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Harmonic manifolds with some specific volume densities

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Abstract. We show that a noncompact, complete, simply connected harmonic manifold (M^d, g) with volume density $\theta_m(r) = \sinh^{d-1} r$ is isometric to the real hyperbolic space and a noncompact, complete, simply connected Kähler harmonic manifold (M^{2d}, g) with volume density $\theta_m(r) = \sinh^{2d-1} r \cosh r$ is isometric to the complex hyperbolic space. A similar result is also proved for quaternionic Kähler manifolds. Using our methods we get an alternative proof, without appealing to the powerful Cheeger–Gromoll splitting theorem, of the fact that every Ricci flat harmonic manifold is flat. Finally a rigidity result for real hyperbolic space is presented.

Keywords. Harmonic manifolds; volume densities; Ricci curvature; second fundamental form.

1. Introduction

A d-dimensional Riemannian manifold (M, g) (supposed to be complete in the rest of the paper) is said to be harmonic if its volume density function $\omega_m = \sqrt{|\det(g_{ij})|}$ is a spherically symmetric function around m. In polar coordinates this density can be written as

$$\theta_m = r_m^{d-1}\, \omega_m,$$

where $r_m(n) = r(m, n)$ is the geodesic distance between m and n. Thus (M, g) is harmonic if θ_m is a radial function around m (see [1], [11] or [12] for details). Two-point homogeneous spaces are harmonic as can be easily seen from their density function. Besides these there were no known non-flat examples of harmonic spaces. Moreover Lichnerowicz proved that up to dimension 4, harmonic spaces are in fact locally isometric to rank one symmetric spaces. This led to the Lichnerowicz conjecture which asserts that *every harmonic space is locally isometric to a two-point homogeneous space.* It should be noted that even higher rank symmetric spaces are not harmonic.

Let (M, g) be a harmonic space. The well known Ledger's formula ([1], see p. 161) gives

$$\omega_p''(r)|_{r=0} = -\frac{1}{3}\mathrm{Ricci}_p.$$

Hence for harmonic manifolds, since ω_p is a function of r alone, the Ricci curvature is a constant, i.e. harmonic spaces are Einstein. Let Ricci $(M) = k$. There arise three cases:

1. $k > 0$. In this case, by Myers–Bonnet theorem, M is compact with finite fundamental group and Szabo [11] proved that compact harmonic manifolds with finite fundamental group are rank one symmetric, thus settling the Lichnerowicz conjecture.
2. $k = 0$. Here one appeals to the powerful Cheeger–Gromoll splitting theorem to conclude that M is isometric to the Euclidean space, i.e. *Ricci-flat harmonic manifolds are flat.*

3. $k<0$. The Lichnerowicz conjecture is not true in this case. Damek and Ricci [3] constructed a family of nonsymmetric harmonic spaces. These spaces are called the NA spaces. In this family there are harmonic manifolds with the same density function as that of quaternionic hyperbolic spaces. All these spaces are homogeneous. Presently it is not known whether there are nonhomogeneous harmonic spaces. So it seems that the classification of harmonic spaces can be achieved only up to the determination of all density functions, i.e, density classification of harmonic spaces. So Szabo [11] asked the following question. "Which harmonic spaces are determined by their density functions?"

We call this notion as weak density equivalence. Szabo [12] defines a stronger notion of density equivalent spaces. In § 2 we answer the above question for three specific cases, namely that of the real, complex and quaternionic hyperbolic spaces. More precisely we prove

Theorem 1.1. *Let (M^d,g) be a non-compact, simply connected harmonic space with density function $\theta(r)=\sinh^{d-1}r$, then (M^d,g) is isometric to the real hyperbolic space.*

Theorem 1.2. *Let (M^{2d},g) be a non-compact, simply connected, Kähler harmonic manifold with density function $\theta(r)=\sinh^{2d-1}r\cosh r$, then M is isometric to the complex hyperbolic space.*

Theorem 1.3. *Let (M^{4d},g) be a noncompact, simply connected, quaternionic Kähler harmonic manifold with volume density $\theta(r)=\sinh^{4d-1}r\cosh^3 r$, then M is isometric to the quaternionic hyperbolic space.*

Theorems (1.1) and (1.2) explain the lack of examples of nonsymmetric (Kähler) harmonic spaces with the same density function as that of the real (complex) hyperbolic space. Using the same methods we also give an alternative proof, without appealing to the powerful Cheeger–Gromoll splitting theorem, of the fact that every Ricci-flat harmonic manifold is isometric to the Euclidean space, which is as follows.

