}$ is asymptotically distributed

$$c_t = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_t(y_n)$$

exists. On the other hand letting

$$B_N = \sum_{t \in Z_r} e(-t\alpha) \sum_{n=0}^{N-1} e(ty_n) = \sum_{n=0}^{N-1} \sum_{t \in Z_r} e(t(y_n - \alpha)).$$

If y_n is in $j + \Lambda_k$ this sum is $A_r |\{y_n \in j + \Lambda_r : 1 \leqslant n \leqslant N\}|$. This means that,

$$\lim_{N \to \infty} \frac{1}{N} |\{y_n \in j + \Lambda_r : 1 \leqslant n \leqslant N\}| = \frac{1}{A_r} \sum_{t \in Z_r} e(-tj) c_t,$$

as required. ■

Lemma 4. *Let* $j(\alpha) = \lim_{k \to \infty} \mu_k(\alpha)$, *which always exists as consequence of the monotonicity of* $\mu_k(\alpha)$ *as a function of* k. *Then if* $(y_n)_{n=0}^{\infty}$ *is asymptotically distributed on* $\Delta_{\mathbf{a}}$

$$\lim_{r \to \infty} \frac{1}{A_r} \sum_{t \in Z_r} |c_t|^2 = \sum_{\alpha \in D} j^2(\alpha),$$

where D *is the countable set where* $j(\alpha) \neq 0$.

Proof. By Lemma 3

$$|c_t|^2 = c_t \bar{c}_t = \sum_{0 \leqslant h \leqslant A_r - 1} e(th) \mu_r(h) \sum_{0 \leqslant m \leqslant A_r - 1} e(-tm) \mu_r(m).$$

Thus

$$\sum_{t \in Z_r} |c_t|^2 = \sum_{0 \leqslant h,\, m \leqslant A_r - 1} \mu_r(h) \mu_r(m) \sum_{t \in Z_r} e(t(h - m)).$$

Now $\Sigma_{t \in Z_r} e(t(h-m))$ is equal to A_r if $h = m$ and 0 otherwise. In addition $\mu_r(m)$ is the constant value of $\mu_r(\alpha)$ on $\alpha + \Lambda_k$. Therefore

$$\frac{1}{A_r} \sum_{t \in Z_r} |c_t|^2 = \sum_{0 \leqslant m \leqslant A_r - 1} \mu_r^2(m) = \int_{\Delta_{\mathrm{a}}} \mu_r(\alpha) \mathrm{d}\bar{\mu}(\alpha),$$

where the measure $\bar{\mu}$ is that whose existence follows by Lemma 3 from the sequence $(c_t)_{t \in \mathbf{Z}(\mathbf{a}^\infty)}$. Letting r tend to infinity, we have

$$\lim_{r \to \infty} \frac{1}{A_r} \sum_{t \in Z_r} |c_t|^2 \int_{\Delta_{\mathbf{a}}} j(\alpha) \mathrm{d}\bar{\mu}(\alpha) = \sum_{\alpha \in D} j^2(\alpha)$$

as required.

Consider $G^{(1)}\colon \mathbf{Z}(\mathbf{a}^\infty)\to\mathbf{C}$ defined by

$$G^{(1)}(\chi_{l/A_r}) = G^{(1)}_{\alpha_0,\ldots,\alpha_k}(\chi_{l/A_r}) = \frac{\chi_{l/A_r}(\alpha_0)}{D_r}\sum_{m=1}^{D_r} e^{2\pi i(\gamma(m)/D_r)},$$

for all l/A_r in $\mathbf{Z}(\mathbf{a}^\infty)$. Here the positive integer D_r and the polynomial γ of degree k with non-negative integer coefficients are described as follows. Let

$$l_j = l(\alpha_j(0) + \alpha_j(1)a_0 + \alpha_j(2)a_0a_1 + \cdots \alpha_j(r-1)a_0\cdots\alpha_{r-2}),\quad j=1,\ldots,k).$$

Here $\alpha_j(r)$ denotes the rth terms of α_j viewed as a sequence. Let m_j/B_j with $(m_j, B_j)=1$ denote l_j/A_r in reduced form. We use D_r to denote the least common multiple of $B_1,\ldots,B_k$ and define $\gamma(x)$ by

$$\frac{\gamma(x)}{D_r} = \frac{m_k}{B_k}x^k + \cdots + \frac{m_1}{B_1}x.$$

Also consider $G^{(2)}\colon\mathbf{Z}(\mathbf{a}^\infty)\to\mathbf{C}$ defined by

$$G^{(2)}(\chi_{l/A_r}) = G^{(2)}_{\alpha_0\cdots\alpha_l}(\chi_{l/A_r}) = \frac{\chi_{l/A_r}(\alpha_0)}{\phi(D_r)}\sum_{\substack{m=1\\(D_r,m)=1}}^{D_r} e^{2\pi i(\gamma(m)/D_r)},$$

where ϕ denotes Euler's totient function.

