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# Substantial Riemannian submersions of $S^{15}$ with 7-dimensional fibres

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**Abstract.** In this paper we show that a substantial Riemannian submersion of $S^{15}$ with 7-dimensional fibres is congruent to the standard Hopf fibration. As a consequence we prove a slightly weak form of the diameter rigidity theorem for the Cayley plane which is considerably stronger than the very recent radius rigidity theorem of Wilhelm.

**Keywords.** Riemannian submersions; Euler class; connections; holonomy groups; second fundamental form.

## 1. Introduction

The study of Riemannian foliations of Euclidean spheres with no restrictions on the geometry of leaves was initiated in [8] and a complete classification was obtained for all the foliations having leaves of dimensions less than or equal to three. Earlier, the classification under the very strong hypothesis of the leaves being totally geodesic was accomplished in [5, 13] and [9]. A remarkable discovery in [8] was the fact that for substantial or highly nonflat (see [8] for definition) Riemannian foliations of $S^n$, the horizontal holonomy group reduces to a Lie group. The low dimensional Riemannian foliations could then be shown to be the orbits of this group. Thus proving substantiality was a crucial step and it wascarried out for the leaves of dimensions less than or equal to three. As a result it is known in particular that if $\pi: S^n \to M$ is a Riemannian submersion with connected fibres, then either it is congruent to a Hopf fibration or we have $\pi: S^{15} \to M^8$. Now the reduction to a Lie group is valid for substantial foliations without restrictions on leaf dimension. Hence it is natural to ask the following question:

If the Riemannian submersion $\pi: S^{15} \to M^8$ is substantial along every leaf, is it congruent to the Hopf fibration ? Since the question is algebraic in nature, this author was under the impression that it must have been settled by now. However looking at a recent paper [14] it was realised that such was not the case. In [14] the following lemma occurs.

*Lemma. Let $\pi : S^{15}(1) \to V$ be a Riemannian submersion with connected $\gamma$-dimensional fibres, and let G be the set of points $v \in V$ so that $\pi^{-1}(v)$ is totally geodesic. Theneither G is discrete or G is totally geodesic and an isomertrically embedded copy of $S^l(1/2)$ for some $1 \leqslant l \leqslant 8$.*

Now in [8] itself it is implicit that if $G \neq \phi$, then $\pi$ will be substantial along every leaf and it should be the case that $G$ is all of $V$. This would simplify much of the subsequent work in [14]. Therefore, in the paper we prove following:

**Theorem 1.1.** *If $\pi: S^{15} \to M^8$ is a Riemannian submersion with connected fibres which is substantial along each leaf, then it is congruent to the Hopf fibration.*

As a corollary we also prove theorem 1.2.

**Theorem 1.2.** (*Almost diameter rigidity for $CaP^2$*). *Let $M$ have sectional curvatures $\geqslant 1$, diam $(M) = \pi/2$. Moreover, let $M$ admit an equilateral triangle of sides $\pi/2$ and have the integral cohomology ring same as that of Cayley plane. Then $M$ is isometric to the standard $CaP^2$ with $1 \leqslant K \leqslant 4$.*

This almost ties up the loose end in [7] and both the radius rigidity theorem in [14] and the corollary II in [4] are considerably improved.

## 2. The group of the submersion

In this section we determine the Lie group which could possible occur as the structure group (holonomy group) of the Riemannian submersion $\pi: S^{15} \to M^8$ when we assume it to be substantial.

**Theorem 2.3.** *If $\pi: S^{15} \to M^8$ is a Riemannian submersion which is substantial along each leaf, then the holonomy group $G$ of $\pi$ is such that Lie $G = so(8)$. Hence $G$ is either $SO(8)$ or* Spin (8).

*Proof.* By [8], the holonomy group $G$ of $\pi$ is a Lie group of dimension $\leqslant 28$ and it acts transitively on the fibres. The fibres of $\pi$ are well known to be homotopy 7-spheres (see [3]). We conclude therefore that a maximal compact subgroup of $G$ can only be one of the following: $SO(8)$, Spin(8), Spin(7), $SU(4)$, $U(4)$ and $Sp(2)$ (see [1], p. 195).