Theorem 1.4. *Every simply connected Ricci-flat harmonic manifold is flat, i.e, it is isometric to the Euclidean space.*

Let (M^d,g) be a noncompact harmonic manifold with density function $\theta(r)$. In § 3 we show that $\theta(r)$ satisfies

$$\frac{\theta(r)}{\sinh^{d-1}r}\to 0 \quad \text{as} \quad r\to\infty$$

or

$$\frac{\theta(r)}{\sinh^{d-1}r}\to c>0 \quad \text{as} \quad r\to\infty.$$

In the second case we show that M is locally isometric to the real hyperbolic space.

2. Weak density equivalence

Proof of theorem (1.1). Choose m in M. Since $\theta(r)=\sinh^{d-1}r$, $\omega_m(r)=(1/r^{d-1})\sinh^{d-1}r$. Hence from the formula

$$\omega_m''(0)=-\tfrac{1}{3}\,\mathrm{Ricci}_m$$

we see that Ricci $(M) = -(d-1)$. Let γ be a geodesic starting at m and $S_{m,r}$ be the distance sphere around m of radius r. Again, since $\theta(r) = \sinh^{d-1} r$, the mean curvature $\sigma_m(r)$ of $S_{m,r}$ is

$$\sigma_{m,r} = \frac{\theta'_m(r)}{\theta_m(r)} = (d-1)\coth r.$$

Let $L = L_{m,r}$ be the second fundamental form of $S_{m,r}$, then

$$\operatorname{trace} L = \sigma_{m,r} = (d-1)\coth r.$$

Moreover L satisfies the Riccati equation

$$L' + L^2 + R(.,\gamma')\gamma' = 0,$$

where R is the curvature tensor of M [8]. Taking traces we get

$$\operatorname{trace} L' + \operatorname{trace} L^2 + \operatorname{Ricci}(\gamma',\gamma') = 0.$$

But trace $L = (d-1)\coth r$. Therefore,

$$\operatorname{trace} L^2 = (d-1)\coth^2 r = \frac{1}{d-1}(\operatorname{trace} L)^2.$$

Now let us use the Cauchy–Schwarz inequality for the linear map L. We have

$$\operatorname{trace} L^2 \geqslant \frac{1}{d-1}(\operatorname{trace} L)^2$$

and equality holds if and only if L is a scalar operator. In our case equality holds for L, hence L is a scalar operator. But trace $L = (d-1)\coth r$, thus $L_{m,r} = \coth r Id$ for all points m in M and all $r > 0$. Thus M has constant sectional curvature equal to -1. Finally because M is simply connected it is isometric to the real hyperbolic space. ■

Now let (M^{2d}, g) be a Kähler harmonic manifold. Denote its complex structure by J. Assume that M has density function $\theta(r) = \sinh^{2d-1} r \cosh r$. The main difficulty in using the method of proof of the above theorem in proving that M is isometric to the complex hyperbolic space is the following: in this case the Riccati equation may not split into two equations along the plane spanned by $\{\gamma', J\gamma'\}$. The Riccati equation will split if and only if $J\gamma'$ is an eigen direction of $R(.,\gamma')\gamma'$. Equivalently this will happen if and only if for any vector field X, the subbundle generated by $\{X, JX\}$ is integrable. To overcome this difficulty we compute the index form along γ.

Proof of theorem (1.2). Let m be a arbitrary point of M. Take any geodesic $\gamma(t)$ starting at m. Denote the complex structure on M by J and let I be the index form of M. Choose any real number $T > 0$. Let $E_2(t), E_3(t), \ldots, E_{2d}(t)$ be unit orthogonal parallel fields along $\gamma(t)$ normal to $\gamma'(t)$ with $E_2(t) = J\gamma'(t)$. Choose Jacobi fields $\{J_i(t)\}_{i=2}^{2d}$ along $\gamma(t)$ such that

$$J_i(0) = 0,\ J_i(T) = E_i(T),\ i = 2, \ldots, 2d.$$

Let

$$X_2(t) = \frac{\sinh 2t}{\sinh 2T} E_2(t)$$

and

$$X_i(t) = \frac{\sinh t}{\sinh T} E_i(t), i = 3, \ldots, 2d$$

be vector fields along $\gamma(t)$. Note that $J_i(0) = X_i(0)$ and $J_i(T) = X_i(T)$ for all i. We compare the index forms of J_i and X_i for corresponding i.