Lemma 5. *For all* χ_{l/A_r}, *we have*

$$G^{(1)}(\chi_{l/A_r}) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{l/A_r}(\rho(n)) \tag{1}$$

and

$$G^{(2)}(\chi_{l/A_r}) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{l/A_r}(\rho(p_n)). \tag{2}$$

Proof. We first prove (1). Note that

$$\begin{aligned}\chi_{l/A_r}(\rho(n)) &= \chi_{l/A_r}(\alpha_0 + \alpha_1 n + \cdots + \alpha_k n^k)\\ &= \chi_{l/A_r}(\alpha_0)\prod_{j=1}^{k}(\chi_{l/A_r}(\alpha_j))^{n^j}\\ &= \chi_{l/A_r}(\alpha_0)\prod_{j=1}^{k}(e^{2\pi i(l/A_r)n^j(\alpha_j(0)+\alpha_j(1)a_0+\cdots+\alpha_j(r-1)a_0\cdots a_{r-2})}),\\ &= \chi_{l/A_r}(\alpha_0)\prod_{j=1}^{k}(e^{2\pi i(l_j/A_r)n^j}) = \chi_{l/A_r}(\alpha_0)\prod_{j=1}^{k}(e^{2\pi i(m_j/B_j)n^j}) = \chi_{l/A_r}(\alpha_0)e^{2\pi i(\gamma(n)/D_r)}.\end{aligned}$$

If we now partition the integers into their residue classes modulo D_r it follows immediately that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{l/A_r}(\rho(n)) = \frac{\chi_{l/A_r}(\alpha_0)}{D_r}\sum_{m=1}^{D_r} e^{2\pi i(\gamma(m)/D_r)},$$

proving (1). We now show (2). Arguing as in (1)

$$\frac{1}{N}\sum_{n=1}^{N}\chi_{l/A_r}(\rho(p_n)) = \frac{\chi_{l/A_r}(\alpha_0)}{\pi_N}\sum_{1\leqslant p\leqslant N} e^{2\pi i(\gamma(p)/D_r)}. \tag{3}$$

Let $\Lambda : \mathbf{N} \mapsto \mathbf{R}$ be the Von Mangoldt function, defined by $\Lambda(n) = \log_e p$ if $n = p^l$ for some prime p and positive integer l, and zero otherwise. Using partial summation we readily see that

$$\frac{1}{\pi_N} \sum_{1 \leqslant p \leqslant N} e^{2\pi i(\gamma(p)/D_r)} = \frac{1}{N} \sum_{1 \leqslant n \leqslant N} \Lambda(n) e^{2\pi i(\gamma(n)/D_r)} + O((\log N)^{-1}). \tag{4}$$

The Siegel–Walfisz prime number theorem for arithmetic progressions [D, p. 133] says that for fixed positive u, if $1 \leqslant D_r \leqslant (\log N)^u$, and $(m, D_r) = 1$, then for some $C > 0$,

$$\sum_{\substack{1 \leqslant n \leqslant N \\ n \equiv m (\mathrm{mod}\, D_r)}} \Lambda(n) = \frac{N}{\phi(D_r)} + o(N e^{-C(\log N)^{1/2}}).$$

Now note that

$$\sum_{n=1}^{N} \Lambda(n) e^{2\pi i(\gamma(n)/D_r)} = \left(\sum_{\substack{m=1 \\ (m, D_r) = 1}}^{D_r} e^{2\pi i(\gamma(m)/D_r)} \right) \left(\sum_{\substack{1 \leqslant n \leqslant N \\ n \equiv m (\mathrm{mod}\, D_r)}} \Lambda(n) \right)$$
$$+ O\left(\sum_{p^l \leqslant N;\ p | D_r} \Lambda(p^l) e^{2\pi i(\gamma(p^l)/D_r)} \right).$$

Using the fact that the third sum on the right is $O((\log N)(\log\log N))$ and the Siegel–Walfisz theorem

$$\frac{1}{N} \sum_{n=1}^{N} \Lambda(n) e^{2\pi i(\gamma(n)/D_r)} = \left(\frac{1}{\phi(D_r)} \sum_{\substack{m=1 \\ (m, D_r) = 1}}^{D_r} e^{2\pi i(\gamma(m)/D_r)} \right) + O\left(\frac{(\log N)(\log\log N)}{N} \right). \tag{5}$$

Combining (3), (4) and (5) and letting N tend to infinity (4) is proved. ∎

To complete the proof of Theorem 1 we need Weyl's inequality [Va].

