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Weakly analytic sets for function spaces

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Dedicated to Prof. M. H. Vasavada on his 60th birthday

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Abstract. We define and study weakly analytic sets for function spaces. This concept generalizes the concept of weak analyticity for function algebras. We also define weak analyticity for the space of affine functions using the ideas of convexity. We show that it coincides with the same concept regarding it as a function space. We give several examples to illustrate the concepts.

Keywords. Weakly analytic set; function space; function algebra; space of affine functions.

1. Introduction

A function space A on a compact Hausdorff space X is a closed subspace of the space $C(X)$ of all continuous, complex-valued functions on X separating points and containing constants. If A is an algebra, it is called a function algebra.

It is well-known that the Bishop and Šilov decompositions play an important role in characterizing function algebras. Later, some other decompositions finer than the Bishop and Šilov decompositions have been defined and studied [4, 7, 8], for example, the decompositions of weakly prime sets, weakly analytic sets etc. We studied some of these decompositions for function spaces ([9, 10]). In the first part of this paper, we generalize the concept of a weakly analytic set for a function space and study its properties. We give several examples connected with the results.

Ellis [4] has studied weakly prime sets for $A(K)$, the space of continuous ffine functions on a compact convex set K of a locally convex topological vector space, using the idea of convexity. We have shown in [10] that this idea coincides with the abstract definition of weakly prime sets for function spaces. In § 2, we define weak analyticity for $A(K)$ using convexity which is finer than the idea of weak primeness according to Ellis. Then, we show that both definitions of weak analyticity for $A(K)$ coincide. Finally, we give some examples to illustrate the concepts.

2. Function space

Arenson [2] has defined a weakly analytic set for a function algebra as follows:

DEFINITION 2.1 [2]

Let A be a function algebra on a compact Hausdorff space X. A closed subset E of X is called a *weakly analytic set* for A if every peak set for $(A_{|E})^-$ either coincides with E or it is nowhere dense in E.

Equivalently, E is a weakly analytic set for A if $E = G \cup H$, with G a peak set for $(A_{|E})^{-}$ and H a closed set in E, then either $G = E$ or $H = E$.

Let A be a function space on X. For a closed subset E of X, we define

$$N(A_{|E}) = \{f \in C(E): fg \in A_{|E} \text{ for all } g \in A_{|E}\}.$$

Then $N(A_{|E})$ is an algebra. Also, $N(A_{|E}) \subseteq A_{|E}$ and if A is an algebra, then $N(A_{|E}) = A_{|E}$.

DEFINITION 2.2

A closed subset E of X is called a *weakly analytic set* for a function space A if $E = G \cup H$, with G a generalized peak set (i.e. intersection of peak sets) for $N(A_{|E})$ and H a closed subset of E, then either $G = E$ or $H = E$.

Equivalently, E is a weakly analytic set for A if G is a proper generalized peak set for $N(A_{|E})$, then the interior of G, $G^0 = \phi$.

The function space A is called weakly analytic if X is a weakly analytic set for A.

Remark 2.3. (i) It can be shown that each weakly analytic set is contained in a maximal weakly analytic set for A. The collection of all maximal weakly analytic sets for A is denoted by $\mathscr{W}_\alpha(A)$. In general, the members of $\mathscr{W}_\alpha(A)$ are not disjoint (see Example 3.2). (ii) It is easy to check that $\mathscr{W}_\alpha(A)$ is finer than $\mathscr{P}(A)$, the family of maximal weakly prime sets [10] for A. (iii) It is known that if A is an analytic function algebra, then it is antisymmetric [6]. It can be easily checked that if A is analytic, then it is weakly analytic and hence weakly prime. But there is a weakly analytic function algebra which is not analytic (see Example 3.4).

The importance of all the decompositions is due to their (D)-property. First, we define this property.

DEFINITION 2.4 [5]

Let $\mathscr{F}$ be a family of closed subsets of X and A be a function space on X. (i) We say that $\mathscr{F}$ has the *(D)-property* for A if whenever $f \in C(X)$ and $f_{|F} \in (A_{|F})^{-}$ for all $F \in \mathscr{F}$, then $f \in A$. (ii) We say that $\mathscr{F}$ has the *(GA)-property* for A if $\mu \in b(A^{\perp})^e$, then Supp $\mu \subset F$ for some $F \in \mathscr{F}$, where $b(A^{\perp})^e$ denotes the set of extreme points of the closed unit ball of the annihilator $A^{\perp}$ of A.

