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Surface instability in a finite thickness fluid saturated porous layer

N RUDRAIAH, B S KRISHNAMURTHY* and T MASUOKA†
UGC-DSA Centre in Fluid Mechanics, Department of Mathematics, Central College Campus, Bangalore University, Bangalore 560 001, India
* Department of Physics, Gulbarga University, Gulbarga 585 106, India
† Department of Mechanical Engineering, Kitakushu Institute of Technology, Japan

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Abstract. The Rayleigh–Taylor (RT) instability at the interface between fluid and fluid saturated sparsely packed porous medium has been investigated making use of boundary layer approximation and Saffmann [8] boundary condition. An analytical solution for dispersion relation is obtained and is numerically evaluated for different values of the parameters. It is shown that RT instability can be controlled by a suitable choice of the thickness of porous layer, ratio of viscosities and the slip parameter.

Keywords. RT instability; flow past porous media; Brinkmann model; Saffmann boundary condition.

1. Introduction

Rayleigh–Taylor instability (RT) in viscous fluids has been extensively investigated in the last few years [3], because of its importance in inertial fusion target design, astrophysics and geophysics. Diffusional instabilities of the form first discussed by Saffmann and Taylor [9] which are normally encountered in viscous creeping flow limit are also studied because of their applications in areas of failure of materials. Recently, Brown [2] has studied RT instability in a finite thickness layer of a viscous fluid under the assumption of creeping flow. Rudraiah *et al* [7] have investigated the effect of oblique magnetic field on the RT instability of a finite conducting fluid layer. However, there is a rather different subject area of interest, namely, instability between fluid and fluid saturated porous media where RT instability is relevant. Examples of this may be found in the areas of the failure of metallic glasses, grain boundaries, failure in metals, failure of polymers and so on.

It is also important for the study of motion of contact line where the effective slip of the interface reverses the singularity in the rate of strain which was otherwise introduced by the no-slip condition. In the present situation the interface may be a smooth surface or a rough surface depending upon the solid matrix of the porous material. In the former case the surface can be considered as a nominal surface (postulated by Beaver and Joseph [1] hereafter called as BJ) at which BJ slip condition exists due to transfer of momentum from fluid to fluid saturated porous media. Later Saffmann [8] used a modified version of BJ slip condition to study the flow past a fluid saturated porous media. The BJ slip condition is independent of the thickness of the porous layer and hence valid when the thickness of the porous layer is very much larger than the thickness of the fluid layer. In many industrial and biochemical applications the thickness of the porous layer is comparable to that of fluid layer and hence the slip condition should involve the thickness of the layer. Later Rudraiah [6] has derived the slip condition known as BJR slip condition involving the thickness of the layer. This BJR slip condition reduces to BJ condition for large thickness of the layer. In the case of rough interface the effective slip condition can replace the rough boundary by a smooth surface and this introduces an effective slip condition applied in the mean position of the interface. In this situation the velocity slip is proportional to tangential stress along the surface; the constant of proportionality is called the slip coefficient. This slip coefficient can be determined using the asymptotic analysis as in the case of fluid in the absence of porous material [4].

In this paper, we study the RT instability at the interface between fluid layer and porous layer of finite thickness using Saffmann slip condition [8]. It is shown that the thickness of the layer, the slip coefficient and the ratio of viscosities of fluid greatly influence the RT instability.

2. Formulation of the problem

The physical configuration considered in this investigation is shown in figure 1. It consists of two regions 1 and 2. The region 1 (i.e., $y > h$) is concerned with a viscous fluid of viscosity μ_{eff}, bounded below at $y = 0$ by a rigid impermeable material with an interface at $y = h$. To derive the basic equations, for this physical configuration considered here, we make the following approximations:

i) The fluid in region 1 is quiescent.

ii) The fluid saturated porous medium in the region 2 is incompressible and performs steady, two-dimensional boundary layer motion, governed by Brinkmann's equation.