Now by the long exact sequence of homotopy of $\pi$ (or directly from [3]) $M^8$ is a homotopy 8-sphere and by the Thom–Gysin sequence of $\pi$, the Euler class of this 7-sphere fibration is the generator of $H^8(M, \mathbb{Z}) \approx \mathbb{Z}$. This enables us to rule out Spin(7), since the Euler class of a Spin(7) bundle must be torsion (this is because $H^*(B\,\mathrm{Spin}(7),\mathbb{Q}) = \mathbb{Q}[p_1, p_2, p_3]$ is generated by Pontrjagin classes only). Since $Sp(2) \subset SU(4) \subset U(4)$, if we rule out $U(4)$ then the other two will be ruled out automatically. Now $H^*(BU(4), \mathbb{Z}) = \mathbb{Z}[c_1, c_2, c_3, c_4]$ is generated by Chern classes and $c_4$ also represents the Euler class $\chi$. Since $M$ is a homotopy sphere of dimension 8, by Bott's integrality theorem (see [11], p. 279), $c_4$ is divisible by 3! in $H^8(M, \mathbb{Z})$. But then $\chi$ cannot be a generator–a contradiction! Therefore, we conclude that $G =$ Spin (8) or $SO(8)$ due to the dimension constraint stated above. q.e.d.

*Remark.* Since there are no Whitney classes $w_1$ and $w_2$, we can fix the group to be Spin(8) which we do from now on.

COROLLARY 2.1

*Let $\chi(N)$ denote the Lie algebra of smooth vector fields on a smooth manifold $N$. For $b \in M$, the map*

$$A: \wedge^2(T_b M) \to \chi(\pi^{-1}(b))$$

*given by $A(x \wedge y) = A_x y$ is injective and its image is a Lie subalgebra isomorphic to so (8).*

*Proof.* See [8].

## 3. The associated principal bundle

Having found the holonomy group $\pi: S^{15} \to M^8$ we cannot apply the methods of [8] directly at this stage. In fact the standard Hopf fibration $S^{15} \to S^8$ (1/2) does not occur via orbits of any Lie group action on $S^{15}$. We therefore adopt the strategy of looking at the associated principal Spin(8) bundle

$$\tilde{\pi}: E \to M$$

and study the induced connection on it. To construct the associated principal bundle we first note that the holonomy Lie algebra $so(8) \subset \chi(\pi^{-1}(b))$ for each $b \in M$ gives an additional geometric structure on the fibres which is preserved under the horizontal holonomy displacement ([8], Lemma 2.12, and Propobition 2.13). Moreover, the fibres being homogeneous under Spin(8) action (though not via isometries *a priori*) become diffeomorphic to $S^7(1)$. Now Spin(8) can act transitively on $S^7$ in three ways: via $\rho$, $\sigma_+$ and $\sigma_-$, where $\rho$ is the natural double cover representation in $\mathbb{R}^8$ and the other two are the two spin representations of Spin(8). However, from the geometrical point of view, the three are equivalent as all three are two sheeted covers of $SO(8)$ and one can pass from one to any other using covering transformations. These are naturally outer automorphisms of Spin(8). To construct the principal bundle we take any one of these representations, say $\rho$. The resulting Spin(8) action on $S^7$ gives rise to a copy of $so(8)$ inside $\chi(S^7(1))$. Let for each $b \in M$, $F_b$ denote the fibre $\pi^{-1}(b)$ and define $E_b$ to be the set of all diffeomorphisms $\phi: S^7(1) \to F_b$ which send this copy of $so(8)$ onto the holonomy Lie algebra sitting in $\chi(F_b)$ isomorphically. Clearly $E_b$ is a copy of the group Aut $(so(8))$ whose identity component is just $PSO(8)$. We now let $E$ to be the disjoint union of these $E_b$ as $b$ runs through the set $M$. The set $E$ has a natural smooth structure and an obvious smooth projection $\tilde{\pi}$ onto $M$ with fibre over any point $b$ being exactly $E_b$. The group Aut$(so(8))$ acts on $E$ from the right freely and the orbits are the fibres $E_b$. We now do two things for the sake of definiteness: (i) reduce the group to the identity component $PSO$ (8), this is possible due to simple connectivity of $M$ and (ii) reduce the group further to Spin(8). This is possible due to the absence of the first two Whitney classes. We continue to denote the reduced space by $E$ only. Thus $E$ is now the associated principal Spin(8) bundle we were looking for. If Spin(7) is the isotropy group at some chosen base point of our model sphere $S^7(1)$, then $E$/Spin(7) is our $S^{15}$ that we started with and we have a tower of smooth fibrations

$$E \to S^{15} \to M.$$

The composite is the map $\tilde{\pi}$.

