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A theorem concerning a product of two general classes of polynomials and the multivariable *H*-function

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Abstract. A theorem concerning a product of two general classes of polynomials and the multivariable *H*-function is established. Certain integrals and expansion formulae have also been derived by the application of this theorem. This general theorem yields a number of new, interesting and useful theorems, integrals and expansion formulae as its particular cases.

Keywords. Multivariable *H*-function; general class of polynomials; integrals; expansion formulae.

1. Introduction and the main result

Srivastava [5; p.l, eq. (1)] considered the general class of polynomials as

$$S_n^m[x] = \sum_{\alpha=0}^{[n/m]} \frac{(-n)_{m\alpha}}{\alpha!} A_{n,\alpha} x^\alpha, n = 0, 1, 2, \ldots, \tag{1}$$

where m is an arbitrary positive integer and the coefficients $A_{n,\alpha}$ $(n, \alpha \geqslant 0)$ are arbitrary constants real or complex. Also $(\lambda)_\alpha$ denotes the Pochhammer symbol and defined as

$$(\lambda)_\alpha = \frac{\Gamma(\lambda+\alpha)}{\Gamma(\lambda)} = \begin{cases} 1 \text{ if } \alpha = 0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1), \\ \forall \alpha = 1, 2, 3, \cdots \end{cases} \tag{2}$$

By suitably specializing the coefficients $A_{n,\alpha}$, polynomials $S_n^m[x]$ can be reduced to the classical orthogonal polynomials i.e. Jacobi, Hermite, Laguerre polynomials etc. (see [3]).

The following theorem can be derived for the *H*-function of several complex variables defined by Srivastava and Panda [6; also see 8, p. 251]

Main Theorem. *If*

$$(1-t)^{u+v-w} {}_2F_1(2u, 2v; 2w; t) = \sum_{k=0}^{\infty} a_k t^k \tag{3}$$

then

$$\int_0^1 {}_2F_1(u, v; w+1/2; t)\, {}_2F_1(w-u, w-v; w+1/2; t)$$
$$\cdot S_n^m[t^k]\, S_{n'}^{m'}[t^{k'}]\, H(z_1 t^{h_1}, \ldots, z_r t^{h_r})\,\mathrm{d}t$$

$$= \sum_{\alpha=0}^{[n/m]} \sum_{\beta=0}^{[n'/m']} \sum_{k=0}^{\infty} \frac{(-n)_{m\alpha}}{\alpha!} A_{n,\alpha} \frac{(-n')_{m'\beta}}{\beta!} A_{n',\beta} \frac{(w)_k}{(w+1/2)_k} a_k$$

$$H^{O,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [-k-h\alpha-h'\beta:h_1,\ldots,h_r], \\ [-k-h\alpha-h'\beta-1:h_1,\ldots,h_r], \end{matrix} \right.$$

$$\left. \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}] : [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(c):\psi',\ldots,\psi^{(r)}] : [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} \begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \right), \tag{4}$$

where

$$h>0, \quad h'>0, \quad h_2>0, \quad \mathrm{Re}\left(1+\sum_{i=1}^{r} h_2 d_j^{(i)}/\delta_j^{(i)}\right)>0,$$

$$-\tfrac{1}{2}<(w-u-v)<\tfrac{1}{2}, \quad |\arg(z_i)|<T_i\pi/2, \quad T_i>0,$$

$i=1,\ldots,r; j=1,\ldots,u^{(i)}$ and m and m' are arbitrary positive integers and the coefficients $A_{n,\alpha}$ and $A_{n',\beta}$ $(n,n',\alpha,\beta\geqslant 0)$ are arbitrary constants, real or complex.

2. Proof of the main theorem

We start with [4, p. 75]

$${}_2F_1(u,v;w+1/2;t)\,{}_2F_1(w-u,w-v;w+1/2;t)$$

$$= \sum_{k=0}^{\infty} \frac{(w)_k}{(w+1/2)_k} a_k t^k, \tag{5}$$

where a_k is given by (3). Now, multiplying both sides of (5) by $S_n^m[t^{h'}]\, S_{n'}^{m'}[t^{h'}]\, H(z_1 t^{h_1},\ldots,z_r t^{h_r})$ and integrating with respect to t between the limits 0 and 1, we obtain

$$\int_0^1 {}_2F_1(u,v;w+1/2;t)\,{}_2F_1(w-u,w-v;w+1/2;t)$$

$$\cdot S_n^m[t^h]\, S_{n'}^{m'}[t^{h'}]\, H(z_1 t^{h_1},\ldots,z_r t^{h_r})\,\mathrm{d}t$$

$$= \int_0^1 \sum_{k=0}^{\infty} \frac{(w)_k}{(w+1/2)_k} a_k t^k S_n^m[t^h]\, S_{n'}^{m'}[t^{h'}]\, H(z_1 t^{h_1},\ldots,z_r t^{h_r})\,\mathrm{d}t. \tag{6}$$

