Proc. Indian Acad. Sci. (Math. Sci.), Vol. 107, No. 2, May 1997, pp. 201–212.


# Random commutation

F MARTÍNEZ and A R VILLENA*

Departamento de Estadística e I.O. Universidad de Granada, 18071 Granada, Spain
*Departamento de Análisis Matemático, Universidad de Granada, 18071 Granada, Spain
E-mail: avillena@goliat.ugr.es

MS received 3 April 1996; revised 5 December 1996

**Abstract.** We investigate the commutation between a continuous linear random operator and a continuous linear deterministic operator on a Banach space. From this we obtain probabilistic versions of theorems by Fuglede and Putnam, both of them dealing with the commutation between continuous linear operators with continuous normal operators on a Hilbert space.

**Keywords.** Random operator; normal operator; compact normal operator.

## 1. Introduction

Operator commutators are a classical topic appearing in several branches of functional analysis and operator theory [1,6].

In this paper we investigate the commutation of a continuous linear random operator $\Phi$ and a continuous linear deterministic operator $F$ both of them acting on a Banach space $X$. This commutation may be understood in a very broad sense; namely, for each element $x$ in $X$ and each continuous linear functional $f$ on $X$, the event $[f(\Phi Fx)=f(F\Phi x)]$ can happen, that is the set$\{\omega: f((\Phi Fx)(\omega))=f(F(\Phi x)(\omega))\}$ has a positive probability which depends upon the element $x$ and the functional $f$. In such a case, there exists a measurable set $\Delta$ with a positive probability in such a way that, for every $x\in X$, the random variables $\Phi Fx$ and $F\Phi x$ coincide almost surely.

Fuglede solved in [3] an outstanding problem in functional analysis, proposed by von Neumann, namely that every continuous linear operator $T$ commuting with a continuous normal operator $F$ on a Hilbert space, also commutes with the adjoint operator of $F$. Also the study of the commutator $[T, F]$ when $[[T, F], F]$ equals zero, and the operators $T$ and $F$ act on a Banach space, has received considerable attention. A fundamental result in this area was the theorem of Kleinecke [4] and Sirokov [9] who proved independently that in such a case $[T, F]$ is generalized nilpotent operator. A special case of this with the additional assumption that $F$ is a continuous normal operator on a Hilbert space, was solved by Putnam [5] by stating that, in such a case the operator $[T, F]$ equals zero. We show several probabilistic versions of those theorems by Fuglede and Putnam.

Finally we show that if there is a sufficiently large probabilistic commutation between a random operator $\Phi$ and a compact normal operator on a Hilbert space, then $\mathbb{P}[\Phi\equiv\varphi(K)]>0$ for a suitable random function $\varphi$ acting on the spectrum of $K$.

## 2. On the continuity of linear random operators

Throughout the paper, $(\Omega,\Sigma,\mathbb{P})$ denotes a complete probability space. Every measurable subset $\Delta$ of $\Omega$ is considered as a new probability space with the inherited structure from $\Omega$, whose induced probability on $\Delta$ is the conditional probability $\mathbb{P}_\Delta$. As it is usual the *expectation* of a given real random variable $\xi$, denoted by $\mathbb{E}(\xi)$, is defined to be the number $\int_\Omega \xi d\mathbb{P}$.

A mapping $\xi$ from $\Omega$ into a given Banach space $Y$, over either the real or the complex field, denoted by $\mathbb{K}$, is said to be a *Bochner random variable* on $Y$ if it is the almost sure limit of a sequence of *simple random variables* on $Y$. We denote, by $\mathscr{L}_0(\mathbb{P}, Y)$ the linear space of all $Y$-valued Bochner random variables, which with the almost sure identification becomes a metrizable complete linear topological space $L_0(\mathbb{P}, Y)$ for the convergence in probability. This topology can be derived from the paranorm given by $\|\xi\|_0 = \mathbb{E}(\|\xi\|/(1+\|\xi\|))$. Relevant subspaces of $\mathscr{L}_0(\mathbb{P}, Y)$ are the spaces $\mathscr{L}_r(\mathbb{P}, Y) = \{\xi \in \mathscr{L}_0(\mathbb{P}, Y): \mathbb{E}\|\xi\|^r < \infty\}$ of all $Y$-valued Bochner random variables having $r$th moment. Besides the inherited topology from $\mathscr{L}_0(\mathbb{P}, Y)$, the space $\mathscr{L}_r(\mathbb{P}, Y)$ has its appropriate topology, namely that associated to convergence in $r$-mean, which can be derived from the paranorm given by $\|\xi\|_r = \mathbb{E}\|\xi\|^r$, when $0 < r < 1$, while if $1 \leqslant r$ it can be derived from the seminorm given by $\|\xi\|_r = (\mathbb{E}\|\xi\|^r)^{1/r}$. Given $\xi \in \mathscr{L}_0(\mathbb{P}, Y)$, $[\xi]$ denotes the equivalence class of $\xi$ for the usual almost sure identfication. The space $L_r(\mathbb{P}, Y) = \{[\xi]: \xi \in \mathscr{L}_r(\mathbb{P}, Y)\}$ becomes a metrizable complete linear space.

Given Banach spaces $X$ and $Y$denote by $BL(X, Y)$ the linear space of all continuous linear operators from $X$ into $Y$ endowed with the usual operator norm given by $\|F\| = \sup_{\|x\|=1} \|Fx\|$. To shorten notation we write $BL(X)$ instead of $BL(X, X)$ and $X'$ instead of $BL(X, \mathbb{K})$.