Lemma 6. *Suppose that the integers a and q are coprime and that $|\alpha_k - (a/q)| \leqslant q^{-2}$ where*

$$\phi(x) = \alpha_k x^k + \cdots + \alpha_1 x + \alpha_0.$$

Then

$$\sum_{x=1}^{Q} e(\phi(x)) \ll Q^{1+\varepsilon}(q^{-1} + Q^{-1} + qQ^{-k})^{1/K},$$

where $K = 2^{k-1}$.

We now complete the proof of Theorem 1. As a consequence of Lemma 6 we see that there exists a $\delta > 0$ such that $|G^{(1)}| \ll A_r^{-\delta}$. Also

$$\sum_{\substack{1 \leqslant m \leqslant q \\ (m,q) = 1}} e(a\gamma(m)q^{-1}) = \sum_{1 \leqslant m \leqslant q} e(a\gamma(m)q^{-1}) - \sum_{p/q} \sum_{\substack{1 \leqslant m \leqslant q \\ p/m}} e(a\gamma(m)q^{-1}).$$

Hence we also have $|G^{(2)}| \ll A_r^{-\delta}$ for a possibly different δ. This means that in either case

$$\frac{1}{A_r} \sum_{t \in Z_r} |c_t|^2 \leqslant \frac{1}{A_r} \sum_{s=0}^{r} A_s^{1-2\delta},$$

which tends to zero as r tends to infinity because $A_r \geqslant 2^r (r = 1, 2, \ldots)$, completing the proof of Theorem 1. ■

3. Proof of Theorem 2 (i)

We now turn to the proof of Theorem 2. We begin with two classical inequalities.

Let $(r_n(x))_{n=0}^{\infty}$ denote the sequence of Rademacher functions. That is we define $r_0(x)$ to be 1 on $(0, \frac{1}{2}]$, $r_0(x)$ to be -1 on $(\frac{1}{2}, 1]$, extend this definition periodically to the whole of $\mathbf{R}$ and define $r_n(x)$ to be $r_0(2^n x)$ $(n = 1, 2, \ldots)$. We have the following well-known inequality of Khinchine [Z1, p. 213].

Lemma 7. Suppose $q \geqslant 1$. Then for each set of complex scalars $(a_n)_{n=0}^{\infty}$, we have positive constants A_q and B_q, such that

$$A_q\left(\sum_{n=0}^{N} |a_n|^2\right)^{1/2} \leqslant \left(\int_0^1 \left|\sum_{n=0}^{N} a_n r_n(x)\right|^q \mathrm{d}x\right)^{1/q} \leqslant B_q\left(\sum_{n=0}^{N} |a_n|^2\right)^{1/2}.$$

Further A_q is absolute (that is independent of q) and $B_q \leqslant c q^{1/2}$, where c is absolute.

The following inequality is due to Bernstein [Z2, p. 11].

Lemma 8. There is an absolute constant $A > 0$ such that

$$\|f'\|_\infty \leqslant AN\|f\|_\infty$$

for every positive integer N, and every trigonometric polynomial f with spectrum in $[-N, N]$.

To prove Theorem 2 we need a series of lemmas, which in their turn need some additional ideas to be stated. For a real valued function F defined on the probability space $P = (\Lambda, \beta, \pi)$ we let $m_{F,P}$ denote its distribution function. That is

$$m_{F,P}(x) = \pi\{\lambda \in \Lambda : F(\lambda) \leqslant x\}.$$

We say $m_{F,P}$ is symmetric if $m_{F,P}(x) + m_{F,P}(-x) = c$ for some constant independent of x and that the function F is symmetric with respect to probability space P if the corresponding distribution $m_{F,P}$ is symmetric. Note that the function $F^s(\lambda, \lambda') = F(\lambda) - F(\lambda')$, called the symmetrization of F, is symmetric with respect to the product probability space $P \otimes P = (\Lambda \otimes \Lambda,\ \beta \otimes \beta,\ \pi \otimes \pi)$. Let $f_n(\lambda) = I_{c_n}(\lambda) - q_n$, where I_{c_n} is the characteristic function of the cylinder