Using the definitions for the general class of polynomials [5, p. 1, eq. (1)] and of the multivariable H-function [8, p. 251] on the right of (6) and then interchange the order of integration and summations which is permissible under the conditions needed in (4) and evaluating with the following result

$$\int_0^1 t^l S_n^m[t^h]\, S_{n'}^{m'}[t^{h'}]\, H(z_1 t^{h_1},\ldots,z_r t^{h_r})\,\mathrm{d}t$$

$$= \sum_{\alpha=0}^{[n/m]} \sum_{\beta=0}^{[n'/m']} \frac{(-n)_{m\alpha}}{\alpha!} \frac{(-n')}{\beta!} A_{n,\alpha} \cdot A_{n',\beta}$$

$$\cdot H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [-l-h\alpha-h'\beta: h_1,\ldots,h_r], \\ [-l-h\alpha-h'\beta-1: h_1,\ldots,h_r], \end{matrix} \right.$$

$$\left. \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}] : [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(c):\psi',\ldots,\psi^{(r)}] : [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} \begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \right), \tag{7}$$

where

$$h>0, \quad h'>0, \quad h_i>0, \quad \mathrm{Re}\left(l+1+\sum_{i=1}^{r} h_i d_j^{(i)}/\delta_j^{(i)} \right) > 0,$$

$|\arg(z_i)| < T_i\pi/2, T_i > 0, i=1,\ldots,r; j=1,\ldots,u^{(i)}$ and m and m' are arbitrary positive integers and the coefficients $A_{n,\alpha}$ and $A_{n',\beta}(n, n', \alpha, \beta \geqslant 0)$ are arbitrary constants, real or complex, we get the required result.

3. Applications

Taking $u = w$ in the main theorem, the value of a_k in (3) becomes equal to $(v)_k$ and the result (4) gives the following interesting integral

$$\int_0^1 {}_2F_1(u,v;u+1/2;t)\, S_n^m[t^h]\, S_{n'}^{m'}[t^{h'}]\, H(z_1 t^{h_1},\ldots,z_r t^{h_r})\, \mathrm{d}t$$

$$= \sum_{\alpha=0}^{[n/m]} \sum_{\beta=0}^{[n'/m']} \sum_{k=0}^{\infty} \frac{(-n)_{m\alpha}}{\alpha!} \frac{(-n')_{m'\beta}}{\beta!} A_{n,\alpha} \cdot A_{n',\beta} \frac{(u)_k (v)_k}{(u+1/2)_k\, k!}$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [-k-h\alpha-h'\beta: h_1,\ldots,h_r], \\ [-k-h\alpha-h'\beta-1: h_1,\ldots,h_r], \end{matrix} \right.$$

$$\left. \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}] : [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(c):\psi',\ldots,\psi^{(r)}] : [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} \begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \right), \tag{8}$$

where $h>0$, $h'>0$, $h_i>0$, Re $(v) > 1/2$, $\mathrm{Re}(1+\sum_{i=1}^{r} h_i d_j^{(i)}/\delta_j^{(i)}) > 0, |\arg(z_i)| < T_i\pi/2$, $T_i > 0, i=1,\ldots,r; j=1,\ldots,u^{(i)}$ and m and m' are arbitrary positive integers and the coefficients $A_{n,\alpha}$ and $A_{n',\beta}(n, n'\, \alpha, \beta \geqslant 0)$ are arbitrary constants, real or complex.

Setting $v = u+1/2$ and $u = -e$(e is a non-negative integer), we have

$$\int_0^1 {}_1F_0(-e;t)\, S_n^m[t^h]\, S_{n'}^{m'}[t^{h'}]\, H(z_1 t^{h_1},\ldots,z_r t^{h_r})\, \mathrm{d}t$$

$$= \sum_{\alpha=0}^{[n/m]} \sum_{\beta=0}^{[n'/m']} \sum_{k=0}^{e} \frac{(-n)_{m\alpha}}{\alpha!} \frac{(-n')_{m'\beta}}{\beta!} A_{n,\alpha} \cdot A_{n',\beta} \frac{(-e)_k}{k!}$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [-k-h\alpha-h'\beta: h_1,\ldots,h_r], \\ [-k-h\alpha-h'\beta-1: h_1,\ldots,h_r], \end{matrix} \right.$$