A map $\Phi: X \times \Omega \to Y$ is said to be a random operator from $X$ to $Y$ if, for each $x \in X$, the map $\omega \mapsto \Phi(x, \omega)$, noted $\Phi x$, lies in $\mathscr{L}_0(\mathbb{P}, Y)$. For a full discussion of random operators the reader is referred to [2] and [10]. Such an operator is said to have $r$th *moment* if the maps $\omega \mapsto \Phi(z, \omega)$ lie in $\mathscr{L}_r(\mathbb{P}, Y)$ and is said to be *linear* if $\mathbb{P}[\Phi(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \Phi x_1 + \alpha_2 \Phi x_2] = 1$, for all $x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \in \mathbb{K}$. The *continuity in probability* of $\Phi$ at $x_0 \in X$ means that $\lim_{x \to x_0} \mathbb{P}[\|\Phi x - \Phi x_0\| > \varepsilon] = 0, \forall \varepsilon > 0$. If $\Phi$ has $r$th moment, then there is a notion of continuity more convenient than the continuity in probability introduced above; namely the *continuity in r-mean*, which means that $\lim_{x \to x_0} \mathbb{E}\|\Phi x - \Phi x_0\|^r = 0$. We define the *conditional operator*, $\Phi_\Delta$, of $\Phi$ as the restriction of $\Phi$ to $X \times \Delta$.

**Theorem 2.1.** *Let $X$ and $Y$ be Banach spaces and $\Phi$ be a linear random operator from $X$ to $Y$. Then the following assertions are equivalent:*

1. *$\Phi$ is continuous in probability at every point in $X$.*
2. $\lim_{x \to 0} \mathbb{P}[\|\Phi x\| > \varepsilon] = 0$ *for every* $\varepsilon > 0$.
3. $\lim_{(x,f) \to (0,0)} \mathbb{P}[|f(\Phi x)| > \varepsilon] = 0$ *for every* $\varepsilon > 0$.

*Furthermore, if in addition $\Phi$ has rth moment, then the preceding conditions are equivalent to the following ones:*

4. *$\Phi$ is continuous in r-mean at every point in $X$.*
5. $\lim_{x \to 0} \mathbb{E}\|\Phi x\|^r = 0$.
6. $\lim_{(x,f) \to (0,0)} \mathbb{E}|f(\Phi x)|^r = 0$.
7. $\exists M > 0: \mathbb{E}\|\Phi x\|^r \leqslant M\|x\|^r \quad \forall x \in X$.

*Proof.* It is clear that 1 implies 2, 4 implies 5, and 6 implies 3.
Since $|f(\Phi x)| \leqslant \|f\| \, \|\Phi x\|$, we have $\lim_{(x,f)\to(0,0)}|f(\Phi x)| = 0$ whenever 2 is fulfilled.

Assume that $\Phi$ satisfies condition 3. To deduce assertion 1, in view of the closed graph theorem, it suffices to show that the graph of $\Phi$ is closed in $X \times \mathscr{L}_0(\mathbb{P}, Y)$. To this end, consider a sequence $\{x_n\}$ converging to 0 in $X$ with $\{\Phi x_n\}$ converging in probability to $\xi$. For every $f \in Y'$ we have $f(\xi) = f(\lim_{n\to\infty} \Phi x_n) = \lim_{n\to\infty} f(\Phi x_n) = 0$. This gives $\mathbb{P}[\xi \in \ker f] = 1$. Since $\{\ker f : f \in Y'\}$ is a family of closed subsets in $Y$, we can apply [11; Lemma 3.1] in order to obtain

$$\mathbb{P}\left[\xi \in \bigcap_{f\in Y'} \ker f\right] = \inf\{\mathbb{P}[\xi \in \ker f_1 \cap \cdots \cap \ker f_k] : k \in \mathbb{N},\ f_1,\ldots,f_k \in Y'\} = 1.$$

By Hahn–Banach theorem, $\cap_{f\in Y'} \ker f = 0$, and thus we get $\mathbb{P}[\xi = 0] = 1$. This establishes 1. To prove 4 we only need to show that $\Phi$ has closed graph in $X \times \mathscr{L}_r(\mathbb{P}, Y)$. Let $\{x_n\}$ be a sequence in $X$ converging to zero with $\{\Phi x_n\}$ converging in $r$-mean to $\xi$. For every $f \in Y'$, we have $\mathbb{E}|f(\xi)|^r = \lim_{n\to\infty} \mathbb{E}|f(\Phi x_n)|^r = 0$. Therefore $\mathbb{P}[\xi \in \ker f] = 1$. Arguing as above we get $\mathbb{P}[\xi = 0] = 1$ and this gives 4.

Assume that $\Phi$ satisfies 5. Since $\mathbb{E}|f(\Phi x)|^r \leqslant \|f\|^r \mathbb{E}\|\Phi x\|^r$, it may be concluded that $\lim_{(x,f)\to(0,0)} \mathbb{E}|f(Tx)|^r = 0$.

It is clear that 5 follows from 7. Conversely, assume that 5 holds and we claim that there are $N, \delta > 0$ such that $\mathbb{E}\|\Phi x\|^r \leqslant N$ if $\|x\| = \delta$. Otherwise for every $n \in \mathbb{N}$ we choose $x_n \in X$ with $\|x_n\| = 1/n$ and $\mathbb{E}\|\Phi x_n\|^r > n$ which contradicts assertion 5. If $x \in X\setminus\{0\}$, then $\mathbb{E}\|\Phi(\delta\|x\|)^{-1}x\|^r \leqslant N$ and therefore $\mathbb{E}\|\Phi x\|^r \leqslant \delta^r N \|x\|^r$, as desired.

We leave it to the reader to state the notion of continuous multilinear random operator.

## 3. Commutation probabilities

To investigate the commutation properties between a random operator and a deterministic one we first require an improvement of [11, Theorem 4.1]. To do this we establish a technical result in the following.