$$c_n = \{\lambda \in \Lambda : \lambda_n = 1\}.$$

Lemma 9. If $(r_k(x))_{n=1}^{\infty}$ is the sequence of Rademacher functions and $(f_k^s(\lambda, \lambda'))_{n=1}^{\infty}$ is the sequence of symmetrizations of the sequence of functions $(f_k(\lambda))_{n=1}^{\infty}$ then for $p \geqslant 1$,

$$\left(\int_0^1 \int_{\Lambda\otimes\Lambda} \left|\sum_{k=1}^{N} r_k(t) f_k^s(\lambda, \lambda')\right|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda')\right)^{1/p}$$
$$= \left(\int_0^1 \int_{\Lambda\otimes\Lambda} \left|\sum_{k=1}^{N} f_k^s(\lambda, \lambda')\right|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda')\right)^{1/p}.$$

Proof. First note that for any two distinct numbers t_1 and t_2 in $[0, 1)$ which are not diadic rationals, the functions $r_k(t_1) f_k^s(\lambda, \lambda')$ and $r_k(t_2) f_k^s(\lambda, \lambda')$ have the same distribution function, that is

$$m_{r_k(t_1)f_k^s,\pi\otimes\pi} = m_{r_k(t_2)f_k^s,\pi\otimes\pi}, \quad (k = 1, 2, \ldots). \tag{6}$$

For an arbitrary measurable subset T of $\mathbf{R}$ let

$$A = \{(x_1, \ldots, x_N) \in \mathbf{R}^N : x_1 + \cdots + x_N \in T\}.$$

In addition for non-diadic t_1 in $[0, 1)$, the sequence of functions $(r_k(t_1)f_k^s(\lambda, \lambda'))_{n=1}^{\infty}$ is independent on $P \otimes P$, which means that

$$\pi\otimes\pi(\{(\lambda,\lambda') : (r_1(t_1)f_1^s(\lambda,\lambda'), \ldots, r_N(t_1)f_N^s(\lambda,\lambda') \in A\})$$

$$= \int \cdots \int_A \mathrm{d}m_{r_1(t_1)f_1^s,\pi\otimes\pi} \cdots \mathrm{d}m_{r_N(t_1)f_N^s,\pi\otimes\pi}.$$

But by (6) this is

$$\int \cdots \int_A \mathrm{d}m_{r_1(t_2)f_1^s,\pi\otimes\pi} \cdots \mathrm{d}m_{r_N(t_2)f_N^s,\pi\otimes\pi}$$

which in consequence tells us that

$$\pi\otimes\pi(\{(\lambda,\lambda') : (r_1(t_1)f_1^s(\lambda,\lambda'), \ldots, r_N(t_1)f_N^s(\lambda,\lambda') \in A\})$$
$$= \pi\otimes\pi(\{(\lambda,\lambda') : (r_1(t_2)f_1^s(\lambda,\lambda'), \ldots, r_N(t_2)f_N^s(\lambda,\lambda') \in A\}).$$

This tells us that

$$\int_{\Lambda\otimes\Lambda} \left| \sum_{k=1}^{N} r_k(t_1) f_k^s(\lambda,\lambda') \right|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda)\mathrm{d}t$$
$$= \int_{\Lambda\otimes\Lambda} \left| \sum_{k=1}^{N} r_k(t_2) f_k^s(\lambda,\lambda') \right|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda')$$

and hence that

$$\int_0^1 \int_{\Lambda\otimes\Lambda} \left| \sum_{k=1}^{N} r_k(t) f_k^s(\lambda,\lambda') \right|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda')\mathrm{d}t$$
$$= \int_{\Lambda\otimes\Lambda} \left| \sum_{k=1}^{N} r_k(t_2) f_k^s(\lambda,\lambda') \right|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda').$$

This means that choosing t_2 such that $r_k(t_2) = 1$ for all k in $[1, N] \cap \mathbf{Z}$ completes the proof of the lemma. ∎

