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From Tanaka's formula to Ito's formula: The fundamental theorem of stochastic calculus

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Abstract. In this article we give a new proof of Ito's formula in $\mathbb{R}^n$ starting from the one-dimensional Tanaka formula. The proof is algebraic and does not use any limiting procedure. It uses the integration by parts formula, Fubini's theorem for stochastic integrals and essential properties of local times.

Keywords. Semi-martingales; Ito formula; Tanaka formula; local times.

1. Introduction

In this note we give a proof of Ito's formula starting from Tanaka's formula. Our approach was motivated by the proof of Ito's formula given in [1], and the results of [5], where we have proved and generalized the Tanaka formula, starting from first principles. In the case of dimension one, the basic method is well known, involving an integration in the space variable and then using the occupation density formula. We use a variation of this method, to go from Tanaka's formula to Ito's formula. In effect, we introduce an increasing process viz. the space integral of the local time and prove the occupation density formula and Ito's formula with this increasing process, which is then identified as the quadratic variation process by taking $f(x) = x^2$ in Ito's formula. The surprising fact is that the same method works in higher dimensions. Indeed, our formalism also yields a proof of the fundamental theorem of ordinary calculus, though only for twice differentiable functions: Let $x(t)$, $y(t)$ be continuous functions on $[0, \infty)$, of bounded variation on finite intervals and first let f be a C^2 function on $\mathbb{R}$ with compact support. Then

$$f(x_t) = f(x_0) + \int_0^t f'(x_s)\,\mathrm{d}x_s.$$

To prove this we first note that the 'ordinary' Tanaka formula holds for (x_t): If $a \in \mathbb{R}$, then

$$(x_t - a)^+ = (x_0 - a)^+ + \int_0^t I_{\{x_s > a\}}\mathrm{d}x_s.$$

This equation follows upon noting that the set $\{x_s > a\}$ is a union of disjoint open intervals of $[0, \infty)$. The integral in the RHS then reduces to a sum which together with the first term adds up to $(x_t - a)^+$. Integrating the ordinary Tanaka formula with respect to $f''(a)\,\mathrm{d}a$ and using Fubini's theorem we get the above expression for $f(x_t)$. We can then extend this equation for $f(x_t)$ to the arbitrary C^2 functions in the usual manner. Taking $f(x)=x^2$ we get

$$x_t^2 = x_0^2 + 2\int_0^t x_s\,\mathrm{d}x_s.$$

Polarization then yields

$$x_t y_t = x_0 y_0 + \int_0^t x_s\,\mathrm{d}y_s + \int_0^t y_s\,\mathrm{d}x_s.$$

Applying this equation to the functions $(x_t - a)^+, (y_t - b)^+$ we get

$$(x_t - a)^+(y_t - b)^+ = (x_0 - a)^+(y_0 - b)^+ + \int_0^t (x_s - a)^+ I_{\{y_s > b\}}\,\mathrm{d}y_s + \int_0^t (y_s - b)^+ I_{\{x_s > a\}}\,\mathrm{d}x_s.$$

Next if we let $f(x, y)$ be a C^2 function with compact support, then integrating the above equation with respect to $\partial_x^2\partial_y^2 f(a,b)\,\mathrm{d}a\mathrm{d}b$ we arrive at

$$f(x_t, y_t) = f(x_0, y_0) + \int_0^t \partial_x f(x_s, y_s)\,\mathrm{d}x_s + \int_0^t \partial_y f(x_s, y_s)\,\mathrm{d}y_s.$$

Our proof of Ito's formula in higher dimensions is an exact analogue of the above computations. The local time terms in the stochastic analogue of the equation for $(x_t - a)^+(y_t - b)^+$, integrated with $\partial_x^2\partial_y^2 f(a,b)\,\mathrm{d}a\mathrm{d}b$, yield the quadratic variation terms of the Ito's formula, via the occupation density formula.