*Lemma* 3.1. *Let $X$ and $Y$ be Banach spaces and $\Phi$ a continuous linear random operator from $X$ to $Y$. Then the set $C_\delta = \{x \in X : \mathbb{P}[\Phi x = 0] \geqslant \delta\}$ is closed, for every $0 \leqslant \delta \leqslant 1$.*

*Proof.* Let $\{x_n\}$ be a sequence in $C_\delta$ converging to an element $x$ in $X$. The sequence $\{\Phi x_n\}$ converges in probability to $\Phi x$ and consequently $\lim_{n\to\infty} \mathbb{E}(\|\Phi x_n\|/(1+\|\Phi x_n\|)) = \mathbb{E}(\|\Phi x\|/(1+\|\Phi x\|))$. Let $\Omega_n = \{\omega \in \Omega : \Phi x_n \neq 0\}$. For every $n \in \mathbb{N}$, we have $\mathbb{E}(\|\Phi x_n\| /(1+\|\Phi x_n\|)) = \int_{\Omega n} \|\Phi x_n\|/(1+\|\Phi x_n\|)d\mathbb{P} \leqslant \mathbb{P}[\Omega_n] \leqslant 1-\delta$. Therefore, $\mathbb{E}(\|\Phi x\|/(1 + \|\Phi x\|)) \leqslant 1-\delta$. Given $k \in \mathbb{N}$, it is clear that $kx_n \in C_\delta$ for every $n \in \mathbb{N}$ and $\{kx_n\}$ converges in probability to $kx$. Accordingly $\mathbb{E}(k\|\Phi x\|/(1+k\|\Phi x\|)) \leqslant 1-\delta$. Letting $k \to \infty$, we deduce that $\mathbb{P}[\Phi x \neq 0] \leqslant 1-\delta$ and $x \in C_\delta$. □

**Theorem 3.2.** *Let $X_1,\ldots,X_N$, $Y$ be Banach spaces and $\Phi$ a continuous multilinear random operator from $X_1 \times \cdots \times X_N$ to $Y$. If, for all $x \in X_1 \times \cdots \times X_N$ and $f \in Y'$*

$$\mathbb{P}[f(\Phi x) = 0] > 0,$$

*then the set*

$$\{\mathbb{P}[f(\Phi x)=0]: x\in X_1 \times \cdots \times X_N, f\in Y'\}$$

*has a nonzero minimum, say* $\delta$, *and there exists a measurable set* $\Delta$ *with* $\mathbb{P}[\Delta]=\delta$ *in such a way that, for every* $x\in X_1 \times \cdots \times X_N$,

$$\Phi x = 0 \text{ almost surely on } \Delta.$$

*Proof.* For simplicity we assume $N=1$. For each $n\in\mathbb{N}$, let $C_n$ be the closed subset of $X_1 \times Y'$ given by $C_n=\{(x,\rho)\in X_1 \times Y': \mathbb{P}[f(\Phi x)=0]\geqslant 1/n\}$. Then $X_1 \times Y' = \cup_{n=1}^{\infty} C_n$. From Baire theorem $C_n$ has an interior point, say $(x',f')$, for a suitable natural number $n$. Note that, if $(x,f)\in X_1 \times Y'$ and $\lambda$, $\mu\in\mathbb{K}\setminus\{0\}$ with $|\lambda|$ and $|\mu|$ small enough, then

$$\mathbb{P}[(f'-\mu f)\Phi(x'-\lambda x)=0]\geqslant\frac{1}{n}.$$

Therefore

$$\begin{aligned}\frac{1}{n}&\leqslant \mathbb{P}[f'-\mu f)\Phi(x'-\lambda x)=0]=\mathbb{P}[f'\Phi(x'-\lambda x)-\mu f\Phi(x'-\lambda x)=0]\\ &=\mathbb{P}[f'\Phi(x'-\lambda x)=f\Phi(x'-\lambda x)=0]\\ &\quad+\mathbb{P}[f'\Phi(x'-\lambda x)=\mu f\Phi(x'-\lambda x), f'\Phi(x'-\lambda x)\neq 0],\end{aligned}$$

and letting $\mu\to 0$ we have

$$\begin{aligned}\frac{1}{n}&\leqslant \mathbb{P}[f'\Phi(x'-\lambda x)=f\Phi(x'-\lambda x)=0]\\ &\leqslant \mathbb{P}[f'\Phi x'=f'\Phi x=0, \quad f\Phi x'=f\Phi x=0]\\ &\quad+\mathbb{P}[f'\Phi x'=\lambda f'\Phi x, f'\Phi x\neq 0]+\mathbb{P}[f\Phi x'=\lambda f\Phi x, f\Phi x\neq 0].\end{aligned}$$

Further, letting $\lambda\to 0$ we get

$$\frac{1}{n}\leqslant \mathbb{P}[(f'\Phi x'=f'\Phi x=0, f\Phi x'=f\Phi x=0]\leqslant \mathbb{P}[f\Phi x=0].$$

This shows that $\inf\{\mathbb{P}[f\Phi x=0]: x\in X_1, f\in Y'\}>0$. Therefore we can apply [11, Theorem 4.1] to show that there exists a measurable set $\Delta$ with

$$\begin{aligned}\mathbb{P}[\Delta]&=\min\{\mathbb{P}[f\Phi x=0]: x\in X_1, f\in Y'\}\\ &=\max\{\mathbb{P}[\Delta]: f\Phi x=0 \text{ almost surely on } \Delta\forall x\in X_1, f\in Y'\}.\end{aligned}$$

For each $x\in X_1 \times \cdots \times X_N$ we have

$$\mathbb{P}_\Delta[\Phi_\Delta x=0]=\mathbb{P}_\Delta\left[\Phi_\Delta x\in \bigcap_{f\in Y'}\ker f\right].$$

Since $\{\ker f: f\in Y'\}$ is a family of closed subsets of $Y$ we can apply [11, Lemma 3.1] to deduce that $\mathbb{P}_\Delta[\Phi_\Delta x=0]$ equals $\inf\{\mathbb{P}_\Delta[f_1(\Phi_\Delta x)=\cdots=f_k(\Phi_\Delta x)=0]: k\in\mathbb{N}, f_1,\ldots,f_k\in Y'\}=1$. Consequently $\Phi x=0$ almost surely on $\Delta$. □

The preceding theorem is used in the following to compute the equivalence of two continuous linear random operators.