## 4. The Riemannian structure on $E$

In this section we describe a metric on $E$ which comes naturally due to the geometric considerations and which makes the above tower, a tower of Riemannian submersions. This is done in several stages.

First we note that for each $\phi \in E$, there is a decomposition of the tangent space into two complementary subspaces . The first of these is the tangent space to the orbit of Spin(8) through $\phi$ and the other is the space generated by the holonomy of $\pi$. To elaborate this, let $\phi \in E_b$ and $\gamma$ a path starting from $b$. Then we get a one parameter family of diffeomorphisms $\tau_{\gamma(t)}$ from $F_b$ to $F_{\gamma(t)}$ generated by the holonomy displacement along $\gamma$. This gives a path $\tau_{\gamma(t)}\circ\phi$ in E. Its derivative at $t=0$ gives a member of the second complementary subspace. As $\gamma$ varies the full subspace is obtained. We naturally declare the break-up as orthogonal and metrize the second part by the inner product on $T_bM$ to which it projects isomorphically under $\tilde{\pi}_*$. Henceforth we will denote the second subspace $\mathscr{H}_\phi$ and call it the horizontal space (relative to $\tilde{\pi}$). We remark that this collection of horizontal spaces gives exactly the connection on the principal bundle which is associated to the connection given by the horizontal spaces in $S^{15}$. We also denote the space complementary to $\mathscr{H}_\phi$ by $\mathscr{V}_\phi$. It remains to define a metric on this vertical part. This is more delicate. We proceed as follows: We already have a copy of $so(7)$ inside $so(8)$ as the isotropy Lie algebra of a chosen base point in our model $S^7$. This naturally breaks $so(8)$ in a unique manner as

$$so(7)\oplus\mathbb{R}^7,$$

each part being an irreducible $so(7)$ module. This is also the Cartan decomposition (see [10]). Corresponding to each $X\in so(8)$, we have a vertical vector field $\bar{X}$ on $E$ (see [2], p. 39). Thus we also have a decomposition

$$\mathscr{V}_\phi=\mathscr{V}'_\phi\oplus\mathscr{V}''_\phi$$

This too we declare to be orthogonal. On the first part we put the metric coming from the bi-invariant inner product of $so(7)$ and on the second part the metric pulled back from $S^{15}$ to which it goes injectively under $\bar{\pi}_*$. Note that its image is precisely the vertical space at $\bar{\pi}(\phi)$ in $S^{15}$. This completes the description of the Riemannian structure on $E$. With this metric on $E$, we have a tower of Riemannian submersions

$$E\to S^{15}\to M^8.$$

The first one is a Spin(7) principal bundle with totally geodesic fibres, each isometric to Spin(7) with the Cartan–Killing metric while the composite is the principal Spin(8) bundle whose fibres *a priori* have varying metrics. The decomposition

$$TE=\mathscr{V}\oplus\mathscr{H}$$

is preserved under the Spin(8) action while the finer decomposition

$$TE=\mathscr{V}'\oplus\mathscr{V}''\oplus\mathscr{H}$$

is preserved under the Spin(7) action. Moreover, the Spin(7) action is evidently via isometries.

## 5. The metric on the fibres of $\tilde{\pi}$

In this section we will see that though the metric on the fibres $E_b$, $b\in M$ could be different, the variation is very limited in nature. More precisely we have the following:

**Theorem 5.4** *The fibres $E_b$, $b\in M$ of $\tilde{\pi}$ are isometric to* Spin(8) *furnished with a left invariant metric which is also right invariant under* Spin(7) *action.*

To prove this result we need to analyse the integrability tensor of $\tilde{\pi}$. So let $\tilde{A}$ denote the integrability tensor of $\tilde{\pi}$ and $\bar{A}$ that of $\bar{\pi}$.