We remark on two other aspects of these techniques: Firstly, we wish to emphasize the role of local time techniques in a multidimensional context. It is well-known that the point local times need not exist in higher dimensions. The resuls of this paper show that this fact need not necessarily restrict the use of local times in higher dimensional situations. Secondly, we would like to point out that each term in the expression for $(x_t - a)^+(y_t - b)^+$, for fixed t, is a locally integrable function in (a, b) and hence can be read as an equation for the corresponding Schwartz distribution. Applying the distributional derivative $\partial_x^2\partial_y^2$ to both sides we get the fomal expression

$$\delta_{(x_t, y_t)} = \delta_{(x_0, y_0)} - \int_0^t \partial_x \delta_{(x_s, y_s)}\,\mathrm{d}x_s - \int_0^t \partial_y \delta_{(x_s, y_s)}\,\mathrm{d}y_s$$

which is like a distributional analogue of the fundamental theorem. The point here is that there is a duality relation embedded in the statement of the fundamental theorem (for test functions) in both the ordinary and stochastic cases. In a forthcoming article, we hope to follow up this aspect in the stochastic case.

The paper is organized as follows. After the preliminaries in § 2, we prove the formula for continuous semi-martingales in § 3, and discuss the necessary modifications for arbitrary (discontinuous) semi-martingales in § 4.

2. Preliminaries (Continuous case)

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ be a filtered probability space with the filtration $(\mathscr{F}_t)$ satisfying usual conditions. Let (X_t) be a continuous semi-martingale viz. $X_t = X_0 + M_t + V_t$ where (M_t) is a continuous $\mathscr{F}_t$ local martingale, (V_t) is an $\mathscr{F}_t$ - adapted process of finite variation and $M_0 = V_0 \equiv 0$.

We recall Tanaka's formulas: $\forall\, a \in \mathbb{R}$,

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t I_{\{X_s > a\}} \mathrm{d}X_s + \frac{1}{2} L(t,a), \tag{1}$$

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t I_{\{X_s \leqslant a\}} \mathrm{d}X_s + \frac{1}{2} L(t,a),$$

where for each a, $(L(t,a))_{t \geqslant 0}$ is a continuous, increasing adapted process, $L(0,a) \equiv 0$ and satisfying

$$L(t,a) = \int_0^t I_{\{X_s = a\}} \mathrm{d}L(s,a). \tag{2}$$

We shall assume, following the results of [7], that the mapping $(t,a,w) \to L(t,a,w)$ is $\mathscr{B}[0,\infty) \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{F}_t / \mathscr{B}(\mathbb{R})$ measurable where $\mathscr{B}(\cdot)$ is the Borel σ-algebra on $(\cdot)$. Recall that for two continuous semi-martingales (X_t), (Y_t), a unique, continuous, adapted process of finite variation $([X, Y]_t)_{t \geqslant 0}$, (the quadratic co-variation process of X and Y) is defined by the following equation:

$$[X, Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t X_s \mathrm{d}Y_s - \int_0^t Y_s \mathrm{d}X_s. \tag{3}$$

We will denote by $\mathbb{D}(\mathbb{R}^n)$ (or simply by $\mathbb{D}$ when the context is clear) the space of infinitely differentiable functions on $\mathbb{R}^n$ with compact support. For $x, y \in \mathbf{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, let

$$\alpha(x,y) = \prod_{k=1}^{n} (x_k - y_k)^+;$$

$$\alpha_i(x,y) = I_{\{x_i > y_i\}} \prod_{k \neq i} (x_k - y_k)^+$$

$$\alpha_{ij}(x,y) = I_{\{x_i > y_i, x_j > y_j\}} \prod_{k \neq i,j} (x_k - y_k)^+$$

and

$$\bar{\alpha}_i(x,y) = \prod_{k \neq i} (x_k - y_k)^+.$$

3. Ito's formula and occupation density formula: The continuous case

Let (X_t) be a continuous semi-martingale as in § 2 and $(L(t,x))_{t \geqslant 0}$ its local time process at $x \in \mathbb{R}$.

