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A new proof of Suzuki's formula

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Abstract. We give a different proof of a formula of Suzuki and its strengthening by Zaidenberg for the topological Euler characteristic of an affine surface fibered over a curve. We deduce this formula using the ideas for the proof of an analogous formula for a proper morphism.

Keywords. Fibration; Euler characteristic.

1. Introduction

In [5] and [6] the following result has been proved.

Theorem. *Let V be a smooth irreducible quasi-projective surface and let $f\colon V \to B$ be a locally Stein (with respect to B) morphism onto a smooth quasi-projective curve B such that a general fiber of f is irreducible. Then we have:*

(a) $\chi(V) = \chi(B)\cdot\chi(F) + \Sigma_s(\chi(F_s) - \chi(F))$ *where, χ denotes topological Euler characteristic, F is a general fiber of f and F_s are the finitely many singular fibers of f (cf. § 1 for the definitions). Further, $\chi(F_s) \geqslant \chi(F)$ for all s.*
(b) *Suppose that for some singular fiber F_s there is an equality $\chi(F_s) = \chi(F)$. Then F is isomorphic to either $\mathbf{C}$ or $\mathbf{C}^* := \mathbf{C} - \{one\ point\}$ and $(F_s)_{\mathrm{red}}$ is isomorphic to F.*

Part (a) of the Theorem was proved by Suzuki in [5] and its strengthening in part (b) was proved by Zaidenberg in [6, Lemma 3.2].

Suzuki's formula together with the above strengthening by Zaidenberg are very useful in the theory of non-complete algebraic surfaces. But the proof given in [5] is rather terse and involves the use of plurisubharmonic functions and Milnor's results on the topology of isolated singularities, etc. Suzuki has proved this formula for a slightly more general situation. But the version we give here is enough for algebraic geometric applications. In [6], a more detailed argument is given by developing the notion of subharmonic partitions (and also Milnor's result). Therefore this result is not as accessible to algebraic geometers as is desirable. Suzuki's formula for a proper morphism f and its strengthening is well-known. Our only observation is that using some arguments from [4, ch. 4, § 4], it is easy to deduce Suzuki's formula for the non-complete case. This is the aim of this short note. We will also give a proof of its strengthening by Zaidenberg, as it is not very well-known. We believe that our proof is elementary and transparent.

2. Preliminaries

Let f: $V \to B$ be as above. It is well-known that there exists a unique smallest finite subset $S \subset B$ such that f: $V - f^{-1}(S) \to B - S$ is a C^∞- locally trivial fiber bundle. Any fiber $f^{-1}(s)$ for $s \in S$ is called a singular fiber of f. A fiber of f which is not a singular fiber is called a general fiber.

Remark. If f is locally affine with respect to B then using the improvement of Suzuki's formula in [6] mentioned above, we can easily see that there is a unique smallest subset S such that for any $s \in S$ either F_s is not reduced or if it is reduced then F_s is not homeomorphic to F.

For any complex analytic variety W, unless otherwise stated, by a component of W, we mean an irreducible component of W.

For any simplicial complex K, its topological Euler characteristic is denoted by $\chi(K)$. The ith Betti number of K is denoted by $b_i(K)$.

The cardinality of a finite set T is denoted by $\# T$.

For any smooth algebraic surface Y and a (not necessarily reduced) divisor $C := \Sigma a_i C_i$ on Y where all the irreducible curves C_i are complete, the *arithmetic genus* of C is defined by $p_a(C) = C^2 + C \cdot K/2 + 1$.

The following result is a special case of a rather general result proved in [3, Lemma 1.5].

Lemma 1. *Let g: $W \to C$ be a morphism from a smooth irreducible algebraic surface W onto a smooth quasi-projective curve C such that a general fiber of g is irreducible.*
(i) *If every fiber of g contains a reduced component, then we have an exact sequence*

$$\pi_1(F) \to \pi_1(W) \to \pi_1(C)$$

where, F is a general fiber of g.
(ii) *If U is a small contractible neighborhood of a point in C then the image of the map $\pi_1(F) \to \pi_1(g^{-1}(U))$ has finite index in $\pi_1(g^{-1}(U))$.*

Part (ii) easily follows from part (i) by taking a suitable ramified cover $V \to U$ of U and considering the normalized fiber product $\tilde{V} := V \times_U g^{-1}(U)$ with the natural map $\tilde{g}: \tilde{V} \to V$. ($\tilde{V}$ can be assumed to be a finite topological covering of $g^{-1}(U)$ such that the special fiber of $\tilde{g}$ contains a reduced irreducible component. Then (ii) applies.)

