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Some inequalities for the polar derivative of a polynomial

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Abstract. Let $P(z)$ be a polynomial of degree n which does not vanish in $|z| < 1$. In this paper, we estimate the maximum and minimum moduli of the kth polar derivative of $P(z)$ on $|z| = 1$ and thereby obtain compact generalizations of some known results, which among other results, yields interesting refinements of Erdos–Lax theorem and a theorem of Ankeny and Rivlin.

Keywords. Bernstein's inequality; polar derivative of a polynomial; inequalities in the complex domain; growth of maximum modulus.

1. Introduction and statement of results

If $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)| \tag{1}$$

and

$$\max_{|z|=R>1} |P(z)| \leqslant R^n \max_{|z|=1} |P(z)|. \tag{2}$$

Inequality (1) is an immediate consequence of Bernstein's theorem on the derivative of a trigonometric polynomial. (For reference, see [13, 6].) Inequality (2) is a simple deduction from the maximum modulus principle (see [12, p. 346], [10, p. 158, problem 269]).

Recently Aziz and Dawood [5] have shown that if $P(z)$ has all zeros in $|z| \leqslant 1$, then

$$\min_{|z|=1} |P'(z)| \geqslant n \min_{|z|=1} |P(z)| \tag{3}$$

and

$$\min_{|z|=R>1} |P(z)| \geqslant R^n \min_{|z|=1} |P(z)|. \tag{4}$$

Ineqalities (1), (2), (3) and (4) are sharp, with equality for the polynomial $P(z) = m\, e^{i\alpha} z^n$, that is, when $P(z)$ has all its zeros at the origin.

Inequalities (1) and (2) can be sharpened, if we restrict ourselves to the class of polynomials having no zero in $|z| < 1$. In fact, if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)| \tag{5}$$

and

$$\max_{|z|=R>1} |P(z)| \leqslant \left(\frac{R^n+1}{2}\right) \max_{|z|=1} |P(z)|. \tag{6}$$

Both these estimates are sharp, with equality for the polynomial $P(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta| = \frac{1}{2}$. Inequality (5) was conjectured by Erdös and later verified by Lax [7] (see also [4]), whereas Ankeny and Rivlin [1] used (5) to prove (6).

Inequalities (5) and (6) were further improved in [5], where under the same hypothesis, it was shown that

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2}\left\{\max_{|z|=1}|P(z)| - \min_{|z|=1}|P(z)|\right\}. \tag{7}$$

and

$$\max_{|z|=R>1} |P(z)| \leqslant \left(\frac{R^n+1}{2}\right)\max_{|z|=1}|P(z)| - \left(\frac{R^n-1}{2}\right)\min_{|z|=1}|P(z)|. \tag{8}$$

In both (7) and (8) equality holds for the polynomial $P(z) = \alpha z^n + \beta$, where $|\beta| \geqslant |\alpha|$.

Let $D_\alpha P(z)$ denote the polar differentiation of the polynomial $P(z)$ of degree n with respect to the point α. Then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ of the polynomial $P(z)$ in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha - z} = P'(z).$$

Now corresponding to a given nth degree polynomial $P(z)$, we construct a sequence of polar derivatives

$$D_{\alpha_1}P(z) = nP(z) + (\alpha_1 - z)P'(z),$$

$$D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z) = (n-k+1)D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_{k-1}}P(z) + (\alpha_k - z)(D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_{k-1}}P(z))',$$

$$k = 2, 3, \ldots, n.$$

The points $\alpha_1, \alpha_2, \ldots, \alpha_k$; $k = 1, 2, \ldots, n$ may be equal or unequal. Like the kth ordinary derivative $P^{(k)}(z)$ of $P(z)$, the kth polar derivative $D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)$ of $P(z)$ is a polynomial of degree at most $n-k$.

More recently, Aziz [2] extended (5) and (6) to the polar derivative of a polynomial.

In the present paper we shall first extend (3) and(4) to the polar derivative and thereby present a compact generalization of these resuslts as well. Thus we prove the following.

Theorem 1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for $|z| \geqslant 1$,*

$$\min_{|z|=1} |D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)|$$

$$\geqslant n(n-1)\cdots(n-k+1)\,|\alpha_1, \alpha_2\cdots\alpha_k|\,|z|^{n-k}\min_{|z|=1}|P(z)|,$$

where α_i are real or complex numbers with $|\alpha_i| \geqslant 1$, $i = 1, 2, \ldots, k$; $k \leqslant n-1$. The result is best possible and equality holds for the polynomial $P(z) = m\,e^{i\beta} z^n$, $m > 0$.