PROPOSITION 1

Let (X_t) be a continuous, bounded semi-martingale. Then a.s.,

$$\int_{\mathbb{R}} L(t,x) \mathrm{d}x < \infty \quad \forall\, t \geqslant 0.$$

Proof. Since (X_t) is a bounded process, $\exists M > 0 \ni |X_t(w)| \leqslant M \;\forall t, w$. Let $K = [-M, M]$. By (2) and our boundedness assumption, $\forall\, x \notin K$, $L(t,x) = 0$ a.s. Hence by Fubini's theorem, a.s., $L(t,x)=0$ for almost all $x\in K^c$. Hence,

$$\int_{\mathbb{R}} L(t,x)\,\mathrm{d}x=\int_K L(t,x)\,\mathrm{d}x.$$

Since K is compact, $\int_K\{(X_t-x)^+-(X_0-x)^+\}\,\mathrm{d}x<\infty$. Further the process $f(s,w)=\int_K I_{\{X_s(w)>x\}}\,\mathrm{d}x$ is bounded and previsible. Hence by Fubini's theorem for stochastic integrals (see [6], Chap. V, p. 165)

$$\int_K \mathrm{d}x\int_0^t I_{\{X_s>x\}}\,\mathrm{d}X_s=\int_0^t f(s)\,\mathrm{d}X_s<\infty \quad \text{a.s.} \quad \forall t\geqslant 0.$$

Integrating eq. (1) over K with respect to $\mathrm{d}x$, it follows that a.s., $\int_K L(t,x)\,\mathrm{d}x<\infty$. This completes the proof. □

PROPOSITION 2

Let (X_t) be a bounded semi-martingale. Then $\Lambda_t:=\int_{\mathbb{R}}L(t,x)\,\mathrm{d}x$ is an adapted continuous and non-decreasing process. Moreover we have a.s., $\forall f\geqslant 0$ measurable

$$\int_0^t f(X_s)\,\mathrm{d}\Lambda_s=\int_{\mathbb{R}} f(x)L(t,x)\mathrm{d}x. \tag{4}$$

Proof. From Fubini's theorem it follows that Λ_t is an adapted process. Proposition (1) says that a.s., $\Lambda_t<\infty\,\forall t\geqslant 0$. Since for every x, $(L(t,x))$ is a non-decreasing and continuous process, the same is true of (Λ_t).

By Fubini's theorem applied to the measure $L(\mathrm{d}t,x)\,\mathrm{d}x$, it is immediate that

$$\int_0^t f(X_s)\,\mathrm{d}\Lambda_s=\int_{\mathbb{R}}\int_0^t f(X_s)\,L(\mathrm{d}s,x)\mathrm{d}x$$

for $f\geqslant 0$, measurable. Now the result follows from eq. (2). □

Theorem 1. (*The occupation density formula*). *Let (X_t) be a continuous semi-martingale and $[X,X]_t$, the quadratic variation process defined via eq. (3). Let $f:[0,\infty)\times\mathbb{R}\times\Omega\to[0,\infty]$ be a measurable function. Then a.s.,*

$$\int_0^t f(s,X_s,w)\,\mathrm{d}[X,X]_s=\int_{\mathbb{R}}\mathrm{d}x\int_0^t f(s,x,w)\,\mathrm{d}L(s,x)\quad \forall t\geqslant 0. \tag{5}$$

Proof. It is sufficient to prove the case when f is a function of x above, since the general case reduces to this case by standard monotone class arguments. Further, it is sufficient to consider the case when X_0 is bounded since replacement of X_t by $I_A(X_0)X_t$ merely multiples eq. (5) by $I_A(X_0)$. Again if eq. (5), is true when $[X,X]_t$, $L(t,x)$ are replaced by $[X,X]_{t\wedge T_n}$, $L(t\wedge T_n,x)$ respectively, for a sequence of stopping times T_n, increasing to ∞, then letting $n\to\infty$, eq. (5) is true for any $t\geqslant 0$. Thus, by stopping X if necessry we can assume that it is a bounded process. By Proposition 2, it therefore suffices to show, that if (X_t) is bounded then, $\Lambda_t\equiv[X,X]_t$. But if $f\in\mathbb{D}(\mathbb{R})$, then integrating eq. (1) with respect to the measure $f''(a)\,\mathrm{d}a$ we get, using proposition 2,

$$f(X_t)=f(X_0)+\int_0^t f'(X_s)\,\mathrm{d}X_s+\frac{1}{2}\int_0^t f''(X_s)\,\mathrm{d}\Lambda_s.$$

It follows by standard localizing arguments that

$$\Lambda_t = X_t^2 - X_0^2 - 2\int_0^t X_s \, dX_s \equiv [X,X]_t.$$

This completes the proof. □

Let $X_t = (X_t^1, \ldots, X_t^n)$ be a continuous vector valued semi-martingale, where each X_t^i is a continuous real valued semi-martingale whose local time process at $x_i \in R$ will be denoted by $(L^i(t, x_i))_{t \geqslant 0}$.