*Lemma* 5.1. *Let $x,y\in T_bM$.* The vertical field *$\tilde{A}_xy$ along the fibre $E_b$ is $\bar{Z}$, for some $Z\in so(8)$. Further, if $Z=Z_1+Z_2$ corresponding to $so(8)=so(7)+\mathbb{R}^7$, then $\bar{Z}_1=\bar{A}_xy$ and $\bar{Z}=A_xy$.*

*Proof.* $\tilde{A}_x y = 1/2[X, Y]^v$ is just the curvature of the connection on the Spin(8) principal bundle alluded to earlier and hence clearly comes from a suitable element of its Lie algebra. Also $[X, Y]^v = [X, Y]^{v'} + [X, Y]^{v''} = 2(\bar{A}_x y + A_x y)$ is also $2(\bar{Z}_1 + \bar{Z}_2)$. By uniqueness of decomposition into components, the result follows. q.e.d.

COROLLARY 5.2

*For* $x, y \in T_b M$, $\tilde{A}_x y$ *is of constant length along* $E_b$.

*Proof.* $\bar{Z}_1$ is of constant length since it comes from an $so(7)$ element and $\bar{Z}_2 = A_x y$ is of constant norm since it is the basic lift of the corresponding field along $F_b \subset S^{15}$. That it is of constant norm there is well known (see [8]). Since the two components are also mutually orthogonal everywhere the corollary follows. q.e.d.

*Lemma* 5.2. *There is an orthonormal framing of* $E_b$ *which is generated by a basis of* $so(8)$.

*Proof.* Choose a basis $\{x_i : 0 \leqslant i \leqslant 7\}$ of $T_b M$ so that $\{A_{x0} x_i : 1 \leqslant i \leqslant 7\}$ forms an orthonormal framing of $F_b$. Set $v_i = A_{x0} x_i$, and consider the fields

$$\{v_i : 1 \leqslant i \leqslant 7\} \cup \{\bar{A}_{v_j} v_k : 1 \leqslant j < k \leqslant 7\}.$$

That the first set of 7 vector fields is an orthonormal frame along $E_b$ is obvious. That it is generated by $so(8)$ elements is one of the contention of the lemma 5.1 above. As for the second set of 21 fields we note that if $v_i = \bar{X}_i$ for suitable $X_i$ in the $\mathbb{R}^7$ component of $so(8)$, then $\bar{A}_{v_j} v_k$ is generated by $[X_j, X_k]$. There is no need to project to the $so(7)$ part since it is already there. (This is a well-known property of Cartan decomposition). That these have pairwise constant inner-products now follows from the very way the metric was defined on the $\mathscr{V}''$ part of $TE$. The two sets are clearly mutually orthogonal. We also note that $\{[X_i, X_j], 1 \leqslant i < j \leqslant 7\}$ is a basis of $so(7)$. An application of Gram–Schmidt orthonormalization on the second set now gives the required framing. q.e.d.

*Proof of the theorem.* $E_b$ can be identified with Spin(8) after choosing some base point on it. The left invariant vector fields of Spin(8) are mapped to the $\mathbb{R}$-span of the above mentioned basis. Left invariance of the metric of $E_b$ now follows exactly as in [8]. Right invariance under Spin(7) is from construction. q.e.d.

*Caution.* Left and right in our situation are opposite to those in [8].

COROLLARY 5.3

*Each fibre* $F_b$ *of* $\pi$ *is a sphere of constant sectional curvature.*

*Proof.* Each $F_b$ is isometric to the quotient of $E_b$ under the group Spin(7) of isometries of $E_b$ acting from the right. Hence Spin(8) acts on $F_b$ via isometries from the left making it a homogeneous Riemannian space. Since the isotropy group Spin(7) acts transitively on tangent two-planes the sectional curvatures are constant pointwise. It follows they are the same constant everywhere (see [12]). q.e.d.

## 6. Left invariant metrics on Spin(8) invariant under AdSpin(7)

Let

$$so(8) = so(7) \oplus \mathbb{R}^7$$

be the Cartan decomposition which we note is also an $so(7)$-module decomposition into its *isotypical* components. With this notation we have the following:

**Theorem 6.5.** *Let $g$ denote the Cartan Killing metric on $so(8)$. Then $g = g_1 \oplus g_2$ corresponding to the Cartan decomposition and any* AdSpin(7) *invariant metric is of the type*

$$g = g_1 \oplus c g_2$$

*(upto an overall scalar multiple).*

*Proof.* The first claim is well-known (see [10]). The second claim follows since under an AdSpin(7)-invariant metric, $so(7)^{\perp}$ must be an $so(7)$ module. This forces the above decomposition to be orthogonal. Further any Spin(7) invariant metric on the natural module $\mathbb{R}^7$ is a scalar multiple of the standard Euclidean metric. q.e.d.