Lemma 10. *Suppose $p \geqslant 1$ and that for the function F is defined on the probability space $P = (\Lambda, \beta, \pi)$. Suppose also that $\int_\Lambda |F(\lambda)|^p \mathrm{d}\pi(\lambda) = 1$. Suppose further that $\int_\Lambda |F(\lambda)| \mathrm{d}\pi(\lambda) \leqslant \frac{1}{4}$. Then $(\int_{\Lambda\otimes\Lambda} |F^s(\lambda,\lambda')|^p \mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda'))^{1/p} \geqslant c > 0$ for some absolute constant c.*

Proof. By Minkowski's inequality

$$\left(\int_{[|F|\leqslant 1]} |F^s|^p \mathrm{d}\pi(\lambda)\pi(\lambda')\right)^{1/p} \geqslant \left(\int_{[|F|\leqslant 1]} |F(\lambda)|^p \mathrm{d}\pi(\lambda)\pi(\lambda')\right)^{1/p}$$
$$- \left(\int_{[|F|\leqslant 1]} |F(\lambda')|^p \mathrm{d}\pi(\lambda)\pi(\lambda')\right)^{1/p}.$$

Note that $F(\lambda)$ and $F(\lambda')$ are independent on $P\otimes P$, hence

$$\left(\int_{[|F|\leqslant 1]}|F(\lambda')|^p\mathrm{d}\pi(\lambda)\pi(\lambda')\right)=\pi([|F|\leqslant 1])\left(\int_\Lambda |F(\lambda')|^p\pi(\lambda')\right)^{1/p}$$
$$=\pi([|F|\leqslant 1])\geqslant\frac{3}{4}.$$

This means that

$$\left(\int_{[|F|\leqslant 1]}|F^s|^p\mathrm{d}\pi(\lambda)\pi(\lambda')\right)^{1/p}\geqslant\left(\frac{3}{4}\right)^{1/p}-\left(\frac{1}{4}\right)^{1/p},$$

which completes the proof of the lemma. ∎

Lemma 11. *Suppose $p\geqslant 1$ and that for the function $F(\lambda)$ defined on the probability space $P=(\Lambda,\beta,\pi)$ we have $\int_\Lambda|F(\lambda)|^p\mathrm{d}\pi(\lambda)=1$ and $\int_\Lambda F(\lambda)\mathrm{d}\pi(\lambda)=0$ Suppose also that $\int_\Lambda|F(\lambda)|\mathrm{d}\pi(\lambda)\geqslant\frac{1}{4}$. Then $(\int_{\Lambda\otimes\Lambda}|F^s(\lambda,\lambda')|^p\mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda'))^{1/p}\geqslant c>0$ for some constant c dependent only on p.*

Proof. Note that

$$\int_{\Lambda\otimes\Lambda}|F^s(\lambda,\lambda')|^p\mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda')\geqslant\int_{[F(\lambda)\leqslant 0]\cap[F(\lambda')>0]}|F^s(\lambda,\lambda')|^p\mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda')$$

and that using the independence of $F(\lambda)$ and $F(\lambda')$ on $P\otimes P$ we have

$$\geqslant\int_{[F(\lambda)\leqslant 0]\cap[F(\lambda')>0]}F(\lambda')^p\mathrm{d}\pi(\lambda')=\pi[F(\lambda)\leqslant 0]\int_{[F(\lambda')>0]}F(\lambda')^p\mathrm{d}\pi(\lambda').$$

Now suppose $\pi[F(\lambda')>0]\geqslant\frac{1}{2}$ then noting that $\int_{[F(\lambda')>0]}F(\lambda')^p\mathrm{d}\pi(\lambda')$ is not less than 1/2 $(1/8)^p$ we have

$$\int_{\Lambda\otimes\Lambda}|F^s(\lambda,\lambda')|^p\mathrm{d}\pi(\lambda)\mathrm{d}\pi(\lambda')\geqslant\frac{1}{2}\left(\frac{1}{8}\right)^p.$$

If $\pi[F(\lambda')>0]<\frac{1}{2}$, the same estimate follows by reversing the roles of λ and λ'. This completes the proof of the lemma.

We now complete the proof of Theorem 2 (i). The following argument is adapted from [S, p. 523–4]. See also [Bo] for a similar argument.