Theorem 2. (*Ito's formula*). *Let* (X_t) *be a continuous n-dimensional semi-martingale. Then for all* $f \in \mathbb{D}(\mathbb{R}^n)$ *we have*

$$f(X_t) = f(X_0) + \sum_{i=1}^{n} \int_0^t \frac{\partial f}{\partial x_i}(X_s)\, dX_s^i + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\, d[X^i, X^j]_s. \tag{6}$$

Proof. Recall the functions $\alpha(x,y)$, $\alpha_i(x,y)$, $\alpha_{ij}(x,y)$ and $\bar{\alpha}_i(x,y)$ defined in § 2. Then using Tanaka's formula (eq. (1)) and the integration by parts formula (eq. (3)), we get by induction on n,

$$\alpha(X_t, y) = \alpha(X_o, y) + \sum_{i=1}^{n} \int_0^t \alpha_i(X_s, y)\, dX_s^i + \frac{1}{2} \sum_{i=1}^{n} \int_0^t \bar{\alpha}_i(X_s, y)\, dL^i(s, y_i)$$
$$+ \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \int_0^t \alpha_{ij}(X_s, y)\, d[X^i, X^j]_s. \tag{7}$$

Let D^2 denote the differential operator $D^2 = \partial^{2n}/(\partial y_1^2, \ldots, \partial y_n^2)$. The following identities are easily verified for $f \in \mathbb{D}(\mathbb{R}^n)$ by integration by parts:

$$f(x) = \int_{\mathbb{R}^n} dy D^2 f(y) \alpha(x,y), \tag{8}$$

$$\frac{\partial f}{\partial x_i}(x) = \int_{\mathbb{R}^n} dy D^2 f(y) \alpha_i(x,y), \tag{9}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \int_{\mathbb{R}^n} dy D^2 f(y) \alpha_{ij}(x,y). \tag{10}$$

Note that by Fubini's theorem and the occupation density formula (eq. (5)), we have

$$\int_{\mathbb{R}^n} dy D^2 f(y) \int_0^t \bar{\alpha}_i(X_s, y)\, dL^i(s, y_i) = \int_0^t \frac{\partial^2 f}{\partial x_i^2}(X_s)\, d[X^i, X^j]_s. \tag{11}$$

Integrating eq. (7) with respect to $D^2 f(y)\, dy$ and using eq. (8), we get

$$f(X_t) = f(X_0) + \sum_{i=1}^{n} \int_{\mathbb{R}^n} dy D^2 f(y) \int_0^t \alpha_i(X_s, y)\, dX_s^i$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \int_{\mathbb{R}^n} dy D^2 f(y) \int_0^t \bar{\alpha}_i(X_s, y)\, dL^i(s, y_i)$$
$$+ \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \int_{\mathbb{R}^n} dy D^2 f(y) \int_0^t \alpha_{ij}(X_s, y)\, d[X^i, X^i]_s.$$

Using Fubini's theorem for stochastic integrals (see [6], Chap. IV, p. 165) and eq. (9) for terms of the 1st summation, eq. (11) for terms of the 2nd summation, the Fubini's theorem for ordinary integrals and eq. (10) for terms of the 3rd summation, the theorem follows. □

4. Ito's formula and occupation density formula: The general case

We now consider the case of a general semi-martingale $X_t = X_0 + M_t + V_t$ where (M_t) is a local martingale and (V_t) an adapted process of finite variation with $M_0 \equiv V_0 \equiv 0$. The Tanaka formula for (X_t), for $a \in \mathbb{R}$, is given by