It can be checked that the (GA)-property is stronger than the (D)-property.

We shall show that $\mathscr{W}_\alpha(A)$ has the (GA)-property for A.

Theorem 2.5. *$\mathscr{W}_\alpha(A)$ has the (GA)-property for A.*

Proof. Let $\mu \in b(A^{\perp})^e$ and $S = \text{Supp}\mu$. It is enough to show that S is a weakly analytic set for A.

Let G be a generalized peak set for $N(A_{|S})$ and H be a closed subset of S with $G \cup H = S$. Let $\mu_1 = \mu_{|G}$ and $\mu_2 = \mu - \mu_1$. First, we shall show that $\mu_1 \in A^{\perp}$. Let $\varepsilon > 0$. There exists an open set U, containing G, in X such that $|\mu|(U \setminus G) < \varepsilon$. Clearly, $S \cap U$ is open in S and $G \subset S \cap U$. Since G is a generalized peak set for $N(A_{|S})$, there is a peak set T with peaking function f in $N(A_{|S})$ such that $G \subset T \subset S \cap U$. Define h on S by $h = 1$ on T and $h = 0$ on $S \setminus T$. Then f^n converges pointwise and boundedly to h on S. Therefore, for $g \in A$,

$$\int_T g \,\mathrm{d}\mu = \int_S gh \,\mathrm{d}\mu = \int_S g(\lim f^n)\,\mathrm{d}\mu = \lim \int_S g f^n \mathrm{d}\mu.$$

But $f^n \in N(A_{|S})$ implies that $gf^n_{|S} \in A_{|S}$ and we have $\mu \in A^\perp = (A_{|S})^\perp$. Therefore, $\int_T g\,d\mu = 0$. Now,

$$\left|\int g\,d\mu_1\right| = \left|\int_G g\,d\mu\right| = \left|\int_G g\,d\mu - \int_T g\,d\mu\right|$$
$$\leqslant \|g\|(|\mu|(T\backslash G)) \leqslant \|g\|(|\mu|(U\backslash G)) < \varepsilon\|g\|.$$

Since ε is arbitrary, $\int g\,d\mu_1 = 0$. Hence $\mu_1 \in A^\perp$. Consequently, $\mu_2 \in A^\perp$. Also, $\|\mu\| = 1 = \|\mu_1\| + \|\mu_2\|$. Hence $\mu_1 = \mu$ or $\mu_2 = \mu$, as $\mu \in b(A^\perp)^e$, i.e. $G = S$ or $H = S$. Hence S is a weakly analytic set for A.

Unlike maximal weakly prime sets, the maximal weakly analytic sets are, in general, not generalized peak sets, even for function algebras (see Example 3.3). However, if X is metrizable then, Arenson [2] has shown that, for a function algebra A, $A_{|E}$ is closed in $C(E)$ for $E \in \mathscr{D}_\alpha(A)$ (which holds if E is a generalized peak set), where $\mathscr{D}_\alpha(A)$ denotes the decomposition of X corresponding to the family $\mathscr{W}_\alpha(A)$. We prove a stronger result that if X is metrizable, then members of $\mathscr{D}_\alpha(A)$ are p-sets for a function space A. In fact, it is a consequence of the following more general result.

PROPOSITION 2.6

Let $\mathscr{E}$ be a decomposition of a metrizable space X with the (GA)-property for a function space A. Then each $E \in \mathscr{E}$ is a p-set for A (i.e., $\mu \in A^\perp \Rightarrow \mu_E \in A^\perp$, where μ_E is the restriction of μ on E) and hence a generalized peak set for A.

Proof. Let $E \in \mathscr{E}$. It is enough to show that $\mu_E \in A^\perp$ whenever $\mu \in b(A^\perp)$. Let $\mu \in b(A^\perp)^e$. Then, there exists $F \in \mathscr{E}$ such that $\mathrm{Supp}\mu \subset F$. Since $\mathscr{E}$ is a decomposition, either $E = F$ or $E \cap F = \phi$. Hence $\mu_E = \mu$ or $\mu_E = 0$ and so $\mu_E \in A^\perp$.