COROLLARY 3.3

*Let $\Phi$ and $\Psi$ be continuous linear random operators from a Banach space $X$ to a Banach space $Y$. If $\mathbb{P}[f(\Phi x) = f(\Psi x)] > 0$ for all $x \in X$ and $f \in Y'$, then the set $\{\mathbb{P}[f(\Phi x) = f(\Psi x)]: x \in X, f \in Y'\}$ has a nonzero minimum, say $\delta$, and there exists a measurable set $\Delta$ with $\mathbb{P}[\Delta] = \delta$ in such a way that, for every $x \in X$, $\Phi x = \Psi x$ almost surely on $\Delta$.*

Two random operators $\Phi$ and $\Psi$ from a Banach space $X$ to a Banach space $Y$ are said to be *equivalent*, written $\Phi \equiv \Psi$, if $\mathbb{P}[f(\Phi x) = f(\Psi x)] = 1 \ \forall x \in X, \forall f \in Y'$. The preceding result shows that $\Phi \equiv \Psi$ if and only if, $\mathbb{P}[\Phi x = \Psi x] = 1 \ \forall x \in X$. The quantity min $\{\mathbb{P}[f(\Phi x) = f(\Psi x)]: x \in X, f \in Y'\}$ is considered as the probability of $\Phi$ and $\Psi$ being equivalent and it is denoted as $\mathbb{P}[\Phi \equiv \Psi]$.

If $F$ and $G$ are continuous linear deterministic operators on Banach spaces $X$ and $Y$, respectively, and $\Phi$ is a continuous linear random operator from $X$ to $Y$, then the operators $(x,\omega) \mapsto \Phi(Fx,\omega)$ and $(x,\omega) \mapsto G(\Phi(x,\omega))$ from $X \times \Omega$ into $Y$ are continuous linear random operators from $X$ to $Y$ and we denote them by $\Phi F$ and $G\Phi$, respectively. Given a continuous linear random operator $\Phi$ and a continuous linear deterministic operator $F$, both of them acting on a Banach space $X$, we define the *commutator* of $\Phi$ and $F$ as the continuous linear random operator $[\Phi, F] = \Phi F - F\Phi$. If $\mathbb{P}[f([\Phi,F]x) = 0] > 0$ for all $x \in X$ and $f \in X'$, then it is reasonable to consider the quantity $\mathbb{P}[\Phi F \equiv F\Phi]$ as a measure of the commutation likelihood between both of the operators. To determine the existence of a measurable set on which $\Phi$ behaves as an operator commuting with $F$, it suffices to check the quantities $\mathbb{P}[f([\Phi,F]x) = 0]$. If all of them are nonzero, then it is likely that $\Phi$ commutes with $F$ and we measure the likelihood that can be expected by computing the minimum of all of them. In such a case, by Corollary 3.3, there exists a measurable set $\Delta$ with $\mathbb{P}(\Delta) = \mathbb{P}[\Phi F \equiv F\Phi]$ and satisfying for every $x \in X$ the equality $\Phi F x = F\Phi x$ almost surely on $\Delta$, that is $\Phi_\Delta F \equiv F\Phi_\Delta$.

In the next we apply Theorem 3.2 to study a large commutation.

COROLLARY 3.4

*Let $\Phi$ be a continuous linear random operator and $\mathscr{F}$ be a norm-closed subspace of continuous linear deterministic operators, all of them acting on a Banach space $X$. If, for all $x \in X$, $f \in X'$, and $F \in \mathscr{F}$, $\mathbb{P}[f(\Phi F x) = f(F\Phi x)] > 0$ then the set $\{\mathbb{P}[f(\Phi F x) = f(F\Phi x)]: x \in X, f \in X', F \in \mathscr{F}\}$ has a nonzero minimum, say $\delta$, and there exists a measurable set $\Delta$ with $\mathbb{P}[\Delta] = \delta$ in such a way that, for all $x \in X$ and $F \in \mathscr{F}$, $\Phi F x = F\Phi x$ almost surely on $\Delta$.*

*Proof.* Consider the continuous bilinear random operator $(x,F) \mapsto [\Phi, F]x$ from $X \times \mathscr{F}$ to $X$ and apply Theorem 3.2. □

## 4. Commutation with normal operators

Commutation theory is concerned mostly with operators acting on a Hilbert space. This has its origin in the commutation relations occurring in quantum mechanics. For a deeper discussion of this topic we refer the reader to [6].

Fuglede theorem (see [8, Corollary 2.2.6]) states that, if a continuous linear operator $T$ on a Hilbert space $H$ commutes with a continuous normal operator $F$ on $H$, then $T$ also commutes with the adjoint operator of $F$, from now on denoted by $F^*$. Unfortunately we don't know whether Fuglede theorem remains true if we replace $T$ by a continuous linear random operator $\Phi$. However we state some relevant results about this. To do this we first require a number of technical results.