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t I_{\{X_{s-} > a\}} \mathrm{d}X_s + l(t,a) + \frac{1}{2} L(t,a), \tag{12}$$

where $l(t,a) = \Sigma_{s \le t}(I_{\{X_{s-} \le a\}}(X_s - a)^+ + I_{\{X_{s-} > a\}}(X_s - a)^-)$ and $(L(t,a))_{t \geq 0}$ is a continuous adapted non-decreasing process satisfying

$$L(t,a) = \int_0^t I_{\{X_s = a\}} \mathrm{d}L(s,a). \tag{13}$$

Further by the results of [7], the map $(t,a,w) \to L(t,a,w)$ can be taken to be jointly measurable. Note that $(l(t,a))_{t \geq 0}$ is a non-negative and non-decreasing process. If (X_t), (Y_t) are two semi-martingales, then the quadratic co-variation process $[X,Y]_t$ defined by

$$[X,Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t Y_{s-} \mathrm{d}X_s - \int_0^t X_{s-} \mathrm{d}Y_s \tag{14}$$

is an adapted process of finite variation such that

$$[X,Y]_t - \sum_{s \leq t} \Delta X_s \Delta Y_s \equiv [X,Y]_t^c \tag{15}$$

is a continuous adapted process of finite variation. If (X_t) is a semi-martingale and T a stopping time, then the pre-stopped process $X^{T-}(t)$ is also a semi-martingale where $X^{T-}(t) \equiv X_t I_{[0,T)}(t) + X_{T-} I_{[T,\infty)}(t)$ with $X_{0-} \equiv X_0$. Moreover, $[X^{T-}, X^{T-}]^c(t) = [X,X]^c(t \wedge T)$.

We refer to [2] for these and other propositions on general semi-martingales.

PROPOSITION 3

Let (X_t) be a bounded semi-martingale. Then a.s.,

$$\int_{\mathbb{R}} (L(t,x) + l(t,x))\,\mathrm{d}x < \infty \quad \forall\, t \geq 0.$$

Proof. Let K be a compact set such that a.s., $X_t \in K\ \forall t \geq 0$. Then it is clear from the definition of $l(t,x)$ that if $x \notin K$, $l(t,x) \equiv 0$. The rest of the argument is the same as in the continuous case (Proposition 1), with $L(t,x)$ replaced by $L(t,x) + l(t,x)$, where for the application of Fubini's theorem for stochastic integrals of general semi-martingales we refer to [3], Chap. V, p. 180 or [4], Chap. IV, p. 157. □

PROPOSITION 4

Let (X_t) be a bounded semi-martingale, then $\Lambda := \int_{\mathbb{R}} L(t,x)\,\mathrm{d}x$ is an adapted process which is continuous and non-decreasing. Moreover we have a.s., for all f non-negative and measurable,

$$\int_0^t f(X_s)\,\mathrm{d}\Lambda_s = \int_{\mathbb{R}} f(x) L(t,x)\,\mathrm{d}x \quad \forall t \geqslant 0. \tag{16}$$

Proof. The first part of the proposition follows from Proposition 3 and the fact that for each $x \in \mathbb{R}$ $(L(t,x))_{t \geqslant 0}$ is a continuous non-decreasing adapted process. The proof of the second part is similar to that of the continuous case (Proposition 2). □

Theorem 3. (*The occupation density formula, general case*). *Let (X_t) be a semi-martingale and $[X,X]_t$ its quadratic variation process defined by eq. (14). Let $f: [0,\infty] \times \mathbb{R} \times \Omega \to [0,\infty]$ be a measurable function. Then a.s.,*

$$\int_0^t f(x, X_s, w)\,\mathrm{d}[X,X]_s^c = \int_{\mathbb{R}} \mathrm{d}x \int_0^t f(s,x,w)\,\mathrm{d}L(s,x) \quad \forall t \geqslant 0. \tag{17}$$

Proof. It is sufficient to consider the case when f is a function of x alone, X_0 is bounded and $[X,X]_t^c$, $L(t,x)$ are replaced by $[X,X]_{t\wedge T_n}^c$, $L(t \wedge T_n, x)$ respectively for a sequence of stopping times T_n increasing to ∞. Note that $[X,X]_{t\wedge T_n}^c$, $L(t\wedge T_n, x)$ are respectively, the continuous part of the quadratic variation process, and local time process at x, of the semi-martingale $X_t^{T_n}$. Thus, by pre-stopping X if necessary we can assume that it is a bounded process.