Let $\mu \in b(A^\perp)$. Since X is metrizable, $C(X)$ is separable and hence the weak* topology on $M(X)$ is metrizable. Since $b(A^\perp)$ is the weak* closed convex hull of $b(A^\perp)^e$ in $M(X)$, by Choquet's theorem [11, p. 19], there exists a regular Borel measure λ supported on $b(A^\perp)^e$ such that $f(\mu) = \int_X f\,d\mu = \int_{b(A^\perp)^e} f(v)\,d\lambda(v)$ for every $f \in C(X)$. Now, E is a G_δ-set and hence

$$\int_E f\,d\mu = \int_X f\chi_E\,d\mu = \int_{b(A^\perp)^e} (f\chi_E)(v)\,d\lambda(v)$$

for every $f \in C(X)$. But for $v \in b(A^\perp)^e$, $v_E = v$ or $v_E = 0$. Hence, for $f \in A$,

$$0 = \int_X f\,dv = \int_E f\,dv = \int_X (f\chi_E)\,dv = (f\chi_E)(v).$$

It follows that $\int_E f\,d\mu = 0$ for each $f \in A$, i.e., $\mu_E \in A^\perp$. Thus E is a p-set for A. Since each p-set is a generalized peak set for a function space, the result follows.

COROLLARY 2.7

Let X be a metrizable space. Then each $E \in \mathscr{D}_\alpha(A)$ is a p-set for A.

Proof. By Theorem 2.5, $\mathscr{W}_\alpha(A)$ has the (GA)-property for A. Since each member of $\mathscr{W}_\alpha(A)$ is contained in some member of $\mathscr{D}_\alpha(A)$, it can be easily checked that $\mathscr{D}_\alpha(A)$ also has the (GA)-property for A. So, by Proposition 2.6, each $E \in \mathscr{D}_\alpha(A)$ is a p-set for A.

3. Examples

In this section, we study maximal weakly analytic sets for some function algebras and function spaces.

Example 3.1. Let X be the union of a line segment F and a sequence of disjoint solid rectangles $\{F_n:n\in\mathbb{N}\}$ converging to F. Let A be the set of all f in $C(X)$ such that $f_{|F_n}$ is a polynomial of degree atmost n. Then A is a function space on X and as in [9, Example 3.5], it can be checked that $\mathscr{W}_\alpha(A)=\{F_n:n\in\mathbb{N}\}\cup\{\{x\}:x\in F\}=\mathscr{D}_\alpha(A)$. Note that, for this function space A, $N(A)=\{f\in C(X):f_{|F_n}$ is constant, for each $n\in\mathbb{N}\}$ and hence $\mathscr{W}_\alpha(N(A))=\{F_n:n\in\mathbb{N}\}\cup\{F\}$.

Example 3.2. Let $D=D_1\cup D_2$, where $D_1=\{z\in\mathbb{C}:|z|\leqslant 1\}$ and $D_2=\{z\in\mathbb{C}:|z-2|\leqslant 1\}$. Let $A=A(D)$, the set of all functions in $C(D)$ which are analytic in the interior of D. Then A is a function algebra on D [4, Example 2]. It can be checked that $\mathscr{W}_\alpha(A)=\{D_1, D_2\}$. But, as $D_1\cap D_2\neq\phi$, $\mathscr{D}_\alpha(A)=\{D\}$.

Next, we give an example which shows that, in general, the members of $\mathscr{W}_\alpha(A)$ are not generalized peak sets.

Example 3.3. [4]. Let D be the set as considered in Example 3.2. Let X denote the quotient space of D by identifying the points 0 and 1; 2 and 3 and let $q:D\to X$ denote the quotient map. Also, let $A=\{f\in C(X);f\circ q\in A(D)\}$. Then A is a function algebra on X. Also, $\mathscr{W}_\alpha(A)=\{q(D_1),q(D_2)\}$. It can be verified that $q(D_1)$ is a peak set for A but $q(D_2)$ cannot be a peak set for A. Further, note that, A is a weakly prime algebra which is not weakly analytic.

Finally, we show that the concept of weak analyticity is really weaker than that of analyticity.

Example 3.4. Let $D=\{z\in\mathbb{C}:|z|\leqslant 1\}$ and $T=\{z\in\mathbb{C}:|z|=1\}$. Also, let $S=T\cup\{0\}$ and $A=A(D)_{|S}$. Then A is a function algebra on S. A is not an analytic algebra, as the function $f(z)=z$ is in A and it is zero on an open subset $\{0\}$ of S. But it can be checked that A is a weakly analytic function algebra.