*Lemma* 4.1. *Let $\Phi$ be a continuous linear random operator from a finite-dimensional Banach space $X$ to a Banach space $Y$. Then $\Phi$ is equivalent to a Bochner random variable on $BL(X,Y)$.*

*Proof.* Let $\{x_1,\ldots,x_n\}$ be a basis in $X$ and, for each $i=1,\ldots,n$, let $\{y_{i,k}\}$ be a sequence of simple random variables on $Y$ converging almost surely to $\Phi x_i$. For each $k\in\mathbb{N}$, we define a simple random variable $\Psi_k$ on $BL(X,Y)$ by

$$\Psi_k\left(\sum_{i=1}^{n}\lambda_i x_i\right)=\sum_{i=1}^{n}\lambda_i y_{i,k}\,\forall\,\lambda_1,\ldots,\lambda_n\in\mathbb{K}.$$

For all $p,q\in\mathbb{N}$ and $\lambda_1,\ldots,\lambda_n\in\mathbb{K}$, we have

$$\left\|(\Psi_p-\Psi_q)\left(\sum_{i=1}^{n}\lambda_i x_i\right)\right\|=\left\|\sum_{i=1}^{n}\lambda_i(y_{i,p}-y_{i,q})\right\|\leqslant\left(\sum_{i=1}^{n}|\lambda_i|\right)\sum_{j=1}^{n}\|y_{j,p}-y_{j,q}\|.$$

Since the map $\lambda_1x_1+\cdots+\lambda_nx_n\mapsto|\lambda_1|+\cdots+|\lambda_n|$ defines a norm on $X$ necessarily equivalent to the original one $\|\cdot\|$, there exists a positive constant $M$ satisfying $|\lambda_1|+\cdots+|\lambda_n|\leqslant M\|\lambda_1x_1+\cdots+\lambda_nx_n\|$, $\forall\lambda_1,\ldots,\lambda_n\in\mathbb{K}$. Therefore

$$\left\|(\Psi_p-\Psi_q)\left(\sum_{i=1}^{n}\lambda_i x_i\right)\right\|\leqslant M\left(\sum_{j=1}^{n}\|y_{j,p}-y_{j,q}\|\right)\left\|\sum_{i=1}^{n}\lambda_i x_i\right\|,$$

and $\|\Psi_p-\Psi_q\|\leqslant M\sum_{j=1}^{n}\|y_{j,p}-y_{j,q}\|$, which shows that the sequence $\{\Psi_k\}$ converges almost surely to a random variable $\Psi$ on $BL(X,Y)$. Moreover, for every $x\in X$ the sequence $\{\Psi_k x\}$ converges almost surely to $\Phi x$ and therefore $\Phi\equiv\Psi$. □

We can now formulate a Fuglede type theorem dealing with the random commutation with a compact normal operator.

**Theorem 4.2** *Let $\Phi$ be a continuous linear random operator and $K$ be a compact normal deterministic operator, both of them acting on a complex Hilbert space $H$. Then*

$$\mathbb{P}[\Phi K\equiv K\Phi]=\mathbb{P}[\Phi K^*\equiv K^*\Phi].$$

*Proof.* Let $\{\lambda_n\}$ denote the sequence of spectral values of $K$ with $\lambda_0=0$ and, for each $n\in\mathbb{N}\cup\{0\}$, $\pi_n$ denote the projection from $H$ onto ker $(\lambda_n I-K)$. By [1, Theorem 3.2], $K=\sum_{n=0}^{\infty}\lambda_n\pi_n$ and $\sum_{n=0}^{\infty}\pi_n=I$ pointwise.

Fix a measurable set $\Delta$ with $\mathbb{P}[\Delta]=\mathbb{P}[\Phi K\equiv K\Phi]$ and $\Phi_\Delta K\equiv K\Phi_\Delta$. Given $n\in\mathbb{N}$, we have $\Phi_\Delta K\pi_n\equiv K\Phi_\Delta\pi_n$ and therefore $(\Phi_\Delta\pi_n)K\equiv K(\Phi_\Delta\pi_n)$. Since $\pi_n$ has finite rank [1, Theorem 2.4], $\Phi_\Delta\pi_n$ is equivalent to a random variable on $BL(H)$. From this it may be concluded that $(\Phi_\Delta\pi_n)K^*\equiv K^*(\Phi_\Delta\pi_n)$, and consequently $[\Phi_\Delta, K^*]\pi_n\equiv 0$. Further we note that $K\pi_0=\pi_0K=0$ and therefore $K(\Phi_\Delta\pi_0)\equiv\Phi_\Delta K\pi_0\equiv 0$. Since $\Phi_\Delta\pi_0$ maps every element almost surely into ker $K$ and ker $K = \ker K^*$, we have $K^*\Phi_\Delta\pi_0 \equiv 0$. Accordingly $[\Phi_\Delta, K^*]\pi_0 \equiv 0$. Since $[\Phi_\Delta, K^*]\pi_n \equiv 0\ \forall n\in\mathbb{N}\cup\{0\}$ and $\sum_{n=0}^{\infty}\pi_n = I$, it follows that $[\Phi_\Delta, K^*] \equiv 0$. Therefore $\mathbb{P}[\Phi K^* \equiv K^*\Phi] \geqslant \mathbb{P}[\Phi K \equiv K\Phi]$. Replacing $K$ by $K^*$ we have $\mathbb{P}[\Phi K \equiv K\Phi] \geqslant \mathbb{P}[\Phi K^* \equiv K^*\Phi]$, as required.

In order to avoid the compactness condition, we require the random operator to have 1st moment.