COROLLARY 6.4

*There is a smooth positive real valued function $c$ on $M$ such that $E_b$ is isometric to* Spin(8) *with the left invariant metric $g_b = g_1 \oplus c(b) g_2$.*

*Proof.* Just observe that there is no change of scale in the $\mathscr{V}'$ part of the tangent space of $E_b$ as the orbits of the subgroup Spin(7) are all isometric to each other. q.e.d.

COROLLARY 6.5

*For any smooth path in $M$ the hononomy displacement of fibres in $E$ preserves the $\mathscr{V} = \mathscr{V}' \oplus \mathscr{V}''$ decomposition. Moreover, it is an isometry on the first part and a dilatation on the second part.*

*Proof.* Let $\gamma$ be a smooth path in $M$ starting from $b_1$ and ending at $b_2$. Let $\phi_1 \in E_{b_1}$ *and* $\phi_2 = \tau(\phi_1)$ be in $E_{b_2}$. On identifying these fibres with Spin(8), it is clear that the holonomy diffeomorphism gets identified with $L_g$ for a suitable $g \in$ Spin(8). Since $g_{b_2}$, the metric on $E_{b_2}$ pulls back under $L_g$ to a left invariant metric which is also right invariant under Spin(7), so it must be of the form $g_1 \oplus (c(b_2)/c(b_1)) \cdot g_2$ (see corollary 6.4 for notation). q.e.d.

## 7. Proof of the main theorem

In this section we complete the proof of Theorem 1.1 stated in the introduction. We observe that a consequence of the last corollary is that the holonomy displacement in $S^{15}$ is via isometries up to scalings factor. This forces all the operators $T^x$ on vertical vectors arising out of the *second fundamental form* of the fibres to be scalar multiples of identity. This in turn forces the largest and the smallest fibres to be great spheres due to isoparametricity. Hence all the fibres are great spheres. Appealing now to the classification of parallel great sphere fibrations as in [13] or [9] we conclude that $\pi$ is a Hopf fibration. q.e.d.

COROLLARY 7.6 (Generalized Wilhelm's Lemma)

*Let*

$$\pi: S^{15}(1) \to M^8$$

*be a Riemannian submersion with connected 7-dimensional fibres and let*

$$G = \{b \in M : \pi^{-1}(b) \text{ is a great sphere}\},$$

*then either G is empty or all of M.*

*Proof.* If $G$ is nonempty let $b \in G$. For any $p \in F_b$ and any $x \in \mathscr{H}_p$, the linear map

$$A_x : \mathscr{H}_p \to \mathscr{V}_p$$

is surjective (see lemma 4.2, [8]). It follows that $A_{c_x'}(t)$ is surjective for all $t$, where $c_x$ denotes the geodesic with initial vector $x \in \mathscr{H}_p$ (see the statement preceding the corollary 2.9 in [8]). Hence $\pi$ is substantial along every fibre. q.e.d.

## 8. Almost diameter rigidity for $CaP^2$

*Proof.* Let $x$, $y$ and $z$ be mutually distance $\pi/2$ apart. In this situation $y$ and $z$ both are in the dual set $\{x\}'$ of $x$. (see [7] and [14] for more details about dual sets). From [7] we know there is a Riemannian submersion

$$\exp_x : S_x \to \{x\}'$$

and similarly for $y$ and $z$. As argued in [14] this forces at least one fibre to be totally geodesic in each case. But then $\exp_p$ is congruent to the Hopf fibration and this makes the space isometric to the standard $CaP^2$ as proved in [7] and [14]. q.e.d.

*Remark.* This rigidity is clearly stronger than even corollary II of [4] where each pair of points separated by a distance of $\pi/2$ is required to be completed into an equilateral triangle.

It is clearly an interesting problem to find reasonable conditions which will force substantiality. One such, as we saw is total geodseity of a single fibre. The author proposes to discuss this elsewhere.

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