Lemma 2. *Let Y be a smooth quasi-projective surface and $C := \Sigma_{i=1}^{i=l} C_i$ be a (reduced) complete, connected curve on Y. Then $\Sigma_{i<j} \#(C_i \cap C_j) \geq l - 1$. If equality holds then the dual graph of C is a tree.*

The proof of this easy result is left to the reader. The next result is standard (cf. [4, ch. 4, Lemma 5]).

Lemma 3. *Let f: $Y \to B$ be a proper morphism from a smooth quasi-projective surface to a smooth curve with connected fibers. Let $F_b := f^*(b)$ and Δ an effective divisor supported on $f^{-1}(b)$. Then $\Delta^2 \leq 0$ and equality occurs iff Δ is a rational multiple of F_b.*

The next result is proved in [4, ch. 4, Lemma 4].

Lemma 4. *Let (Y, C) be as in Lemma 2 above. Then $\chi(C) \geq -C \cdot (C + K) + l - 1$. If equality holds then the dual graph of C is a tree.*

Proof. In [4, Ch. 4, Lemma 4], the inequality $\chi(C) \geq -C \cdot (C + K) + \Sigma_{i<j} \#(C_i \cap C_j)$ is proved. Hence by Lemma 2, $\chi(C) \geq -C \cdot (C + K) + l - 1$, and equality holds implies that the dual graph of C is a tree.

We will implicitly use the following easy result whose proof is left to the reader.

Lemma 5. Let C be a (reduced) complete, connected curve on a smooth quasi-projective surface Y and let E be an exceptional curve of the first kind contained in C. Let $f\colon Y \to Z$ be the contraction of E to a smooth point on the smooth surface Z. Then $b_1(C) = b_1(f(C))$.

The point of this result is that the number of essential loops in C and $f(C)$ is the same.

The only non-trivial result about Stein manifolds that we need in our proof of the theorem is the following property proved first by Bishop.

Lemma 6. A connected Stein manifold of dimension > 1 is connected at infinity.

Proof. We will only give a sketch for a topological proof. Let M be a Stein manifold of dimension $n > 1$. We can find arbitrarily large compact subsets K of M having the properties:

(1) K is a C^∞-manifold of real dimension $2n$ with nice boundary and

(2) the interior K^0 of K is a Stein manifold of dimension n which has the same homotopy type as K.

Since $H_i(K^0; \mathbf{Q}) = (0)$ for $i > n$, it follows that $H^{2n-1}(K; \mathbf{Q}) = (0)$. Now consider the long exact cohomology sequence with compact supports for the pair (M, K). The end piece of the sequence is

$$\cdots \to H_c^{2n-1}(K; \mathbf{Q}) \to H_c^{2n}(M, K; \mathbf{Q}) \to H_c^{2n}(M; \mathbf{Q}).$$

By duality, $H_c^{2n}(M, K; \mathbf{Q}) \simeq H_0(M - K; \mathbf{Q})$ and $H_c^{2n}(M, \mathbf{Q}) \simeq H_0(M; \mathbf{Q}) \simeq \mathbf{Q}$. It follows that $H_0(M - K; \mathbf{Q}) \simeq \mathbf{Q}$ and hence $M - K$ is connected.