Under these approximations the basic equations governing the flow are (Nield [5])

$$\frac{\partial p}{\partial x} = \mu_{\mathrm{eff}}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] - \left(\frac{\mu}{k}\right)u, \tag{2.1}$$

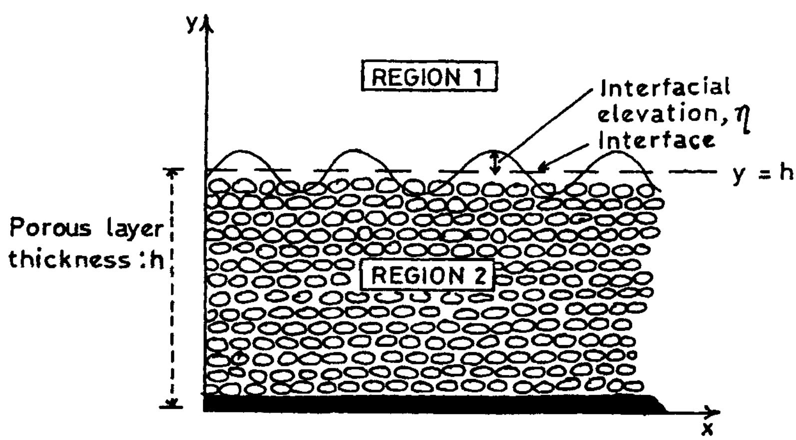

Figure 1. Physical configuration.

and

$$\frac{\partial p}{\partial y} = \mu_{\text{eff}}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] - \left(\frac{\mu}{k}\right)v, \tag{2.2}$$

where p is the pressure and u and v are the two-dimensional velocity vector components of the fluid saturated porous medium with components u and v along x and y directions, respectively; k is the permeability of the porous material. Since the fluid is incompressible the continuity equation is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.3}$$

These equations are governed by the following boundary conditions.

The no slip condition is

$$u = v = 0| \text{ at } y = 0, \tag{2.4}$$

which on using the continuity equation becomes

$$v = Dv = 0| \text{ at } y = 0, \tag{2.5}$$

where $D = \mathrm{d}/\mathrm{d}y$.

At the interface, we use the Saffmann condition [8]

$$\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] = \left[\frac{\Omega}{\sqrt{k}}\right] u \text{ at } y = h. \tag{2.6}$$

Physically, this represents the continuity of the tangential stress at the interface. Here Ω is the slip parameter which depends on the structure of the porous material. The continuity of normal stress at the interface leads to

$$p = -\delta\eta - \gamma\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \mu_{\text{eff}}\left(\frac{\partial v}{\partial y}\right) - \left(\frac{\mu}{\sqrt{k}}\right)v. \tag{2.7}$$

Here, δ is the stress gradient, γ is the surface tension and $\partial^2\eta/\partial x^2$ is the curvature of the interface, η is the elevation of the interface satisfying the dynamic equation

$$\frac{\partial \eta}{\partial t} + (q \cdot \nabla)\eta = 0. \tag{2.8}$$

This leads to the condition

$$v = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}. \tag{2.9}$$

In the regime of linear theory eq. (2.9) may be written as

$$v = \frac{\partial \eta}{\partial t}. \tag{2.10}$$

We invoke solutions for the quantities η, u, v, p varying as

$$[\eta(y), u(y), v(y), p(y)] \exp(i\alpha x + nt), \tag{2.11}$$

respectively, where n is the growth rate of the interfacial disturbances and α is the wavevector. From eqs (2.3), (2.6) and (2.11), we get

$$i\alpha u + Dv = 0, \tag{2.12}$$

$$D^2 v + \alpha^2 v = \left(\frac{\Omega}{\sqrt{k}}\right) Dv \text{ at } y = h. \tag{2.13}$$

Eliminating pressure term between eqs (2.1)–(2.3) and using eq. (2.12), we get the stability equation

$$D^4 v + (2\alpha^2 + \alpha_0^2) D^2 v + \alpha^2(\alpha^2 + \alpha_0^2) v = 0, \tag{2.14}$$

where $\alpha_0^2 = 1/Mk$ and $M = \mu_{\text{eff}}/\mu$.

Equation (2.1), using (2.11) leads to

$$p = (\mu_{\text{eff}}/\alpha^2)(D^3 v - [\alpha^2 + \alpha_0^2] Dv). \tag{2.15}$$

Equating (2.15) with (2.7) and rearranging the terms we get for the growth rate n the expression

$$n = \frac{v[\gamma\alpha^2 - \delta]}{(\mu_{\text{eff}}/\alpha^2)[D^3 v - \{2\alpha^2 + \alpha_0^2\} Dv] + (\mu/\sqrt{k}v)} \text{ at } y = h \tag{2.16}$$