The following corollary, which immediately follows from Theorem 1, is a compact generalization of (3) ad (4).

COROLLARY 1

If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for any real or complex number α, with $|\alpha| \geqslant 1$,

$$|D_\alpha P(z)| \geqslant n|\alpha||z|^{n-1} \min_{|z|=1}|P(z)|, \quad \textit{for} \ |z| \geqslant 1. \tag{10}$$

The result is best possible, with equality for the polynomial having all its zeros at the origin.

Remark 1. Dividing the two sides of (10) by $|\alpha|$, letting $\alpha \to \infty$ and noting that

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z),$$

we get

$$|P'(z)| \geqslant n\,|z|^{n-1} \min_{|z|=1}|P(z)|, \text{ for } |z| \geqslant 1,$$

which in particular gives (3).

Taking $z = \alpha$ in (10) and noting that

$$|D_\alpha P(z)|_{z=\alpha} = n|P(\alpha)|,$$

we obtain

$$|P(\alpha)| \geqslant |\alpha|^n \min_{|z|=1}|P(z)|,$$

for every α with $|\alpha| \geqslant 1$, from which (4) follows immediately.

Next we present the following result which is a refinement of Theorem 1 of [2] and is also a compact generalization of (7) and (8).

Theorem 2. *If $P(z)$ is a polynomial of degree n which does not vanish in the disk $|z| < 1$, then for $|z| \geqslant 1$*

$$\begin{aligned} &|D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z)| \\ &\leqslant \frac{n(n-1)\cdots(n-k+1)}{2}\Big\{(|\alpha_1 \alpha_2 \cdots \alpha_k||z|^{n-k} + 1)\max_{|z|=1}|P(z)| \\ &\qquad - (|\alpha_1 \alpha_2 \cdots \alpha_k||z|^{n-k} - 1)\min_{|z|=1}|P(z)|\Big\}. \end{aligned} \tag{11}$$

where $|\alpha_i| \geqslant 1$ for $i = 1, 2, \ldots, k$. The result is best possible and equality in (11) holds for the polynomial

$$P(z) = \tfrac{1}{2}(z^n + 1).$$

The following corollary which immediately follows from Theorem 2, is a compact generalization of (7) and (8).

COROLLARY 2

If $P(z)$ is a polynomial of degree n which has no zeros in the disk $|z| < 1$, then for every real or complex number α, with $|\alpha| \geqslant 1$,

$$|D_\alpha P(z)| \leqslant \frac{n}{2}\left\{(|\alpha|\,|z|^{n-1}+1)\max_{|z|=1}|P(z)| - (|\alpha|\,|z|^{n-1}-1)\min_{|z|=1}|P(z)|\right\}, \text{ for } |z| \geqslant 1. \tag{12}$$

The result is best possible and equality in (12) holds for the polynomial $P(z) = a\,z^n + b$ where $|a| = |b| = \frac{1}{2}$ and $|\alpha| \geqslant 1$.

Remark 2. By using similar argument as in Remark 1, it follows that if we divide both sides of (12) by $|\alpha|$ and let $|\alpha| \to \infty$, we get (7) and for $z = \alpha$, we get (8).

If we write

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_2 z^2 + a_1 z + a_0,$$

then

$$\begin{aligned} D_\alpha P(z) &= nP(z) + (\alpha - z)P'(z) \\ &= (n\alpha a_n + a_{n-1})z^{n-1} + ((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-2} \\ &\quad + \cdots + (2\alpha a_2 + (n-1)a_1)z + (\alpha a_1 + na_0). \end{aligned}$$

If the polynomial $P(z)$ has no zeros in the disk $|z| < 1$, then by Theorem 2, with $k = 1$, it follows that

$$\begin{aligned} &|(n\alpha a_n + a_{n-1})z^{n-1} + ((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-2} + \cdots + (2\alpha a_2 + (n-1)a_1)z \\ &\qquad + (\alpha a_1 + na_0)| \\ &\leqslant \frac{n}{2}\left\{(|\alpha|\,|z|^{n-1}+1)\max_{|z|=1}|P(z)| - (|\alpha|\,|z|^{n-1}-1)\min_{|z|=1}|P(z)|\right\}, \\ &\qquad \text{for } |z| \geqslant 1, |\alpha| \geqslant 1. \end{aligned}$$

If we divide two sides of this inequality by $|z|^{n-1}$ and let $|z| \to \infty$, we easily obtain

$$|n\alpha a_n + a_{n-1}| \leqslant \frac{n}{2}|\alpha|\left\{\max_{|z|=1}|P(z)| - \min_{|z|=1}|P(z)|\right\},$$

for every α with $|\alpha| \geqslant 1$. By choosing argument of α suitably, we immediately get the following result, which is the refinement of Corollary 2 of [2].