3. Proof of the Theorem

Let V, B, f be as in the statement of the Theorem. With the notation introduced in § 1, we have $\chi(V - f^{-1}(S)) = \chi(B - S)\cdot\chi(F)$. For any complex manifold M and a closed complex subvariety N of M the following relation holds $\chi(M - N) = \chi(M) - \chi(N)$. For the proof of this, it is enough to throw away the singular locus of N from both M and N and then use the long exact sequence for cohomology with compact support for the pair (M, N) and Poincaré duality with compact support for the pair (M, N). Hence we get the formula

$$\chi(V) = \chi(B).\chi(F) + \Sigma_s(\chi(F_s) - \chi(F)).$$

Let F_s be a singular fiber of f. Next we prove the inequality $\chi(F_s) \geqslant \chi(F)$. For this, let $V \subset X$ where X is a smooth quasi-projective surface such that f extends to a proper morphism $\tilde{f}\colon X \to B$. We will also assume that $D := X - V$ is a divisor with simple normal crossings and the closure $\bar{F}_s$ of F_s in X intersects D transversally.

Any (irreducible) component of D which is not contained in any fiber of $\tilde{f}$ is called a *horizontal* component of D. Any (irreducible) component of D which is contained in a fiber of f is called a *vertical* component of D. We denote by $H_1, H_2, \ldots, H_l$ the horizontal components of D and $H := \cup_1^l H_i$.

Let U be a small contractible neighborhood of s in B not containing any other singular point of f. Further, we assume that $(\tilde{F}_s)_{\text{red}}$ is a strong deformation retract of $\tilde{f}^{-1}(U)$ where, $\tilde{F}_s$ is the fiber of $\tilde{f}$ containing F_s. By assumption, $f^{-1}(U)$ is Stein. Let $\tilde{F}$ denote a general fiber of $\tilde{f}$.

By Lemma 1, we have $b_1(\tilde{F}) \geqslant b_1(\tilde{f}^{-1}(U)) = b_1((\tilde{F}_s)_{\text{red}})$. Since $b_2(\tilde{F}) = 1 \leqslant b_2(\tilde{F}_s)$, we get $\chi(\tilde{F}) \leqslant \chi(\tilde{F}_s)$. Equality holds if and only if $b_1(\tilde{F}) = b_1(\tilde{F}_s)$ and $\tilde{F}_s$ is irreducible.

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ be all the connected components of the union of all the vertical components of D contained in $\tilde{F}_s$. We denote by $C_1, C_2, \ldots, C_n$ the irreducible components of $\cup \Gamma_i$. Since f is locally Stein with respect to $B, f^{-1}(U)$ is connected at infinity (Lemma 6). Hence for each i, $\Gamma_i \cap H_j \neq \phi$ for some j, $1 \leqslant j \leqslant l$.

Let D_j be an irreducible component $\Gamma := \cup \Gamma_i$ such that $D_j \cap H \neq \phi$. Let m_j be the multiplicity of D_j in the scheme-theoretic fiber $\tilde{F}_s$. Then for a point $p \in D_j \cap H$, a small neighborhood of p in $\tilde{f}^{-1}(U)$ contains exactly m_j points of $\tilde{F} \cap H$.

Similarly, if C is an irreducible component of F_s occuring with multiplicity m in $\tilde{F}_s$, then in a small neighborhood of any point in $C \cap H$ there are exactly m points of $\tilde{F} \cap H$. With these notations, we get from $\tilde{F}_s \cdot H = \tilde{F} \cdot H$,

$$\Sigma_j m_j D_j \cdot H + \Sigma_C m_C C \cdot H = \tilde{F} \cdot H.$$

Here, the first summation is over the irreducible components of Γ and the second over the irreducible components of F_s. Note that by assumption $\overline{F}_s$ and D have normal crossings and hence $H \cap \Gamma \cap \overline{F}_s = \phi$.