Making eq. (2.16) dimensionless, using $\lambda = \sqrt{(\gamma/\delta)}$ the scale for length

$$\alpha^* = \alpha\lambda, D^* = D\lambda, h^* = h/\lambda, n^* = n\mu_{\text{eff}}(\sqrt{\gamma\delta})$$

we get,

$$n^* = \frac{v\alpha^{*2}(1 - \alpha^{*2})}{[(2\alpha^{*2} + \alpha_0^{*2}) D^* v - D^{*3} v] - \left[\frac{v\alpha_0^* \alpha^{*2}}{M^{1/2}}\right]} \text{ at } y = h, \tag{2.17}$$

where asterisk denotes the dimensionless quantities. The solution of eq. (2.14) is

$$v(y) = A_1 \text{ch}(\alpha y) + A_2 \text{sh}(\alpha y) + A_3 \text{ch}(a y) + A_4 \text{sh}(a y), \tag{2.18}$$

where $a = \sqrt{(\alpha^2 + \alpha_0^2)}$ and for brevity, we have denoted cosh(x) and sinh(x) by ch(x) and sh(x), respectively. The coefficients $A_i (i = 1$ to $4)$ are determined using the boundary conditions (2.4) to (2.6) and (2.13) and they are given by

$$A_1 + A_3 = 0, \tag{2.19}$$

$$A_4 + A_2 \frac{\alpha}{a} = 0, \tag{2.20}$$

$$A_2 + RA_1 = 0, \tag{2.21}$$

$$A_4 - A_1 R(\alpha/a) = 0, \tag{2.22}$$

where

$$R = \frac{R_1}{R_2},$$

$$R_1 = [2\alpha^2 \mathrm{ch}(\alpha h) - (\alpha^2 + a^2)\mathrm{ch}(ah)] + \left(\frac{\Omega}{\sqrt{k}}\right)[a\,\mathrm{sh}(ah) - \alpha\,\mathrm{sh}(\alpha h)], \quad (2.23)$$

$$R_2 = \left[2\alpha^2 \mathrm{sh}(\alpha h) - \left(\frac{\Omega\alpha}{\sqrt{k}}\right)\mathrm{ch}(\alpha h) + \left(\frac{\alpha\Omega}{\sqrt{k}}\right)\mathrm{ch}(ah) - \left(\frac{\alpha}{a}\right)(\alpha^2 + a^2)\,\mathrm{sh}(ah)\right]. \quad (2.24)$$

The constants A_i ($i = 1$ to 4) in (2.19) to (2.24) are expressed in terms of A_1 and obtain,

$$v(h) = A_1\left[(\mathrm{ch}(\alpha h) - \mathrm{ch}(ah)) + R\left(\left[\frac{\alpha}{a}\right]\mathrm{sh}(ah) - \mathrm{sh}(\alpha h)\right)\right], \quad (2.25)$$

$$Dv(h) = A_1\left[(\mathrm{sh}(\alpha h) - a\,\mathrm{sh}(ah)) + R\,\mathrm{ch}(ah) - \alpha\,\mathrm{ch}(\alpha h))\right], \quad (2.26)$$

$$D^3 v(h) = A_1\left[\alpha^3 \mathrm{sh}(\alpha h) - a^3 \mathrm{sh}(ah)) + R(\alpha a^2 \mathrm{ch}(ah) - \alpha^3 \mathrm{ch}(\alpha h))\right]. \quad (2.27)$$

The expression for the growth rate of instability of the interface given by eq. (2.17) may be computed numerically for typical values of thickness h, viscosity ratio M, the porous parameter D_A and slip parameter Ω. This aspect is given in the next section.

3. Numerical computation and discussion

Analytical expressions for the dispersion relation in the case of RT instability past a porous layer is obtained and several interesting conclusions are made.

In the limit of $\Omega \to 0$ and $M \to 1$, the dispersion relation (2.17) tends to that given by Brown [2]. This can be seen from the numerical results depicted in figure 2. For different values of Ω, M and h^*, the dispersion relation (2.17) is numerically evaluated and the results are shown in figure 3, 4.

For $h^* < 1$, $M < 1$, $D_A < 1$ and $\Omega < 1$, the growth rate n^* is always positive for all values of α^* as can be seen in figure 3 (curves 1 and 2) exhibiting instability as in the case of RT instability in the absence of porous medium, discussed by Brown [2]. For these values of parameters although the magnitude of growth is one order less than that of Brown [2], the slip at the porous media has a significant effect on the qualitative nature of RT instability.