COROLLARY 3

If

$$P(z) = \sum_{j=1}^{n} a_j z^j$$

is a polynomial of degree n which does not vanaish in the disk $|z| < 1$, then

$$n|a_n| + |a_{n-1}| \leqslant \frac{n}{2}\left\{\max_{|z|=1}|P(z)| - \min_{|z|=1}|P(z)|\right\}.$$

2. Lemmas

For the proofs of the theorems, we need the following lemmas. The first result is Laguerre's theorem [8, p. 52] (see also [3]).

Lemma 1. *If all the zeros of the nth degree polynomial $P(z)$ lie in a circular region C, and if W is any zero of*

$$D_\alpha P(z) = nP(z) + (\alpha - z)\, P'(z),$$

the polar derivative of $P(z)$, then both points W and α may not lie outside of C.

By repeated applications of Lemma 1, we get the following result, when the circular region C is the circle $|z| \leqslant r$.

Lemma 2. *If all the zeros of an nth degree polynomial $P(z)$ lie in $|z| \leqslant r$ and if none of the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ lie in $|z| \leqslant r$, then each of the polar derivatives $D_{\alpha_1} D_{\alpha_2} \cdots, D_{\alpha_k} P(z)$, $k = 1, 2, \ldots, n-1$ has all its zeros in $|z| \leqslant r$.*

The next lemma is due to Aziz [2].

Lemma 3. *If $P(z)$ is a polynomial of degree n such that*

$$\max_{|z|=1} |P(z)| = M$$

and $\alpha_1, \alpha_2, \ldots, \alpha_k$; $k \leqslant n-1$ are complex numbers with $|\alpha_i| \geqslant 1$ for all $i = 1, 2, \ldots, k$; then for $|z| \geqslant 1$

$$|D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z)| + |D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} Q(z)|$$
$$\leqslant n(n-1) \cdots (n-k+1) \{ |\alpha_1 \alpha_2 \cdots \alpha_k| |z|^{n-k} + 1 \} M, \tag{13}$$

where

$$Q(z) = z^n \overline{P(1/\bar{z})}.$$

3. Proofs of the theorems

Proof of theorem 1. The result is clear if $P(z)$ has a zero on $|z| = 1$, then

$$m = \min_{|z|=1} |P(z)| = 0.$$

We now suppose that all the zeros of $P(z)$ lie in $|z| < 1$, then $m > 0$ and we have $m \leqslant |P(z)|$ for $|z| = 1$. Hence for every β with $|\beta| < 1$, we have $|P(z)| > |m\beta z^n|$ for $|z| = 1$ and therefore, it follows by Rouche's theorem that the polynomial $F(z) = P(z) - m\beta z^n$ has all its zeros in $|z| < 1$. If $\alpha_1, \alpha_2, \ldots. \alpha_k$ are complex numbers with $|\alpha_i| \geqslant 1$, $i = 1, 2, \ldots.$ k; $k \leq n-1$, then by Lemma 2, all the zeros of

$$D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} F(z)$$
$$= D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z) - m\beta n(n-1) \cdots (n-k+1) \alpha_1 \alpha_2 \cdots \alpha_k z^{n-k} \tag{14}$$

lie in $|z| < 1$, which implies, for $|z| \geqslant 1$,

$$mn(n-1) \cdots (n-k+1) |\alpha_1 \alpha_2 \cdots \alpha_k| |z|^{n-k} \leqslant |D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} P(z)|. \tag{15}$$

If inequality (15) is not true, then there is a point $z = z_0$, with $|z_0| \geqslant 1$, such that

$$[mn(n-1)\cdots(n-k+1)|\alpha_1\alpha_2\cdots\alpha_k||z|^{n-k}]_{z=z_0} > [|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)|]_{z=z_0},$$

and we take

$$\beta = \frac{[D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)]_{z=z_0}}{[mn(n-1)\cdots(n-k+1)\alpha_1\alpha_2\cdots\alpha_k z^{n-k}]_{z=z_0}},$$

so that $|\beta| < 1$ and from (14), with this choice of β, we get $[D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}F(z)]_{z=z_0} = 0$, where $|z_0| \geqslant 1$, which contradicts the fact that the zeros of $D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}F(z)$ lie in $|z| < 1$. Therefore from (15) we conclude that

$$\min_{|z|=1}|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)|$$

$$\geqslant n(n-1)\cdots(n-k+1)|\alpha_1\alpha_2\cdots\alpha_k||z|^{n-k}\min_{|z|=1}|P(z)|$$

and this completes the proof of Theorem 1.