**Theorem 4.3.** *Let $\Phi$ be a continuous linear random operator having* 1st *moment and $F$ be a continuous normal deterministic operator, both of them acting on a complex Hilbert space $H$. Then*

$$\mathbb{P}[\Phi F \equiv F\Phi] = \mathbb{P}[\Phi F^* \equiv F^*\Phi].$$

*Proof.* Our proof follows on the same lines as given by Rosenblum (see [8, Theorem 2.2.5]). Let $\Delta$ be a measurable set with $\mathbb{P}[\Delta] = \mathbb{P}[\Phi F \equiv F\Phi]$ and $\Phi_\Delta F \equiv F\Phi_\Delta$. For all $n\in\mathbb{N}$ and $\lambda\in\mathbb{C}$, we have $\Phi_\Delta(\bar{\lambda}F)^n \equiv (\bar{\lambda}F)^n\Phi_\Delta$. Since $\Phi_\Delta$ is continuous in mean, for every $x\in H$, we have the almost surely equalities

$$\begin{aligned}\Phi_\Delta \exp(\bar{\lambda}F)x &= \Phi_\Delta \sum_{n=0}^{\infty}\frac{1}{n!}(\bar{\lambda}F)^n x = \sum_{n=0}^{\infty}\Phi_\Delta\frac{1}{n!}(\bar{\lambda}F)^n x\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(\bar{\lambda}F)^n\Phi_\Delta x = \exp(\bar{\lambda}F)\Phi_\Delta x.\end{aligned}$$

Hence $\Phi_\Delta \exp(\bar{\lambda}F) \equiv \exp(\bar{\lambda}F)\Phi_\Delta$, which implies $\Phi_\Delta \equiv \exp\,(\bar{\lambda}F)\Phi_\Delta\,\exp(-\bar{\lambda}F)$ and $\exp(-\lambda F^*)\,\Phi_\Delta \exp(\lambda F^*) \equiv \exp(\bar{\lambda}F - \lambda F^*)\Phi_\Delta \exp(-\bar{\lambda}F + \lambda F^*)$. Since $\exp(\bar{\lambda}F - \lambda F^*)$ and $\exp(-\bar{\lambda}F + \lambda F^*)$are unitary operators we have $\|\exp.(\bar{\lambda}F - \lambda F^*)\| = \|\exp(-\bar{\lambda}F + \lambda F^*)\| = 1$. Given $x, y \in H$, we define a complex $L_1(\mathbb{P}_\Delta, \mathbb{C})$-valued function $\varphi$ holomorphic on the whole complex plane by

$$\begin{aligned}\varphi(\lambda) &= [(\Phi_\Delta \exp(\lambda F^*)x\,|\exp(-\bar{\lambda}F)y)] = [(\exp(-\lambda F^*)\Phi_\Delta \exp(\lambda F^*)x|y)]\\ &= [(\exp(\bar{\lambda}F - \lambda F^*)\Phi_\Delta \exp(-\bar{\lambda}F + \lambda F^*)x|y)]\\ &= [(\Phi_\Delta\,\exp(-\bar{\lambda}F + \lambda F^*)x|\exp(-\bar{\lambda}F + \lambda F^*)y)].\end{aligned}$$

From Theorem 2.1 we have $\mathbb{E}\|\Phi_\Delta x\| \leqslant M\|x\|\forall x\in X$ for a suitable $M > 0$. Therefore $\mathbb{E}|\varphi(\lambda)| \leqslant M\|\exp(\bar{\lambda}F - \lambda F^*)x\|\,\|\exp(-\bar{\lambda}F + \lambda F^*)y\| \leqslant M\|x\|\,\|y\|$, for a suitable $M > 0$. Liouville theorem [7, Theorem 3.32] assures that the vector-valued function $\varphi$ is constant. Hence $\varphi(\lambda) = \varphi(0) = [(\Phi_\Delta x|y)]$ (which mean that, for every $\lambda\in\mathbb{C}$, $\varphi(\lambda) = (\Phi x|y)$ almost surely on $\Delta$) and $0 = \varphi'(\lambda) = [(\Phi_\Delta \exp(\lambda F^*)F^*\ x|\exp(-\bar{\lambda}F)y)] + [(\Phi_\Delta \exp\lambda F^*)x|\exp(-\bar{\lambda}F)(-F)y)]$. Accordingly $0 = \varphi'(0) = [(\Phi F^* x|y) - (\Phi x|Fy)]$ almost surely, for all $x, y\in H$. This shows that $F^*\Phi \equiv \Phi F^*$. Therefore $\mathbb{P}[\Phi F^* \equiv F^*\Phi] \geqslant \mathbb{P}[\Phi F \equiv F\Phi]$, and replacing $F$ by $F^*$ we have $\mathbb{P}[\Phi F \equiv F\Phi] \geqslant \mathbb{P}[\Phi F^* \equiv F^*\Phi]$, which ends the proof. □

For a continuous linear operator $T$ and a continuous normal operator $F$ on a Hilbert space $H$, for which the commutator $[T, F]$ commutes with $F$, Putnam theorem [8, Corollary 2.2.9] shows that $[T, F] = 0$. It seems to be unknown whether Putnam theorem remains true for a continuous linear random operator $T$. We establish several probabilistic versions of that theorem.

In order to translate Putnam theorem to random operators we first restrict our attention to compact deterministic operators.

**Theorem 4.4.** *Let $\Phi$ be a continuous linear random operator and $K$ be a compact normal deterministic operator, both of them acting on a complex Hilbert space $H$. Then*

$$\mathbb{P}[[[\Phi,K],K]\equiv 0]=\mathbb{P}[[\Phi,K]\equiv 0].$$

*Proof.* As in the proof of Theorem 4.2 we put $K=\sum_{n=0}^{\infty}\lambda_n\pi_n$.

It is clear that $\mathbb{P}[[[\Phi,K],K]\equiv 0]\geqslant\mathbb{P}[[\Phi,K]\equiv 0]$.