For $D_A = 1$, $M = 1$, $h^* = 1$ and $\Omega < 1$, n^* is always positive having minimum value at $\alpha^* = 0{\cdot}5$ and maxima at $\alpha^* = 0{\cdot}2$ or $0{\cdot}9$ as seen in the figure 3 (curve 3). This oscillatory nature of n^* is due to the drag at the interface. The same behaviour is also true for increasing values of h^* and D_A

For $h^* > 1$, $M = 1$, $D_A < 1$ and $\Omega < 1$, n^* takes positive or negative values i.e., the interface is stable or unstable depending on the values of α^* as is seen in figure 4 (curve 1). For an increase in both h^* and M for $D_A < 1$ and $\Omega < 1$, n^* is positive for all α^* as shown in figure 3 (curve 4), indicating the instability.

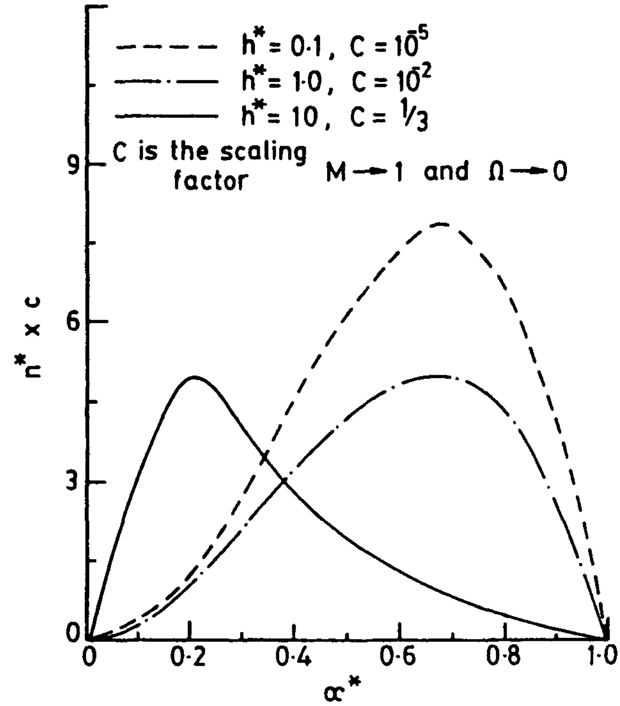

Figure 2. Plot of reduced growth n^* vs. α^*.

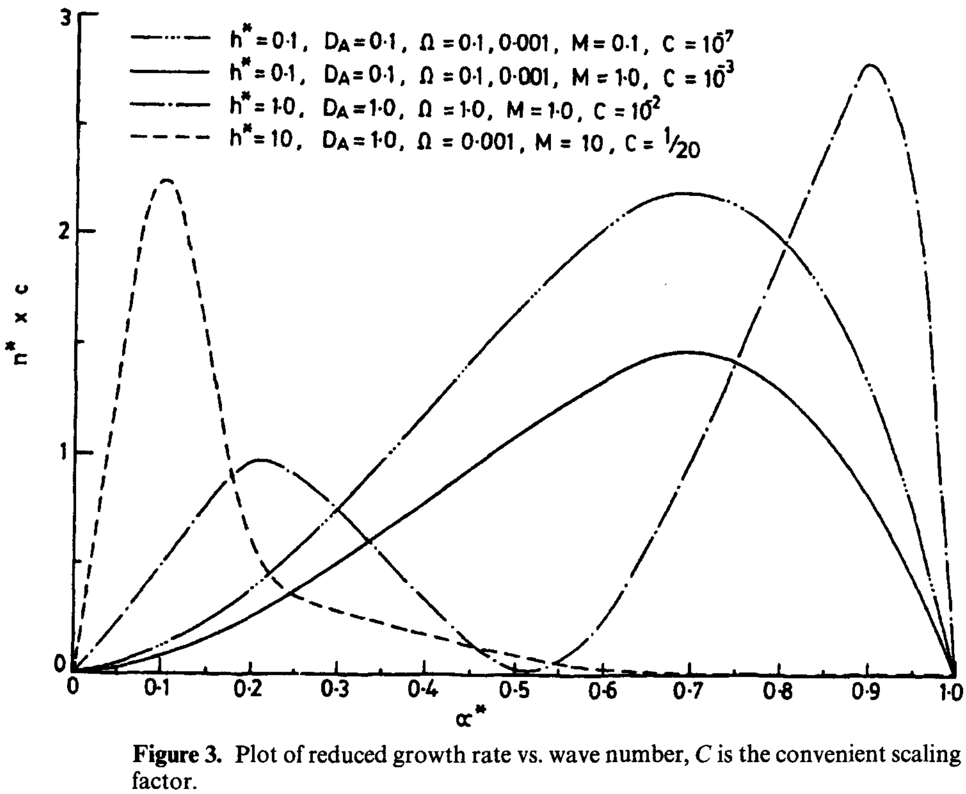

Figure 3. Plot of reduced growth rate vs. wave number, C is the convenient scaling factor.