$$\left. \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}] : [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(c):\psi',\ldots,\psi^{(r)}] : [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} \begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \right), \tag{9}$$

where $h>0$, $h'>0$, $h_i>0$, $\mathrm{Re}(1+\sum_{i=1}^{r} h_i d_j^{(i)}/\delta_j^{(i)})>0$, $|\arg(z_i)|<T_i\pi/2$, $T_i>0$, $i=1,\dots,r; j=1,\dots,u^{(i)}$ and m and m' are arbitrary positive integers and the coefficients $A_{n,\alpha}$ and $A_{n',\beta}(n,n'\,\alpha,\beta\geqslant 0)$ are arbitrary constants, real or complex.

Now we establish the following interesting and useful expansion formula by appealing to the integral on the left of (9) with the help of (7).

$$\sum_{\alpha=0}^{[n/m]}\sum_{\beta=0}^{[n'/m']}\sum_{k=0}^{e}\frac{(-n)_{m\alpha}}{\alpha!}\frac{(-n')_{m'\beta}}{\beta!}A_{n,\alpha}A_{n',\beta}\frac{(-e)_k}{k!}$$

$$H^{0,\lambda+1:(u',v');\dots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\dots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[-k-h\alpha-h'\beta:h_1,\dots,h_r],\\ [-k-h\alpha-h'\beta-1:h_1,\dots,h_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]:[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\dots,\psi^{(r)}]:[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\right),$$

$$=\Gamma(e+h\alpha+h'\beta+1)\sum_{\alpha=0}^{[n/m]}\sum_{\beta=0}^{[n'/m']}\frac{(-n)_{m\alpha}}{\alpha!}\frac{(-n')_{m'\beta}}{\beta!}A_{n,\alpha}A_{n',\beta}$$

$$H^{0,\lambda+1:(u',v');\dots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\dots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[o:h_1,\dots,h_r],\\ [-e-h\alpha-h'\beta-1:h_1,\dots,h_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]:[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}]:\\ [(c):\psi',\dots,\psi^{(r)}]:[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\right)\tag{10}$$

provided that both the sides exist.

4. Special Cases

(i) If we take $m=2=m'$ and $A_{n,\alpha}=(-1)^{\alpha}$, $A_{n',\beta}=(-1)^{\beta}$ in (4), we obtain the following theorem.

Theorem 1 (a). *If*

$$(1-t)^{u+v-w}\,{}_2F_1(2u,2v;2w:t)=\sum_{k=0}^{\infty}a_k t^k$$

then

$$\int_0^1 {}_2F_1(u,v;w+1/2:t)\,{}_2F_1(w-u,w-v;w+1/2:t)\,t^{hn/2}\,t^{h'n'/2}$$

$$H_n\left(\frac{1}{2\sqrt{t^h}}\right)H_{n'}\left(\frac{1}{2\sqrt{t^{h'}}}\right)H(z_1t^{h_1},\dots,z_rt^{h_r})\,\mathrm{d}t$$

$$=\sum_{\alpha=0}^{n/2}\sum_{\beta=0}^{n'/2}\sum_{k=0}^{\infty}\frac{(-n)_{2\alpha}(-1)^{\alpha}}{\alpha!}\frac{(-n')_{2\beta}(-1)^{\beta}}{\beta!}\frac{(w)_k a_k}{(w+1/2)_k}$$

$$H^{0,\lambda+1:(u',v');\dots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\dots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[-k-h\alpha-h'\beta:h_1,\dots,h_r],\\ [-k-h\alpha-h'\beta-1:h_1,\dots,h_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]:[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\dots,\psi^{(r)}]:[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\right),\tag{11}$$

valid under the same conditions needed for (4).

(ii) On taking $m = 1 = m'$ and

$$A_{n,\alpha} = \binom{n+f}{n}\frac{(f+g+n+1)_\alpha}{(f+1)_\alpha},$$

$$A_{n',\beta} = \binom{n'+f'}{n'}\frac{(f'+g'+n'+1)_\beta}{(f'+1)_\beta} \text{ in (4),}$$

we get the following theorem.