We can write

$$\chi(\tilde{F}_s) - \chi(\tilde{F}) = (b_1(\tilde{F}) - b_1(\tilde{F}_s)) + (b_2(\tilde{F}_s) - 1).$$

Since $F_s = \tilde{F}_s - (\overline{F_s} \cap H) \cup \Gamma$, we get

$$\chi(F_s) = \chi(\tilde{F}_s) - \chi(\Gamma) - \# H \cap \overline{F_s}$$

and

$$\chi(F) = \chi(\tilde{F}) - \Sigma_j m_j D_j \cdot H - \Sigma_C m_C C \cdot H.$$

This gives

$$\chi(F_s) - \chi(F) = \chi(\tilde{F}_s) - \chi(\tilde{F}) + (\Sigma_j m_j D_j \cdot H - \chi(\Gamma)) + (\Sigma_C m_C C \cdot H - \# H \cap \overline{F_s}).$$

Clearly, $b_2(\tilde{F}_s) = b_2(\overline{F_s}) + n$. Hence we have

$$\chi(F_s) - \chi(F) = b_1(\tilde{F}) - b_1(\tilde{F}_s) + (b_2(\overline{F_s}) + n - 1) + (\Sigma_j m_j D_j \cdot H - r - n + b_1(\Gamma)) + (\Sigma_C m_C C \cdot H - \# H \cap \overline{F_s}).$$

Since $\Gamma_j \cap H \neq \phi$ for each j, we certainly have $\Sigma_j m_j D_j \cdot H \geqslant r$. Similarly, $\Sigma_C m_C C \cdot H \geqslant \# H \cap \overline{F_s}$. Now it is clear that $\chi(F_s) \geqslant \chi(F)$. This proves part (a) of the Theorem.

Proof of part (b). Suppose $\chi(F_s) = \chi(F)$. Then we get the following equalities:

(1) $b_1(\tilde{F}) = b_1(\tilde{F}_s)$ and $b_2(\overline{F_s}) = 1$. In particular, F_s is irreducible.
(2) $\Sigma_j m_j D_j \cdot H = r$, $m_{\overline{F}_s} \overline{F}_s \cdot H = \# H \cap \overline{F}_s$. In particular, if F_s is not reduced in $\tilde{F}_s$ then $H \cap \overline{F}_s = \phi$.
(3) $b_1(\Gamma) = 0$. Hence each Γ_j is a connected union of smooth rational curves such that the dual graph of Γ is a tree. Further, H intersects each Γ_j transversally in exactly one point and the irreducible component of Γ_j meeting H is reduced in $\tilde{F}_s$.

First we dispose off the case when F_s is not a reduced fiber.

Case 1. Suppose that F_s has multiplicity $m > 1$. Then we consider the ramified map $U \to U$ given by $z \to z^m$ and the normalized fiber product $P := f^{-1}(U) \times_U U$. This is an ètale cover of $f^{-1}(U)$ of degree m. The fiber of the map $P \to U$, say P_s, satisfies by part (a) above $\chi(P_s) = m\chi(F_s) \geqslant \chi(F) = \chi(F_s)$. Hence $\chi(F_s) \geqslant 0$. If $\chi(F_s) = 0 = \chi(F)$, then clearly F is isomorphic to $\mathbf{C}^*$. If $\chi(F_s) = 1 = \chi(F)$, then F is isomorphic to $\mathbf{C}$. In either case $\bar{f}$ is a morphism with general fiber $\mathbf{P}^1$. It is well-known that any irreducible component of any fiber of a $\mathbf{P}^1$-fibration on a smooth projective surface is smooth (cf. [2, ch. 1, Lemma 4.4.1]). Therefore F_s is smooth and hence isomorphic to F, if taken with reduced structure.

Case 2. Now we assume that $\overline{F_s}$ is a reduced irreducible component of $\tilde{F}_s$. Denote by C the reduced divisor $(\tilde{F}_s)_{\text{red}}$. By Lemma 4 and (1) above

$$\chi(C) = 1 - b_1(C) + (1 + n) \geqslant -C\cdot(C + K) + n$$

i.e. $C\cdot(C + K) + 2 \geqslant b_1(C)$ and equality holds only if the dual graph of C is a tree. By Lemma 3, $C^2 \leqslant 0$ and equality occurs iff $\tilde{F}_s$ is reduced (since $\overline{F_s}$ has multiplicity 1).

We write $\tilde{F}_s = \overline{F_s} + \Sigma_1^n m_i C_i$.

Claim. The dual graph of C is a tree and C_i is an exceptional curve of the first kind for some i.