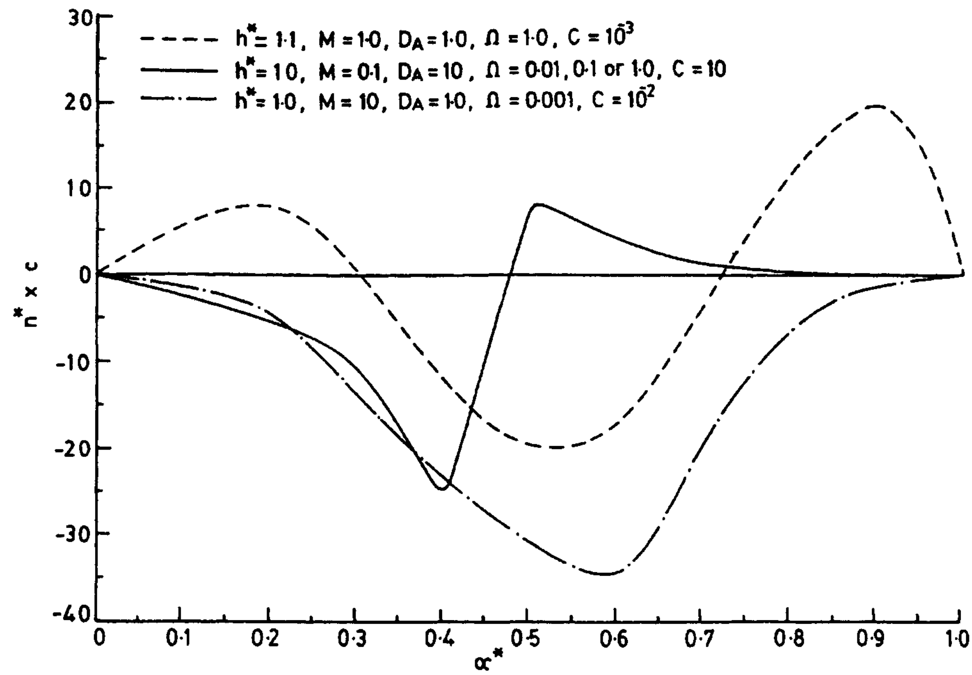

Figure 4. Plot of reduced growth rate vs. wave number.

For $h^* = 10$, $M = 0{\cdot}1$, $D_A = 10$ and $\Omega < 1$, the oscillatory behaviour of n^* prevails for some values of α^* and is shown in figure 4 (curve 2). For a suitable thickness of the porous layer the increase in the value of M make the system stable which was otherwise unstable as seen in figure 4 (curve 3).

From these numerical calculations we conclude that RT instability can be controlled by controlling the thickness of the porous layer and the slip parameter (i.e. structure of porous media). This can be effectively used in materials science processing.

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References

[1] Beavers G S and Joseph D D, Boundary conditions at a naturally permeable wall. *J. Fluid. Mech.* **30** (1967) 197
[2] Brown H C, Rayleigh–Taylor instability in thickness conducting fluid, *Phys. Fluids* **A1** (1989) 895–896
[3] Kull H J, Theory of Rayleigh–Taylor instability, *Phys. Rep.* **206** (1991) 199–325
[4] Michael J, Misksis and Stephen H Davis, Slip over rough and coated surfaces, *J. Fluid Mech.* **273** (1994) 125–139
[5] Nield D A, The boundary correction for the Rayleigh–Darcy problem – Limitations of the Brinkmann equations, *J. Fluid Mech.* **128** (1983) 37–46
[6] Rudraiah N, Coupled parallel flows in a channel and a bounding porous medium of finite thickness, *Trans. ASME* **107** (1985) 322–329

[7] Rudraiah N, Krishnamurthy B S and Mathad R D, The effect of oblique magnetic field on the surface instability of a finite conducting fluid layer, *Acta Mech.* (in press) (1995)
[8] Saffmann P G, On the boundary at the surface of a porous medium. *Stud. Appl. Math.* **50**, (1971) 93–101
[9] Saffmann P G and Taylor G I, The perturbation of fluid into a porous medium or Hele-shaw cell containing a more viscous liquid. *Proc. R. Soc. London* **A243** (1958) 312–320