By Allamigeon's theorem we know that there are no conjugate points along γ [1]. Hence we get $I(J_i, J_i) \leqslant I(X_i, X_i)$ for all i. Sum over all i to get

$$\sum_{i=2}^{2d} I(J_i, J_i) \leqslant \sum_{i=2}^{2d} I(X_i, X_i). \tag{2.1}$$

We know that $\Sigma_{i=2}^{2d} I(J_i, J_i) = (\theta'_m(T)/\theta_m(T))$. Further a simple calculation gives

$$I(X_2, X_2) = \frac{\sinh 4T + 4T}{2\sinh^2 2T} - \int_0^T \frac{\sinh^2 2t}{\sinh^2 2T} H(\gamma'(t))\, dt \tag{2.2}$$

and

$$I(X_i, X_i) = \frac{2T + \sinh 2T}{4\sinh^2 T} - \int_0^T \frac{\sinh^2 t}{\sinh^2 T} K(\gamma'(t), E_i(t))\, dt, i = 3, \ldots, 2d.$$

Here $K(\gamma'(t), E_i(t))$ is the sectional curvature of the plane spanned by $\gamma'(t)$ and $E_i(t)$ and $H(\gamma'(t)) = K(\gamma'(t), J\gamma'(t))$ is the holomorphic sectional curvature. Also, since $\theta(r) = \sinh^{2d-1} r \cosh r$, the Ricci curvature of M is $-2(d+1)$. Now let us calculate the sum $\Sigma_3^{2d} I(X_i, X_i)$. We have

$$\begin{aligned}
&\sum_3^{2d} I(X_i, X_i)\\
&= (2d-2)\frac{2T + \sinh 2T}{4\sinh^2 T} - \int_0^T \frac{\sinh^2 t}{\sinh^2 T}\left(\sum_3^{2d} K(\gamma'(t), E_i(t))\right) dt\\
&= (2d-2)\frac{2T + \sinh 2T}{4\sinh^2 T} - \int_0^T \frac{\sinh^2 t}{\sinh^2 T}(\mathrm{Ricci}\,(\gamma'(t), \gamma'(t)) - H(\gamma'(t)))\, dt\\
&= (2d-2)\frac{2T + \sinh 2T}{4\sinh^2 T} + 2(d+1)\left(\frac{\sinh 2T - 2T}{4\sinh^2 T}\right)\\
&\quad + \int_0^T \frac{\sinh^2 t}{\sinh^2 T} H(\gamma'(t))\, dt\\
&= \frac{4d\sinh 2T - 8T}{4\sinh^2 T} + \int_0^T \frac{\sinh^2 t}{\sinh^2 T} H(\gamma'(t))\, dt.
\end{aligned} \tag{2.3}$$

Thus (2.2) and (2.3) gives

$$\sum_{i=2}^{2d} I(X_i, X_i) = A(T) + \int_0^T B(t) H(\gamma'(t))\, dt,$$

where

$$A(T) = \frac{\sinh 4T + 4T}{2\sinh^2 2T} + \frac{1}{4\sinh^2 T}(4d \sinh 2T - 8T)$$

and

$$B(t)=\frac{\sinh^2 t}{\sinh^2 T}-\frac{\sinh^2 2t}{\sinh^2 2T}.$$

Substituting all this information in (2.1), we get

$$\sum_{i=2}^{2d} I(J_i,J_i)=\frac{\theta_m'(T)}{\theta_m(T)}\leqslant (A(T)-4C(T))+\int_0^T B(t)(H(\gamma'(t))+4)\,\mathrm{d}t$$
$$=\sum_{i=2}^{2d} I(X_i,X_i),$$

where

$$C(T)=\int_0^T B(t)\,\mathrm{d}t$$
$$=\frac{\sinh 2T-2T}{4\sinh^2 T}-\frac{\sinh 4T-4T}{8\sinh^2 2T}.$$