4. Space of affine functions

Let $A(K)$ denote the Banach space of all real-valued continuous affine functions on a compact convex set K in a locally convex topological vector space and ∂K denote the set of extreme points of K.

In this section, we define weak analyticity for $A(K)$ using only ideas of convexity. Then as in the case of weakly prime sets [10], we show that the concept coincides with the one given in Definition 2.2. Finally, we give certain examples to illustrate the concepts.

For the definitions and results regarding compact convex sets and space of affine functions, we refer to [1] and [3].

DEFINITION 4.1

A subset E of ∂K is called a *weakly analytic set for* $A(K)$ if $E=\partial F$, for some closed face F of K and if $F=co(G\cup H)$, where G is a closed split face of F and H is a closed convex subset of F, then either $G=F$ or $H=F$.

We say that $A(K)$ is weakly analytic if ∂K is a weakly analytic set for $A(K)$.

The following proposition can be easily verified.

PROPOSITION 4.2

Each weakly analytic set for $A(K)$ is contained in a maximal weakly analytic set for $A(K)$.

Let $\mathscr{W}_{\alpha c}(A(K))$ denote the family of all maximal weakly analytic sets for $A(K)$. Then, it can be checked that $\mathscr{W}_{\alpha c}(A(K))$ is finer than $\mathscr{W}_{pc}(A(K))$, the family of maximal weakly prime sets for $A(K)$ defined by Ellis [4].

Now, $A(K)$ can also be looked upon as a function space on K. So, we can discuss $\mathscr{W}_{\alpha}(A(K))$ for $A(K)$. But, since the functions in $A(K)$ are determined by their values on ∂K, it is natural to consider the space $A(K)_{|\partial K}$. Also, note that, the members of $\mathscr{W}_{\alpha c}(A(K))$ are subsets of ∂K. So, one would like to compare the families $\mathscr{W}_{\alpha c}(A(K))$ and $\mathscr{W}_{\alpha}(A(K)_{|\partial K})$.

We shall need the following lemma which can be easily proved.

Lemma 4.3. *$Ce(A(K))_{|\partial K} = N(A(K)_{|\partial K})$, where $Ce(A(K)) = \{f \in A(K) : fg_{|\partial K} \in A(K)_{|\partial K}$ for every $g \in A(K)\}$, the centre of $A(K)$.*

As a consequence of the above lemma, we get the following result.

COROLLARY 4.4

A subset E of ∂K is a facially closed subset of ∂K if and only if E is a generalized peak set for $N(A(K)_{|\partial K})$

Now, we show that the families $\mathscr{W}_{\alpha c}(A(K))$ and $\mathscr{W}_{\alpha}(A(K)_{|\partial K})$ coincide, if they consist of only one member ∂K.

PROPOSITION 4.5

$\mathscr{W}_{\alpha c}(A(K)) = \{\partial K\}$ if and only if $\mathscr{W}_{\alpha}(A(K)_{|\partial K}) = \{\partial K\}$, i.e., $A(K)$ is weakly analytic in the sense of Definition 4.1 if and only if the function space $A(K)_{|\partial K}$ is weakly analytic in the sense of Definition 2.2.

Proof. Assume that $A(K)$ is weakly analytic. To show that the function space $A(K)_{|\partial K}$ is weakly analytic, let G be a generalized peak set for $N(A(K)_{|\partial K})$. By Corollary 4.4, G is a facially closed subset of ∂K. Then $G = \partial E$, for some closed split face E of K. Let $H = \overline{co}(\partial K - G^0)$. Then H is a closed convex subset of K. Also, $\overline{co}\ (E \cup H) = \overline{co}(\overline{co}(\partial E) \cup \overline{co}(\partial K - G^0)) = \overline{co}(\partial E \cup (\partial K - G^0)) = \overline{co}(G \cup (\partial K - G^0)) = \overline{co}(\partial K) = K$. Since $A(K)$ is weakly analytic, by Definition 4.1, $E = K$ or $H = K$. If $E = K$, then $\partial E = \partial K$, i.e., $G = \partial K$. If $H = K$, then $\partial H = \partial K$. Now, $\overline{co}(\partial K - G^0) = H = \overline{co}(\partial H)$ and so, $\partial H \subset \partial K - G^0$. Thus, we get $\partial K = \partial H \subset \partial K - G^0 \subset \partial K$ which implies that $G^0 = \phi$. Hence, either $G = \partial K$ or $G^0 = \phi$. Consequently, $A(K)_{|\partial K}$ is weakly analytic in the sense of Definition 2.2.