Let $\Delta$ be a measurable set with $[[\Phi_\Delta,K],K]\equiv 0$ and $\mathbb{P}[\Delta]=\mathbb{P}[[[\Phi,K],K]\equiv 0]$. For every $n\in\mathbb{N}$, we have $[[\Phi_\Delta\pi_n,K],K]\equiv[[\Phi_\Delta,K],K]\pi_n\equiv 0$. Since $\Phi_\Delta\pi_n$ is equivalent to a random variable on $BL(H)$, it may be concluded that $[\Phi_\Delta\pi_n,K]\equiv 0$ and therefore $[\Phi_\Delta,K]\pi_n\equiv 0$. Since $[\Phi_\Delta,K]K\equiv K[\Phi_\Delta,K]$, we have $0\equiv[\Phi_\Delta,K]K\pi_0\equiv K[\Phi_\Delta,K]\pi_0$. Consequently $[\Phi_\Delta,K]\pi_0$ maps every element almost surely into ker $K$, which gives $\pi_n[\Phi_\Delta,K]\pi_0\equiv 0\ \forall\, n\in\mathbb{N}$. Further $\pi_0[\Phi_\Delta,K]\pi_0\equiv 0$ and therefore $[\Phi_\Delta,K]\pi_0\equiv 0$, since $\sum_{n=0}^{\infty}\pi_n=I$. Accordingly $[\Phi_\Delta,K]\equiv 0$ and so $\mathbb{P}[[\Phi,K]\equiv 0]\geqslant\mathbb{P}[[[\Phi,K],K]\equiv 0]$, which concludes the proof. □

Now compactness condition may be omitted by requiring the expectation property on the random operator.

*Lemma* 4.5. *Let $\Phi$ be a continuous linear random operator having* 1*st moment and $F$ be a continuous normal deterministic operator, both of them acting on a complex Hilbert space $H$. Assume that $F=\int z\,dE(z)$ for a suitable spectral measure $E$ on the spectrum* sp $(F)$ *of $F$. Then $\Phi F\equiv F\Phi$* and *$\Phi F^*\equiv F^*\Phi$ if and only if, $\Phi E(\Lambda)\equiv E(\Lambda)\Phi$ for every measurable set $\Lambda$ in* sp $(F)$.

*Proof.* If $\Phi F\equiv F\Phi$ and $\Phi F^*\equiv F^*\Phi$, then $\Phi p(F,F^*)\equiv\Phi p(F,F^*)$ for every complex polynomial $p$ in $F$ and $F^*$. Given a measurable subset. $\Lambda$ of sp$(F)$, the projection $E(\Lambda)$ can be obtained as the strong limit in $BL(H)$ of a suitable sequence $\{p_n(F,F^*)\}$ of complex polynomials $p_n(F,F^*)$ in $F$ and $F^*$ with $p_n(F,F^*)^*=p_n(F,F^*)$. Therefore, for every $x\in H$, we have

$$\Phi E(\Lambda)x=\|\cdot\|_1-\lim_{n\to\infty}\Phi p_n(F,F^*)x=\|\cdot\|_1-\lim_{n\to\infty}p_n(F,F^*)\Phi x=E(\Lambda)\Phi x$$

(where $\|\cdot\|_1-\lim$ denotes the limit in mean) and thus $\Phi E(\Lambda)\equiv E(\Lambda)\Phi$.

We now assume that, $\Phi E(\Lambda)\equiv E(\Lambda)\Phi$ for every measurable subset $\Lambda$ of sp$(F)$. Let $\varepsilon$ be a positive number, $\Lambda_1,\ldots,\Lambda_n$ measurable subsets of sp$(F)$, and $z_1\in\Lambda_1,\ldots,z_n\in\Lambda_n$ such that $\|F-\sum_{k=1}^{n}z_kE(\Lambda_k)\|<\varepsilon$. Then

$$\begin{aligned}\|[\Phi,F]x\|&=\left\|\Phi\left(F-\sum_{k=1}^{n}z_kE(\Lambda_k)\right)x-\left(F-\sum_{k=1}^{n}z_kE(\Lambda_k)\right)\Phi x\right\|\\&\leqslant\left\|\Phi\left(F-\sum_{k=1}^{n}z_kE(\Lambda_k)\right)x\right\|+\left\|\left(F-\sum_{k=1}^{n}z_kE(\Lambda_k)\right)\Phi x\right\|,\end{aligned}$$

almost surely, for every $x\in H$. Since $[\Phi,F]$ is a continuous linear 1st order random operator, Theorem 2.1 shows that there exists $M>0$ such that $\mathbb{E}\|[\Phi,F]x\|\leqslant M\|x\|\ \forall\, x\in X$.

Accordingly

$$\mathbb{E}\|[\Phi, F]x\| \leqslant 2M \left\| F - \sum_{k=1}^{n} z_k E(\Lambda_k) \right\| \|x\| \leqslant 2M\varepsilon \|x\|,$$

for a suitable $M > 0$. Letting $\varepsilon \to 0$ we have $\mathbb{E}\|[\Phi, F]x\| = 0$ and therefore $[\Phi, F] \equiv 0$.

In the previous argument, just replace $z_1, \ldots, z_n$ by $\bar{z}_1, \ldots, \bar{z}_n$, and it follows that $[T, F^*] \equiv 0$. □

**Theorem 4.6.** *Let $\Phi$ be a continuous linear random operator having 1st moment and $F$ be a continuous normal deterministic operator, both of them acting on a complex Hilbert space $H$. Then*

$$\mathbb{P}[[[\Phi, F], F] \equiv 0] = \mathbb{P}[[\Phi, F] \equiv 0].$$

*Proof.* It is clear that $\mathbb{P}[[[\Phi, F], F] \equiv 0] \geqslant \mathbb{P}[[\Phi, F] \equiv 0]$.
Let $\Delta$ be a measurable set with $\mathbb{P}[\Delta] = \mathbb{P}[[[\Phi, F], F] \equiv 0]$ and $[[\Phi_\Delta, F], F] \equiv 0$. Since $[\Phi_\Delta, F]\ F \equiv F[\Phi_\Delta, F]$, Theorem 4.3 shows that $[\Phi_\Delta, F]F^* \equiv F^*[\Phi_\Delta, F]$. Let $E$ be a spectral measure on sp $(F)$ such that $F = \int z dE(z)$. By applying the preceding lemma, for every measurable subset $\Lambda$ in sp$(F)$, we have $[\Phi_\Delta, F]\ E(\Lambda) \equiv E(\Lambda)[\Phi_\Delta, F]$. Since