Note that $B(t)\geqslant 0$, $t\in[0,T]$. Finally, we have

$$A(T)-4C(T)$$
$$=\frac{\sinh 4T+4T}{2\sinh^2 2T}+\frac{1}{4\sinh^2 T}(4d\sinh 2T-8T)$$
$$-\frac{\sinh 2T-2T}{\sinh^2 T}-\frac{\sinh 4T-4T}{2\sinh^2 2T}$$
$$=\frac{\sinh 4T}{\sinh^2 2T}+\frac{(d-1)\sinh 2T}{\sinh^2 T}$$
$$=2\coth 2T+2(d-1)\coth T$$
$$=(2d-1)\coth T+2\coth 2T-\coth T$$
$$=(2d-1)\coth T+\tanh T$$
$$=\frac{\theta_m'(T)}{\theta_m(T)}.$$

Thus it follows that

$$\int_0^T B(t)(H(\gamma'(t))+4)\,\mathrm{d}t\geqslant 0. \tag{2.4}$$

Now suppose that $H(\gamma'(t))<-4$ for some $t>0$, say $t=s$. Then by continuity of H we see that $H(\gamma'(t))<-4$ in a small neighbourhood of s, say $(s-\varepsilon, s+\varepsilon)$. But the point $m=\gamma(0)$ and $0<T$ we started with, are arbitrary. Hence if we take $m=\gamma(s)$ and $T=\varepsilon$ in the above computation, then we will get a contradiction to the inequality (2.4). Thus, since $B(t)\geqslant 0$, $t\in[0,T]$, we get $H(\gamma'(t))\geqslant -4$ for all $t>0$. Again the geodesic γ is arbitrary. We conclude that

$$H(v)\geqslant -4$$

for all unit vectors v. An algebraic calculation ([1], see p. 150) yields

$$\int_{U_m M} H(v)\,\mathrm{d}v = \frac{\mathrm{Vol}(U_m M)}{d(d+1)} \mathrm{Scal}_m M,$$

where Scal is the scalar curvature of M. In our case, $\mathrm{Ricci}(M) = -2(d+1)$, hence $\mathrm{Scal}(M) = -4d(d+1)$. Thus

$$\int_{U_m M} H(v)\mathrm{d}v = -4\,\mathrm{Vol}(U_m M).$$

Combined with the conclusion $H(v) \geqslant -4$, we get $H(v) \equiv -4$ for all unit vectors v, i.e. M has constant holomorphic sectional curvature. The simple connectivity of M shows that M is isometric to the complex hyperbolic space. ∎

Remark. The above theorems explain the lack of nonsymmetric (Kähler) harmonic spaces with the same density function as the real or complex hyperbolic space. Along the same lines we prove a similar theorem for quaternionic Kähler harmonic manifolds.

Proof of Theorem (1.3). Consider a chart (U, m), $m \in M$ with two almost complex structures J_2, J_3 such that the Levi–Civita derivatives of J_2, J_3 are linear combinations of J_2, J_3 and $J_4 = J_2 J_3$. Such a chart exists because M is quaternionic Kähler [2]. Let $\gamma(t)$ be a geodesic starting at m. Choose $T > 0$, such that $\gamma[0, T] \subset U$. Let I denote the index form on γ. Consider unit orthogonal parallel fields $E_2(t), \ldots, E_{4d}(t)$ along γ which are normal to $\gamma'(t)$ such that $E_2(t)$, $E_3(t)$, $E_4(t)$ belong to the three-dimensional subbundle spanned by $\{J_2(\gamma'(t)), J_3(\gamma'(t)), J_4(\gamma'(t))\}$.