For $p\geqslant 1$, let

$$B_p=\left(\int_\Lambda\left|\sum_{n=1}^N I_{c_n}\right|^p\mathrm{d}\pi(\lambda)\right)^{1/p}.$$

Then by Minkowski's inequality

$$B_p\leqslant\sum_{k=1}^N q_k+\left(\int_\Lambda\left|\sum_{k=1}^N(I_{c_k}-q_k)\right|^p\mathrm{d}\pi(\lambda)\right)^{1/p},$$

which by lemmas 10 and 11 is

$$\leqslant\sum_{k=1}^N q_k+2\left(\int_0^1\int_\Lambda\left|\sum_{k=1}^N r_k(t)(I_{c_k}-q_k)\right|^p\mathrm{d}\pi(\lambda)\mathrm{d}t\right)^{1/p}.$$

Using Lemma 7 this is

$$\leqslant \sum_{k=1}^{N} q_k + 2\sqrt{p}\left(\int_\Lambda \left(\sum_{k=1}^{N} |I_{c_k} - q_k|^2\right)^{p/2} \mathrm{d}\pi(\lambda)\right)^{1/p},$$

which is

$$\leqslant \sum_{k=1}^{N} q_k + 4\sqrt{p} B_p^{1/2}.$$

It therefore follows that there exists a positive constant C such that

$$B_p \leqslant C \max\left(p, \sum_{k=1}^{N} q_k\right) \tag{7}$$

and that

$$\left(\int_\Lambda \left|\sum_{k=1}^{N} \chi_{I/A_r}(k\eta)(I_{c_k} - q_k)\right|^p \mathrm{d}\pi(\lambda)\right)^{1/p} \leqslant C \max\left(p, p^{1/2}\left(\sum_{k=1}^{N} q_k\right)^{1/2}\right). \tag{8}$$

Let

$$E(\lambda) = \frac{1}{S_N} \sum_{n \in S_N} \chi_{I/A_r}(n\eta) - \frac{1}{\sum_{l=1}^{N} q_l} \sum_{k=1}^{N} q_k \chi_{I/A_r}(k\eta).$$

By the triangle inequality this satisfies

$$|E(\lambda)| \leqslant \frac{1}{\sum_{k=1}^{N} p_k}\left(\left|\sum_{k=1}^{N} (I_{c_k} - q_k)\right| + \left|\sum_{k=1}^{N} \chi_{I/A_r}(k\eta)(I_{c_k} - q_k)\right|\right).$$

Now by the mean value theorem there exists θ_0 in the interval $[\theta_1, \theta_2]$ such that

$$E(\lambda, \theta_0) - |\theta_1 - \theta_2||E'(\lambda, \theta_0)| \leqslant |E(\lambda, \theta_1)| \leqslant E(\lambda, \theta_0) + |\theta_1 - \theta_2||E'(\lambda, \theta_0)|.$$

We therefore have by Lemma 8

$$E(\lambda, \theta_0) - |\theta_1 - \theta_2| N \sup_{\theta \in [0, 2\pi)} |E(\lambda, \theta)| \leqslant |E(\lambda, \theta_1)|$$

$$\leqslant E(\lambda, \theta_0) + |\theta_1 - \theta_2| N \sup_{\theta \in [0, 2\pi)} |E(\lambda, \theta)|.$$

Hence, for an appropriate choice of the positive constant c, if $|\theta_1 - \theta_2| \leqslant c/N$, there exist positive constants c_1 and c_2 such that

$$c_1 |E(\lambda, \theta_1)| \leqslant |E(\lambda, \theta_2)| \leqslant c_2 |E(\lambda, \theta_1)|.$$

This means that if for some θ in $[0, \pi)$ we let

$$D = \left\{\theta, \theta + \frac{c}{N}, \ldots, \theta + \frac{c(N-1)}{N}\right\}$$

and hence we have

$$\left(\int_\Lambda \sup_{\theta \in [0, 2\pi)} |E(\lambda, \theta)|^p \mathrm{d}\pi(\lambda)\right)^{1/p} \leqslant C\left(\int_\Lambda \sup_{\theta \in D} |E(\lambda, \theta)|^p \mathrm{d}\pi(\lambda)\right)^{1/p}$$

$$\leqslant 2C \sup_{\theta \in D}\left(\int_\Lambda |E(\lambda, \theta)|^p \mathrm{d}\pi(\lambda)\right)^{1/p}$$

which by (7) and (8) is

$$\leqslant C\left\{\frac{\log N}{\Sigma_{k=1}^{N} q_n}\right\}^{1/2}.$$

Applying Markov's inequality we have

$$\pi\left(\left\{\lambda:\left|\frac{1}{|S_N|}\sum_{n\in S_N}\chi_{l/A_r}(n\eta)-\frac{1}{\Sigma_{l=1}^{N}q_l}\sum_{k=1}^{N}q_k\chi_{l/A_r}(k\eta)\right|\geqslant B\left\{\frac{\log N}{\Sigma_{l=1}^{N}q_l}\right\}^{1/2}\right)$$

$$\leqslant c^{-p}\left\{\frac{\Sigma_{l=1}^{N}q_l}{\log N}\right\}^{-p/2}\int_{\Lambda}\left(\sup_{\theta\in[0,2\pi)}|E(\lambda,\theta)|\right)^{p}\mathrm{d}\pi(\lambda).$$