Conversly, assume that the function space $A(K)_{|\partial K}$ is weakly analytic. Let $K = co(G \cup H)$, with G a closed split face of K and H a closed convex subset of K. Then ∂G is facially closed and so it is a generalized peak set for $N(A(K)_{|\partial K})$, by Corollary 4.4. Also, $H \cap \partial K$ is closed in ∂K. Now, we shall show that $\partial G \cup (H \cap \partial K) = \partial K$.

Since $\partial G = G \cap \partial K$, it is clear that $\partial G \cup (H \cap \partial K) \subset \partial K$. Let $x \in \partial K$. If $x \notin \partial G$, then $x \notin G$. As $K = co(G \cup H)$, $x = \Sigma_i r_i x_i$ with $x_i \in (G \cup H)$. Since $x \in \partial K$, $x = x_i$ for all i. But, if $x \notin G$, then $x_i \in H$ for each i and so, $x \in H$. Thus, if $x \notin \partial G$, then $x \in H \cap \partial K$, i.e. $\partial K \subset \partial G \cup (H \cap \partial K)$. Hence $\partial K = \partial G \cup (H \cap \partial K)$.

Thus, we have $\partial K = \partial G \cup (H \cap \partial K)$, where ∂G is a generalized peak set for $N(A(K)_{|\partial K})$ and $H \cap \partial K$ is closed in ∂K. Since $A(K)_{|\partial K}$ is weakly analytic, either $\partial G = \partial K$ or $H \cap \partial K = \partial K$. If $\partial G = \partial K$, then $G = K$. If $H \cap \partial K = \partial K$, then $\partial K \subset H$ and so, $K \subset H$ as H is convex, i.e. $K = H$. Thus, either $G = K$ or $H = K$. Hence $A(K)$ is weakly analytic in the sense of Definition 4.1.

We do not know, how $\mathscr{W}_{\alpha c}(A(K))$ and $\mathscr{W}_{\alpha}(A(K)_{|\partial K})$ are related in general.

Examples 4.6. (i). Let K be a triangle in $\mathbb{R}^2$ with vertices $\{a, b, c\}$. Then, it is clear that $\partial K = \{a, b, c\}$ and each vertex is a closed split face of K. Let $E = \{a, b\}$. Then $F = coE$ is a closed face of K. Also, $F = co(G \cup H)$, where $G = \{a\}$ and $H = \{b\}$ are closed split faces of F. Thus, a set containing more than one point cannot be a weakly analytic set for $A(K)$, according to Definition 4.1. Hence we get, $\mathscr{W}_{\alpha c}(A(K)) = \{\{a\}, \{b\}, \{c\}\}$. Further, since ∂K is a Bauer simplex, $A(K)_{|\partial K} = C(\partial K)$. So, one can easily check that $\mathscr{W}_{\alpha}(A(K)_{|\partial K}) = \{\{a\}, \{b\}, \{c\}\}$. (ii) Let $G = \{(x, 0) \in l^1 x \mathbb{R} : x \geqslant 0, \ ||x|| \leqslant 1\}$, $F = \{(0, t) \in l^1 x \mathbb{R} : |t| \leqslant 1\}$ and let $K = co(F \cup G)$. Then $\partial K = (\partial G \cup \partial F) - \{(0,0)\}$. It is shown in [3, p. 219] that the Bishop decomposition for $A(K)$ consists of all singleton sets. Since the decomposition of weakly prime sets is finer than the Bishop decomposition [4] and the decomposition of weakly analytic sets is finer than the decomposition of weakly prime sets, $\mathscr{W}_{\alpha c}(A(K))$ consists of all singleton sets. (iii) Let K be a square in $\mathbb{R}^2$, with vertices $\{a, b, c, d\}$. Then $\partial K = \{a, b, c, d\}$. Also, it can be checked that K has no closed split face. So, ∂K is trivially a weakly analytic set for $A(K)$. Thus, $\mathscr{W}_{\alpha c}(A(K)) = \{\partial K\}$. Hence by Proposition 4.5, $\mathscr{W}_{\alpha}(A(K)_{|\partial K}) = \{\partial K\}$.

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