Now $\mathrm{Ricci}(M) = -(4n+8)$ since $\theta_m(r) = \sinh^{4d-1} r \cosh^3 r$. The choice of J_2 and J_3 shows that the four-dimensional space spanned by $\{\gamma'(t), J_2\gamma'(t), J_3\gamma'(t), J_4\gamma'(t)\}$ is a quaternionic line parallel along γ which has an Sp(1) action. Therefore we get a family of almost complex structures $J_2(t), J_3(t), J_4(t)$ along γ such that

$$E_i(t) = J_i(t)(\gamma'(t)), J_i(0) = J_i, \quad \text{for} \quad i = 2, 3, 4.$$

The rest of the proof is similar to that of the previous theorem, so we shall be brief. Take vector fields

$$X_i(t) = \frac{\sinh 2t}{\sinh 2T} E_i(t), i = 2, 3, 4$$

and

$$X_i(t) = \frac{\sinh t}{\sinh T} E_i(t), i = 5, \ldots, 4d.$$

Let $J_i(t)$ be Jacobi fields along $\gamma(t)$ such that $J_i(0) = X_i(0)$ and $J_i(T) = X_i(T)$ for all i. By Allamigeon's theorem there are no conjugate points along $\gamma(t)$. Therefore the following inequality holds

$$\sum_{i=2}^{4d} I(J_i, J_i) \leqslant \sum_{i=2}^{4d} I(X_i, X_i) \tag{2.5}$$

and equality holds if and only if $X_i(t) = J_i(t)$ for all i and for all t in $[0, T]$. Now

$$I(X_i, X_i) = \frac{\sinh 4T + 4T}{2\sinh^2 2T} - \int_0^T \frac{\sinh^2 2t}{\sinh^2 2T} K(\gamma'(t), E_i(t))\,\mathrm{d}t, i = 2, 3, 4.$$

and

$$I(X_i, X_i) = \frac{2T + \sinh 2T}{4\sinh^2 T} - \int_0^T \frac{\sinh^2 t}{\sinh^2 T} K(\gamma'(t), E_i(t))\,\mathrm{d}t, \; i = 5, \ldots, 4d.$$

Summing up for $i = 2, \ldots, 4d$ we get

$$\sum_2^{4d} I(X_i, X_i) = (A(T) - 12C(T)) + \int_0^T B(t)\left(\sum_2^4 K(\gamma'(t), E_i(t)) + 12\right)\mathrm{d}t, \qquad (2.6)$$

where $K(x, y)$ stands for the sectional curvature of the plane spanned by vectors x and y. Moreover

$$A(T) = \frac{3(\sinh 4T + 4T)}{2\sinh^2 2T} + \frac{((8n+4)\sinh 2T - 24T)}{4\sinh^2 T}$$

$$B(t) = \frac{\sinh^2 t}{\sinh^2 T} - \frac{\sinh^2 2t}{\sinh^2 2T} > 0, t \in [0, T]$$

and

$$C(T) = \int_0^T B(t)\,\mathrm{d}t$$

$$= \frac{\sinh 2T - 2T}{4\sinh^2 T} - \frac{\sinh 4T - 4T}{8\sinh^2 2T}.$$

But we know that $\Sigma_{i=2}^{4d} I(J_i, J_i) = (\theta'_m(T)/\theta_m(T))$ and $E_i(t) = J_i(t)$. Therefore (2.5) and (2.6) gives

$$\frac{\theta'_m(T)}{\theta_m(T)} \leqslant (A(T) - 12C(T)) + \int_0^T B(t)\left(\sum_2^4 K(\gamma'(t), E_i(t)) + 12\right)\mathrm{d}t. \qquad (2.7)$$

Now we use the following relation between the components of the curvature tensor of a quaternionic Kähler manifold [2]

$$\sum_2^4 K(\gamma'(t), J_i(t)(\gamma'(t))) = \frac{3}{n+2}\,\mathrm{Ricci}(M).$$

In our case $\mathrm{Ricci}(M) = -(4d + 8)$, hence

$$\sum_2^4 K(\gamma'(t), J_i(t)(\gamma'(t))) = -12.$$

Substituting this into (2.7) we finally get

$$\frac{\theta'_m(T)}{\theta_m(T)} \leqslant A(T) - 12C(T)$$

and equality holds if and only if X_i is a Jacobi field for all i. A simple computation verifies that

$$\frac{\theta'_m(T)}{\theta_m(T)} = A(T) - 12C(T)$$

$$= (4d - 1)\coth T + 3\tanh T.$$

Thus $X_i(t) = J_i(t)$, i.e, the $X_i(t)$ are Jacobi fields on M for all i. Since the point m is arbitrary and γ is any geodesic starting at m, M is isometric to the quaternionic hyperbolic space. ■

Using the above techniques we give an alternative proof of the fact that Ricci-*flat harmonic manifolds are flat*. The method of proof is same as that of Theorems (1.1) and (1.2). We study the second fundamental form of the horospheres to prove the Theorem. Some basic ideas about horospheres are needed. They are the level sets of Busemann functions. For the definitions and properties of these functions we refer to [8].