For appropriate choice of B this is

$$\leqslant N^{-\alpha}. \tag{9}$$

Because $\Delta_{\mathbf{a}}$ is a countable group to prove Theorem 2(i) it is sufficient to show that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{l/A_r}(k_n\eta)=0$$

for almost all S in Λ. Remember that $\lim_{n\to\infty}q_n n=\infty$ and $\lim_{N\to\infty}\log N/\Sigma_{l=1}^{N}q_l=0$. In addition q_n decreases monotonically to zero and $\Sigma_{k=1}^{\infty}q_k=\infty$ for η not equal to zero, we have

$$\lim_{N\to\infty}\frac{1}{\Sigma_{k=1}^{N}q_k}\sum_{k=1}^{N}q_k\chi_{l/A_r}(k\eta)=0.$$

Hence as a consequence of (9) the proof of Theorem 2 (i) is complete.

4. Proof of Theorem 2 (ii)–(ix)

To demonstrate Theorem 2 (ii)–(ix) we need a number of lemmas. The first appears in [Na3].

Lemma 12. *Suppose for a sequence of real numbers $(a_n)_{n=1}^{\infty}$ and a real number α, for all β in $\{\alpha+m:m\in\mathbf{Z}\}\setminus\{\mathbf{Z}\}$, that we have*

$$E(\beta,N)=\sum_{n=1}^{N}e(\beta a_n)=o(N).$$

Then we have

$$\sum_{n=1}^{N}e(\alpha[a_n])=o(N).$$

We now state the well known uniform distribution results of Weyl [W] and Vinogradov [Vi, p. 287] as in Lemmas 13 and 14 respectively.

Lemma 13. *Let*

$$P(x)=\alpha_k x^k+\alpha_{k-1}x^{k-1}+\cdots+\alpha_0.$$

Suppose the real numbers $\alpha_1, \ldots, \alpha_k$ are not all rational multiples of a fixed real number. Then for each real number β other than zero we have

$$\sum_{n=1}^{N} e(\beta P(n)) = o(N).$$

Lemma 14. *Suppose that the polynomial $P(x)$ satisfies the hypothesis of Lemma 13 and that $(p_n)_{n=1}^{\infty}$ denotes the sequence of rational primes. Then again*

$$\sum_{n=1}^{N} e(\beta P(p_n)) = o(N).$$

Lemmas 15 and 16 which follow appear in [Ba].

Lemma 15. *Suppose $f(z)$ is a non-polynomial entire function which is real valued on the real numbers and such that $|f(z)| \ll e^{(\log|z|)^{\alpha}}$ with $\alpha < \frac{4}{3}$, then if $\beta \neq 0$*

$$\sum_{n=1}^{N} e(\beta f(n)) = o(N).$$

Similarly we have the following lemma.

Lemma 16. *Suppose $f(z)$ is as in Lemma 15 and that $(p_n)_{n=1}^{\infty}$ denotes the sequence of rational primes, then if $\beta \neq 0$*

$$\sum_{n=1}^{N} e(\beta f(p_n)) = o(N).$$

The following lemma is quoted from [Co].

Lemma 17. *Let a, b, k be integers. Suppose that $a < b$, $k \geqslant 2$, and set $s = 2^k$. Suppose the real valued function g is k times differentiable in the interval $[a, b]$ and that $|g^{(k)}(x)| \geqslant \lambda$ for some positive constant λ which is independent of x in $[a, b]$. Then letting*

$$R = \frac{1}{a-b}|g^{(k-1)}(a) - g^{(k-1)}(b)|$$

we have

$$\left|\sum_{u=a}^{b} e(g(u))\right| < C(b-a)\left(\left(\frac{R^2}{\lambda}\right)^{1/(k-2)} + \frac{1}{(\lambda(b-a)^k)^{2/k}} + \left(\frac{R}{\lambda(b-a)}\right)^{2/k}\right),$$

for some $C > 0$.