Suppose that $K.C_i \geqslant 0$ for each i. Then we have

$$b_1(C) \leqslant C^2 + C\cdot K + 2 \leqslant \tilde{F}_s^2 + \tilde{F}_s\cdot K + 2 = \tilde{F}^2 + \tilde{F}\cdot K + 2 = b_1(\tilde{F}).$$

From $b_1(C) = b_1(\tilde{F})$, we see that all the inequalities are equalities and this implies that the dual graph of C is a tree and for any i with $C_i.K \neq 0, m_i = 1$. Since each Γ_i is a tree of smooth rational curves by (3), we deduce easily that $b_1(\overline{F_s}) = b_1(\tilde{F}_s)$. Hence

$$2p_a(\tilde{F}) - 2 = \tilde{F}\cdot K = (\overline{F_s} + \Sigma n_i C_i)\cdot K \leqslant 2p_a(\overline{F_s}) - 2 = \overline{F_s}^2 + \overline{F_s}\cdot K.$$

This equation implies that $(\Sigma m_i C_i)\cdot K = \overline{F_s}^2 < 0$, if $\Gamma \neq \phi$. Thus $C_i\cdot K < 0$ and $C_i^2 < 0$ for some i implies that C_i is an exceptional curve of the first kind.

We contract the exceptional curve C_i to a smooth point on a smooth surface X_1 and consider the new embedding $V \subset X_1$. For the new closure of F_s in X_1, say $(\overline{F_s})_1$, we have $p_a((\overline{F_s})_1) \geqslant p_a(\overline{F_s}) \geqslant p_a(\tilde{F})$.

On the other hand, the first Betti numbers of C and its image in X_1 are the same. Using Lemma 4 we repeat the above argument. This gives a new (-1)-curve in the fiber containing $\overline{F_{s1}}$, not equal to $\overline{F_{s1}}$. Continuing this argument we see that the dual graph of C is a tree. This proves the claim.

After contracting all the exceptional curves as above, we arrive at a smooth quasi-projective surface Y containing V as a Zariski-open subset and with a morphism $g: Y \to B$ extending f. The fiber G_s of g containing F_s is the closure of F_s in Y. Since each Γ_j intersects $\overline{F_s}$ only once, we see that G_s has at worst unibranch singularities. Hence G_s is homeomorphic to its normalization $\tilde{G}_s$. By flatness of g, we get the equality $p_a(G_s) = p_a(\tilde{F})$. The fundamental group of the complete fiber containing F_s remains unchanged in the sequence of blowings-down (this uses the observation that the dual graph of C is a tree). Therefore $b_1(\tilde{G}_s) = b_1(G_s) = b_1(\tilde{F})$ also holds.

Since the arithmetic genus of G_s is at least as big as the genus of $\tilde{G}_s$ = genus of $\tilde{F}$, we deduce that G_s is itself smooth. This implies that the same number of points from G_s and $\tilde{F}$ are removed to obtain F_s and F respectively. Now it follows that F_s is a general fiber of f. This completes the proof of part (b) of the Theorem.

The following result is useful in some applications of Suzuki's result. It follows from the proof of part (b) of the Theorem.

COROLLARY

With the above notations, $\chi(F_s) - \chi(F) = 1$ if and only if exactly one of the quantities $b_1(\tilde{F}) - b_1(\tilde{F}_s)$, $b_2(\overline{F_s}) - 1$, $b_1(\Gamma)$, $\Sigma_j m_j D_j \cdot H - r$, $\Sigma_C m_C C \cdot H - \# H \cap \overline{F_s}$ is equal to 1 and all the other quantities are equal to 0.

Remark. We can give a very short proof of part (a) of the Theorem by using Grauert's contraction theorem (cf. [1]). After contracting all the Γ_i to normal singular points on a normal complex surface, we come to a situation analogous to the surface Y above with the morphism g (but possibly Y may have singularities). Then firstly, $\chi(G_s) \geqslant \chi(\tilde{F})$. Next we see easily that $\#(\tilde{F} - F) \geqslant \#(G_s - F_s)$. From this (a) follows. But Grauert's proof also uses plurisubharmonic functions! Therefore we have given a somewhat longer but elementary argument above.

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