Proof of Theorem (1.4). It is enough to prove the theorem for simply connected harmonic manifolds. Let M^d be a simply connected Ricci-flat harmonic manifold. If M were compact, then by Szabo's theorem M is a rank one symmetric space and hence Ricci $(M) > 0$, a contradiction. Thus M is noncompact. By Allamigeon's theorem M is diffeomorphic to $\mathbb{R}^d$. Moreover every geodesic is a line in M. Let m be a point of M and consider a geodesic ray $\gamma(t)$ starting at m. Choose a real number $T > 0$. Let $\{E_i(t)\}_2^d$ be parallel orthonormal vector fields along γ which are normal to $\gamma'(t)$. Consider normal Jacobi fields $\{J_i(t)\}_2^d$ such that $J_i(0) = 0$ and $J_i(T) = E_i(T)$ for $i = 2, \ldots, d$. Now consider the vector fields

$$X_i(t) = \frac{t}{T} E_i(t), i = 2, \ldots, d.$$

Note that $X_i(0) = J_i(0)$ and $X_i(T) = J_i(T)$. Let us now compare the index forms $I(J_i, J_i)$ and $I(X_i, X_i)$. Since there are no conjugate points along $\gamma(t)$ for $t \in [0, T]$, the following inequality holds:

$$\sum_2^d I(J_i, J_i) \leqslant \sum_2^d I(X_i, X_i). \tag{2.8}$$

Note that

$$\sum_2^d I(J_i, J_i) = \frac{\theta_m'(T)}{\theta_m(T)} \tag{2.9}$$

and

$$I(X_i, X_i) = \frac{1}{T} - \int_0^T \frac{t^2}{T^2} K(\gamma'(t), E_i(t))\, dt. \tag{2.10}$$

Summing over $i = 2, \ldots, d$ and using the fact Ricci $(M) = 0$, we get

$$\sum_{i=2}^d I(X_i, X_i) = \frac{n-1}{T}. \tag{2.11}$$

Substituting (2.9) and (2.11) into (2.8) gives

$$\frac{\theta_m'(T)}{\theta_m(T)} \leqslant \frac{n-1}{T}.$$

But we know that

$$\frac{\theta_m'(T)}{\theta_m(T)} \geqslant 0$$

because the geodesic balls are convex. Therefore

$$\frac{\theta'_m(T)}{\theta_m(T)} \to 0 \quad \text{as} \quad T \to \infty. \tag{2.12}$$

Now let F_γ be the Busemann function relative to γ and consider the horospheres determined by F_γ. Let L be the second fundamental form of the horospheres. Then (2.12) implies that trace $L = 0$. But L satisfies the Riccati equation

$$L' + L^2 + R(\cdot, \gamma')\gamma' = 0.$$

Taking traces and using the fact that Ricci $(M) = 0$, we get

$$\text{trace } L' + \text{trace}(L^2) = 0.$$

But trace $L' = 0$, since trace $L = 0$. Thus trace $L = 0$ and trace $L^2 = 0$. Hence $L = 0$. This shows that all the sectional curvatures $K(\gamma'(t), u) = 0$ for all unit vectors u. But the point m in M and the geodesic γ were arbitrarily chosen. Hence $K(M) \equiv 0$ and the proof is complete. ∎

3. Rigidity of $\mathbf{R}H^d$

In this section we prove that noncompact harmonic manifolds naturally fall into two classes:

$$\frac{\theta(r)}{\sinh^{d-1} r} \to 0$$

or

$$\frac{\theta(r)}{\sinh^{d-1} r} \to c > 0$$

for $r \to \infty$.