Theorem 1(b). *If*

$$(1-t)^{u+v-w}\ {}_2F_1(2u,2v;2w:t) = \sum_{k=0}^{\infty} a_k t^k$$

then

$$\int_0^1 {}_2F_1(u,v;w+1/2:t)\ {}_2F_1(w-u,w-v;w+1/2:t)$$

$$P_n^{(f,g)}(1-2t^h)\ P_{n'}^{(f',g')}(1-2t^{h'})\ H(z_1 t^{h_1},\ldots,z_r t^{h_r})\,\mathrm{d}t$$

$$= \sum_{\alpha=0}^{n}\sum_{\beta=0}^{n'}\sum_{k=0}^{\infty}(-1)^\alpha(-1)^\beta\binom{n+f}{n-\alpha}\binom{n+f+g+\alpha}{\alpha}$$

$$\binom{n'+f'}{n'-\beta}\binom{n'+f'+g'+\beta}{\beta}\cdot\frac{(w)_k a_k}{(w+1/2)_k}$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[-k-h\alpha-h'\beta:h_1,\ldots,h_r],\\ [-k-h\alpha-h'\beta-1:h_1,\ldots,h_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:\ [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\ldots,\psi^{(r)}]:\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ \begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\right), \tag{12}$$

valid under the same conditions as obtainable from (4).

(iii) Letting $m = 1 = m'$ and $A_{n,\alpha} = \binom{n+f}{n}\frac{1}{(f+1)_\alpha}$, $A_{n',\beta} = \binom{n'+f'}{n'}\frac{1}{(f'+1)_\beta}$ in (4), we get the following.

Theorem 1(c). *If*

$$(1-t)^{u+v-w}\ {}_2F_1(2u,2v:2w:t) = \sum_{k=0}^{\infty} a_k t^k$$

then

$$\int_0^1 {}_2F_1(u,v;w+1/2;t)\ {}_2F_1(w-u,w-v;w+1/2;t)$$

$$L_n^{(f)}(t^h)\ L_{n'}^{(f')}(t^{h'})\ H(z_1 t^{h_1},\ldots,z_r t^{h_r})\,\mathrm{d}t$$

$$= \sum_{\alpha=0}^{n}\sum_{\beta=0}^{n'}\sum_{k=0}^{\infty}\frac{(-1)^\alpha}{\alpha!}\frac{(-1)^\beta}{\beta!}\binom{n+f}{n-\alpha}\binom{n'+f'}{n'-\beta}\frac{(w)_k a_k}{(w+1/2)_k}$$

$$H^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[-k-h\alpha-h'\beta:h_1,\ldots,h_r],\\ [-k-h\alpha-h'\beta-1:h_1,\ldots,h_r],\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:\ [(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\ z_1\\ [(c):\psi',\ldots,\psi^{(r)}]:\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\ \vdots\ z_r\end{matrix}\right), \tag{13}$$

valid under the same conditions as obtainable from (4).

(iv) *If* $n' \to 0$, *the theorem given by* (4) *reduces to a theorem recently obtained by Chaurasia and Sharma* [2, *p.* 291, *eq.* (3)].

(v) *Taking* $n' \to 0$, *the results in* (7) *to* (10) *reduce to the known results obtained by Chaurasia and Sharma* [2, *eqs* (6), *p.* 292, (7) *and* (8), *p.* 293 *and* (9), *p.* 294].

(vi) *The main theorem given in* (4) *reduces to a known theorem obtained by Chaurasia* [1, *eq.* (1.2), *p.* 193], *on making* $n \to 0$ *and* $n' \to 0$.

(vii) *When* $n \to 0$ *and* $n' \to 0$ *the result* (7) *through* (10) *reduce to the known results obtained by Chaurasia* [1, *eq.* (2.3), *p.* 194, (3.1) *and* (3.2), *p.* 195 *and* (3.3), *p.* 195].

The importance of our results lies in its manifold generality. In view of its generality of the multivariable H-function, on specializing the various parameters and variables in the multivariable H-function, we can derive, from our theorems, integrals and expansion formulae etc. involving a remarkably wide variety of useful functions (or products of several such functions) which are expressible in terms of E, F, G and H-functions of one and more variables.

Secondly, by specializing the coefficients $A_{n,\alpha}$ and $A_{n',\beta}$ and making a free use of the special cases of $s_n^m[x]$ listed by Srivastava and Singh [7], our results can be reduced to a large number of theorems, integrals and expansion formulae etc. involving the product of generalized Hermite polynomials, Hermite polynomials, Jacobi polynomials and their various special cases, products of Laguerre polynomials, Bessel polynomials, Gould-Hopper polynomials, Brafman polynomials and their various combinations. Thus, the results established in this paper would at once yield a very large number of known and new theorems, integrals and expansion formulae, involving a large variety of polynomials and various special functions appearing in the literature on mathematical analysis, applied mathematics and mathematical physics.

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