Proof of theorem 2. By hypothesis, the polynomial $P(z)$ has all its zeros in $|z| \geqslant 1$. In case it has a zero on $|z| = 1$, then

$$m = \min_{|z|=1}|P(z)| = 0$$

and the result follows by Theorem 1 of [2].

We now suppose that all the zeros of $P(z)$ lie in $|z| > 1$, then $m > 0$ and

$$m \leqslant |P(z)| \quad \text{for} \quad |z| = 1. \tag{16}$$

If β is any complex number such that $|\beta| < 1$, then it follows by Rouche's theorem that the polynomial $F(z) = P(z) - \beta m$ does not vanish on $|z| = 1$ also. Because, if for some $z = z_0$,

$$F(z_0) = P(z_0) - m\beta = 0, \text{ where } |z_0| = 1,$$

then we have

$$|P(z_0)| = |m\beta| = m|\beta| < m,$$

which is a contradiction to (16). Thus for every β, with $|\beta| < 1$, the polynomial $F(z) = P(z) - \beta m$ has all its zeros in $|z| > 1$. If

$$G(z) = z^n\overline{F(1/\bar{z})} = z^n\overline{P(1/\bar{z})} - \bar{\beta}mz^n = Q(z) - \bar{\beta}mz^n,$$

then all the zeros of $G(z)$ lie in $|z| < 1$ and $|G(z)| = |F(z)|$ for $|z| = 1$. Therefore, for every complex number λ with $|\lambda| > 1$, the polynomial $F(z) - \lambda G(z)$ has all its zeros in $|z| < 1$. It follows by Lemma 1, that if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are complex numbers such that $|\alpha_i| \geqslant 1$, $1 \leqslant i \leqslant k$, the polynomial $D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}(F(z) - \lambda G(z))$ has all its zeros in $|z| < 1$. Equivalently, all the zeros of $D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}F(z) - \lambda D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}G(z)$ lie in $|z| < 1$. This clearly implies that for $|z| \geqslant 1$,

$$|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}F(z)| \leqslant |D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}G(z)|,$$

or

$$|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}(P(z) - m\beta)| \leqslant |D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}(Q(z) - m\bar{\beta}z^n)|,$$

equivalently

$$|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)-m\beta n(n-1)\cdots(n-k+1)|$$
$$\leqslant|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}Q(z)-m\bar{\beta}n(n-1)\cdots(n-k+1)\alpha_1\alpha_2\cdots\alpha_k z^{n-k}|. \tag{17}$$

In view of (15) we can choose an argument of β in (17) such that for $|z|\geqslant 1$,

$$|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)|-m|\beta|n(n-1)\cdots(n-k+1)$$
$$\leqslant|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}Q(z)|-m|\beta|n(n-1)\cdots(n-k+1)/\alpha_1\alpha_2\cdots\alpha_k||z|^{n-k}. \tag{18}$$

Letting $|\beta|\to 1$ in (18), we get

$$|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)|$$
$$\leqslant|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}Q(z)|-n(n-1)\cdots(n-k+1)\{|\alpha_1\alpha_2\ldots\alpha_k||z|^{n-k}-1\}m. \tag{19}$$

Inequality (19) in conjuction with Lemma 3 gives for $|z|\geqslant 1$, $|\alpha_i|\geqslant 1$, $i=1,2,\ldots,k$; $k\leqslant n-1$,

$$|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}P(z)|$$
$$\leqslant\frac{n(n-1)\cdots(n-k+1)}{2}\left\{(|\alpha_1\alpha_2\cdots\alpha_k||z|^{n-k}+1)\max_{|z|=1}|P(z)|\right.$$
$$\left.-(|\alpha_1\alpha_2\cdots\alpha_k||z|^{n-k}-1)\min_{|z|=1}|P(z)|\right\}.$$

This proves the desired result.

4. Note

Finally, we mention that the idea of the proof of Theorem 1 comes from a paper of Bernstein [6], where he applied the Gauss–Lucas theorem to obtain (1). The central idea is that if a polynimial $P(z)$ has all its zeros in the unit disk, then so does the derivative $P'(z)$. There are operators other than differentiation for which this is true. Polar derivative with respect to a point outside the unit disk is one such operator. There are many others and just as interesting (for example, see [11, p. 55–59]). It is an interesting problem to characterize all such operators. An interested reader may consult [9] and [11] for other results related to the problem considered in this paper.

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