Further we show that the second class has only one element namely, the real hyperbolic space. Let (M^d, g) be a noncompact harmonic manifold. Since the Ricci curvature of M is constant, let us normalize the metric so that Ricci$(M) = -(d-1)$. The Bishop–Gromov volume comparison theorem (see for example [5] or [10]) gives the density function to be

$$\theta(r) = \alpha(r) \sinh^{d-1} r,$$

where $\alpha(r)$ satisfies

$$0 \leqslant \alpha(r) \leqslant 1; \alpha(0) = 1, \alpha'(0) = 0 \text{ and } \alpha'(r) \leqslant 0.$$

Hence $\alpha(r)$ is a monotonically decreasing function bounded from below by 0. Two cases arise:

1. $\lim_{r \to \infty} \alpha(r) = 0$ and
2. $\lim_{r \to \infty} \alpha(r) = c$ for some constant $c > 0$.

In the second case the density of M is asymptotically the same as that of the real hyperbolic case. In this case we show that, in fact $c = 1$ and hence M is locally isometric to the real hyperbolic space. This can be treated as a rigidity result for the real hyperbolic space.

Theorem 3.5. *Let (M^d, g) be a noncompact harmonic manifold with $\lim_{r\to\infty} \alpha(r) = c > 0$. Then, in fact $c = 1$ and M is locally isometric to the real hyperbolic space of constant sectional curvature -1.*

Proof. Let γ be a geodesic ray in M and let F_γ be the Busemann function of γ. Our hypothesis

$$\lim_{r\to\infty} \alpha(r) = c > 0$$

implies that the density of the horospheres determined by F_γ is

$$\theta(r) = c \sinh^{d-1} r.$$

Hence the mean curvature of these horospheres is

$$\sigma(r) = \frac{\theta'(r)}{\theta(r)}$$

$$= (d-1)\coth r = \text{trace } L$$

where L is the *second fundamental form* of the horospheres. Now the Riccati equation $L' + L^2 + R\,(\cdot,\gamma')\gamma' = 0$ combined with Ricci $(M) = -(d-1)$ gives

$$\text{trace}(L^2) = (d-1)\coth^2 r,$$

but

$$\text{trace } L = (d-1)\coth r.$$

Hence by Cauchy–Schwartz inequality we get $L = (\coth r) Id$. Thus M is locally isometric to the real hyperbolic space of constant curvature -1.

4. Remarks

Let us calculate $\alpha(r)$ for the complex hyperbolic space $\mathbb{C}H^d$. For that normalize the standard metric on $\mathbb{C}H^d$ so that Ricci $(\mathbb{C}H^d) = -(2d-1)$, i.e., take $g = \sqrt{(2d+2)/(2d-1)}\,Std$. Let $\mu = \sqrt{(2d-1)/(2d+2)} < 1$. Thus the density becomes

$$\theta(r) = \sinh^{2d-1} \mu r \cosh \mu r.$$

Hence

$$\alpha(r) = \frac{\theta(r)}{\sinh^{2d-1} r}$$

$$= \left(\frac{\sinh \mu r}{\sinh r}\right)^{2d-1} \cosh \mu r.$$

Letting $r \to \infty$ we get

$$\alpha(r) \sim e^{(2d\mu - (2d-1))r}.$$

But $2d\mu < (2d-1)$. Thus $\alpha(r) \to 0$ as $r \to \infty$.

Let (M^d, g) be a noncompact, Kähler harmonic manifold. Normalize the metric so as to get Ricci $(M) = -(2d+2)$. Then the density θ is given by

$$\theta(r) = \beta(r) \sinh^{2d-1} r \cosh r.$$

The following question is natural

Question. If $\beta(r) \to b > 0$ as $r \to \infty$, is M locally isometric to the complex hyperbolic space?

After this paper was written our attention was drawn by Vanhecke to his paper with Gray [7]. They investigated Riemannian spaces on which the volume of the geodesic ball with suficiently small radius R is same as that of a two-point homogeneous space. Under some extra hypothesis, for e.g. Ricci (M) is constant, they prove that these spaces are in fact locally isometric to the corresponding spaces. Thus our results follow from their theorems which are purely local. Nevertheless, it should be noted that our proofs, using the global character of our hypothesis, are entirely of different nature.

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