By Proposition (4), it therefore suffices to show that if (X_t) is a bounded semi-martingale, then $\Lambda_t \equiv [X,X]_t^c$. If $f \in \mathbb{D}(\mathbb{R})$, then integrating eq. (12) with respect to the measure $f''(a)\mathrm{d}a$, we get using Proposition 4 that

$$f(X_t) = f(X_0) + \int_0^t f(X_{s-})\,\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)\,\mathrm{d}\Lambda_s$$
$$+ \sum_{s \leqslant t} \{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\}.$$

If $X_t \in K\ \forall t \geqslant 0$, where K is a compact set, then choosing $f \in \mathbb{D}(\mathbb{R}), f(x) = x^2, x \in K$ we get, from above

$$\Lambda_t = X_t^2 - X_0^2 - 2\int_0^t X_{s-}\mathrm{d}X_s - \sum_{s \leqslant t} (\Delta X_s)^2$$
$$\equiv [X,X]_t^c.$$

This completes the proof. □

Theorem 4. *Let (X_t) be an $\mathbb{R}^n$-valued semi-martingale with components (X_t^i) $1 \leqslant i \leqslant n$. Then, $\forall f \in \mathbb{D}(\mathbb{R}^n)$ we have a.s.,*

$$f(X_t) = f(X_0) + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(X_{s-})\,\mathrm{d}X_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s-})\,\mathrm{d}[X^i, X^j]_s^c$$
$$+ \sum_{s \leqslant t} \{f(X_s) - f(X_{s-}) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{s-})\Delta X_s^i\}. \tag{18}$$

Proof. For $x, y \in \mathbb{R}^n$, we recall the functions $\alpha(x,y)$, $\alpha_i(x,y)$, $\bar{\alpha}_i(x,y)$, $\alpha_{ij}(x,y)$ of § 2. Let $L^i(s,y_i)$ denote the local time of (X_s^i) at $y_i \in \mathbb{R}$, where $y=(y_1,\ldots,y_n)$. Then using Tanaka formula (12), and the integration by parts formula (14), we get by induction on n, that

$$\begin{aligned}\alpha(X_t,y) &= \alpha(X_0,y) + \sum_{i=1}^{n}\int_0^t \alpha_i(X_{s-},y)\,\mathrm{d}X_s^i \\ &+\frac{1}{2}\sum_{i=1}^{n}\int_0^t \bar{\alpha}_i(X_{s-},y)\,\mathrm{d}L^i(s,y_i) \\ &+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\int_0^t \alpha_{ij}(X_{s-},y)\,\mathrm{d}[X^i,X^j]_s^c \\ &+\sum_{s\leqslant t}\left(\sum_{i=1}^{n}\bar{\alpha}_i(X_{s-},y)\Delta l^i(s,y_i)\right) \\ &+\sum_{s\leqslant t}\left\{\alpha(X_s,y)-\alpha(X_{s-},y)-\sum_{i=1}^{n}\bar{\alpha}_i(X_{s-},y)\Delta(X_s^i-y_i)^+\right\}.\end{aligned}$$

It is easily verified that

$$\bar{\alpha}_i(X_{s-},y)\Delta l^i(s,y_i)-\bar{\alpha}_i(X_{s-},y)\Delta(X_{s-}^i-y_i)^+ = -\alpha_i(X_{s-},y)\,\Delta X_s^i$$

and hence,

$$\begin{aligned}\alpha(X_t,y) &= \alpha(X_0,y) + \sum_{i=1}^{n}\int_0^t \alpha_i(X_{s-},y)\,\mathrm{d}X_s^i \\ &+\frac{1}{2}\sum_{i=1}^{n}\int_0^t \bar{\alpha}_i(X_{s-},y)\,\mathrm{d}L^i(s,y_i) \\ &+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\int_0^t \alpha_{ij}(X_{s-},y)\,\mathrm{d}[X^i,X^j]_s^c \\ &+\sum_{s\leqslant t}\left\{\alpha(X_s,y)-\alpha(X_{s-},y)-\sum_{i=1}^{n}\alpha_i(X_{s-},y)\Delta X_s^i\right\}. \end{aligned} \tag{19}$$

Let $D^2=\partial^{2n}/(\partial y_1^2\ldots\partial y_n^2)$. Integrating eq. (19) with respect to the measure $D^2 f(y)\mathrm{d}y$ and using Fubini's theorem for ordinary and stochastic integrals (for the latter case, see [3] or [4]), the proof is completed as in the continuous case (Theorem 2) with the aid of eqs (8), (9), (10) and the analog of eq. (11) for the discontinuous case. □

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