Setting $g(x) = \beta x^{\alpha}$, $a = 1$, $b = 1$ and $[\alpha] + 1$ we readily see that the following lemma is true.

Lemma 18. *For fixed ε, if $0 < \lambda < k - \alpha$ there exists $t \in (0, \frac{\varepsilon}{2})$ such that if $N^{\varepsilon - \alpha} \leqslant |\beta| \leqslant N^{\varepsilon}$ then*

$$\left|\sum_{u=a}^{b} e(\beta u^{\alpha})\right| \leqslant CN^{1-t},$$

where the constant depends only on α and ε.

For a sequence of real numbers $(x_n)_{n=1}^N$ we call

$$D(x_1,\ldots,x_N) = \sup_{I \subseteq [0,1)} \left| \frac{1}{N} \sum_{n=1}^{N} \chi_I(\langle x_n \rangle) - |I| \right|$$

its discrepancy. Here the supremum is taken over all intervals I contained in $[0,1)$ which are closed on the left and open on the right. The next few lemmas are metrical in nature. Lemmas 19 and 20 in the case where β is an integer other than zero appear in [Na1]. The general case is virtually identical and the proof is therefore forgone.

Lemma 19. *Suppose that* $(a_n)_{n=1}^{\infty}$ is a strictly increasing sequence of integers, β *is a real number other than zero and that we are given* $\varepsilon > 0$. *Then for almost all* x *with respect to Lebesgue measure we have*

$$D(\beta a_1 \cos(a_1 x),\ldots,\beta a_N \cos(a_N x)) = o(N^{-1/4}(\log N)^{(3/2)+\varepsilon}).$$

A refinement is the following lemma.

Lemma 20. *Suppose that* $(a_n)_{n=1}^{\infty}$ *are as in Lemma* 18 *and further that* $a_n = O(n^p)$ *for some* $p > 1$ *then*

$$D(\beta a_1 \cos(a_1 x),\ldots,\beta a_N \cos(a_N x)) = o(1)$$

except on a set of Hausdorff dimension at most $1 - (1/(4p + (1/2)))$.

Lemmas 21 and 22 which follow appear in [Na2].

Lemma 21. *Suppose the sequence* $(g_n(x))_{n=1}^{\infty}$ *of continuously differentiable functions are all defined on the same interval* $[a,b]$ *and for each pair of distinct natural numbers m and n that we have*:

(a) $g_n'(x) - g_m'(x)$ *is monotone on* $[a,b]$ *and*

(b) *there is an absolute constant* λ *such that*

$$|g_n'(x) - g_m'(x)| \geqslant \lambda > 0.$$

Then given $\varepsilon > 0$, *for almost all* x *with respect to Lebesgue measure we have*

$$D(\beta g_1(x),\ldots,\beta g_N(x)) = o(N^{-1/2}(\log N)^{(5/2)+\varepsilon}).$$

Analogous to Lemma 21 is the following lemma.

Lemma 22. *Suppose that the sequence of functions* $(g_n(x))_{n=1}^{\infty}$ *satisfies the hypothesis of Lemma* 22 *and further that the following is true*:

(c) *for all* x *in* $[a,b]$ *we have*

$$|g_n'(x)| = O(n^p)$$

for some $p > 1$ *and with an implied constant independent of* x *and*

(d) *for each pair of distinct positive integers m and n the functions*

$$\frac{g_n'(x)g_m'(x)}{g_m'(x) - g_n'(x)}$$

are monotonic on $[a,b]$, *then*

$$D(\beta g_1(x),\ldots,\beta g_N(x)) = o(1)$$

except on a set of Hausdorff dimension at most $1-(1/p)$.

We have the following lemma due to Koksma [Ko].

Lemma 23. Suppose that the function f *is of bounded variation* $V(f)$ *on* $[0,1)$ *then for a finite set of reals* $(x_n)_{n=1}^N$ *we have*

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(\langle x_n\rangle) - \int_0^1 f(t)\mathrm{d}t\right| \leqslant V(f)D(x_1,\ldots,x_N).$$

Appplying Lemma 12 and quoting Lemmas 13 to 18 we have Theorem 2 in the cases (ii)–(vi). The remaining cases of Theorem 2 follow on applying Lemma 23 to $f(x)=e(x)$ and quoting Lemmas 12 and 19–23.

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