$$\begin{aligned} [\Phi_\Delta, E(\Lambda)]F &\equiv \Phi_\Delta\, E(\Lambda)F - E(\Lambda)\Phi_\Delta\, F \equiv \Phi_\Delta\, FE(\Lambda) - E(\Lambda) - \Phi_\Delta F \\ &\equiv [\Phi_\Delta, F]E(\Lambda) + F\Phi_\Delta\, E(\Lambda) - E(\Lambda)\Phi_\Delta F \\ &\equiv E(\Lambda)[\Phi_\Delta, F] + F\Phi_\Delta E(\Lambda) - E(\Lambda)\Phi_\Delta F \\ &\equiv F\Phi_\Delta\, E(\Lambda) - E(\Lambda)F\Phi_\Delta \equiv\ F[\Phi_\Delta, E(\Lambda)]. \end{aligned}$$

Theorem 4.3 shows that $[\Phi_\Delta,\ E(\Lambda)]F^* \equiv F^*[\Phi_\Delta,\ E(\Lambda)]$ and the preceding lemma gives $[\Phi_\Delta, E(\Lambda)]E(\Lambda) \equiv E(\Lambda)[\Phi_\Delta, E(\Lambda)]$. Accordingly $\Phi_\Delta\, E(\Lambda) - E(\Lambda)\Phi_\Delta\, E(\Lambda) \equiv E(\Lambda)\Phi_\Delta$ $E(\Lambda) - E(\Lambda)\Phi_\Delta$. By multiplying on the left and on the right by $E(\Lambda)$ we obtain $\Phi_\Delta E(\Lambda) \equiv E(\Lambda)\Phi_\Delta$. By Lemma 4.5 $[\Phi_\Delta,\ F] \equiv 0$ and therefore $\mathbb{P}[[\Phi, F] \equiv 0]$ $\geqslant \mathbb{P}[[[\Phi, F], F] \equiv 0]$. □

## 5. Commutation with compact normal operators

Throughout this section, $K$ stands for a compact normal deterministic operator on a complex Hilbert space $H$. It decomposes into a pointwise sum $K = \sum_{n=0}^{\infty} \lambda_n \pi_n$ where $\{\lambda_n\}$ is the sequence of its spectral values with $\lambda_0 = 0$ and, for each $n$, $\pi_n$ is the projection onto ker $(\lambda_n I - K)$.

It is well known that any continuous linear deterministic operator $F$ that commutes with $K$ can be decomposed into a pointwise sum $F = \sum_{n=0}^{\infty} \varphi(\lambda_n)\pi_n$ for a suitable complex valued bounded function acting on sp$(K)$. Actually for every such function the above series converges pointwise to a continuous linear deterministic operator, usually denoted by $\varphi(K)$, satisfying that commutation property.

In this section we study whether the above assertions remain true in the random setting, replacing $F$ by a random operator and $\varphi$ by a random function.

**Theorem 5.1.** *Let $\Phi$ be a continuous linear random operator on $H$. If*

$$\mathbb{P}[([\Phi, F]x|y) = 0] > 0$$

*for every continuous linear deterministic operator F on H commuting with K and all x, y ∈ H, then there exists a sequence bounded in probability $\{\xi_n\}$ of complex-valued random variables such that, for every x ∈ H, the random series $\Sigma \xi_n \pi_n x$ converges in probability and*

$$\mathbb{P}\left[\Phi x = \sum_{n=0}^{\infty} \xi_n \pi_n x\right] > 0.$$

*Furthermore, if Φ has rth moment, then the random variables $\xi_n$ have rth moment, the sequence $\{\xi_n\}$ is bounded in r-mean and the series $\Sigma \xi_n \pi_n x$ converges in r-mean.*

*Proof.* By considering the closed subspace $\mathscr{F} = \{F \in BL(H): FK = KF\}$ of $BL(H)$ we can choose a measurable set $\Delta$ with $\mathbb{P}[\Delta] > 0$ and $[\Phi_\Delta, F] \equiv 0, \forall F \in \mathscr{F}$. For every $\lambda_n$ we fix $u_n \in \ker(\lambda_n I - K)$ with $\|u_n\| = 1$ and we define the random variable $\xi_n$ to be $(\Phi u_n | u_n)$ on $\Delta$ and 0 otherwise. Since $\Phi$ is continuous $\{\xi_n\}$ is bounded in probability. For every $n \in \mathbb{N}$ and every $u \in \ker(\lambda_n I - K)$, we consider the continuous linear operator $F$ on $H$ given by $Fx = (x|u_n)u$. Then $F$ commutes with $K$ and so $\Phi_\Delta F \equiv F\Phi_\Delta$. For every $y \in H$, $(\Phi_\Delta u|y) = (\Phi_\Delta F u_n|y) = (F\Phi_\Delta u_n|y) = (\Phi_\Delta u_n|F^* y) = \xi_{n|\Delta}(u|y)$. Therefore $\Phi_\Delta$ is equivalent to $\xi_{n|\Delta}\pi_n$ on $\ker(\lambda_n I - K)$. For every $x \in H$, $\Sigma_{n=0}^\infty \pi_n x$ converges in probability (in $r$-mean if $\Phi$ has $r$th moment) to $x$ and consequently $\Phi_\Delta x = \Sigma_{n=0}^\infty \Phi_\Delta \pi_n x$. From this we deduce $\Phi_\Delta x = \Sigma_{n=0}^\infty \xi_{n|\Delta}\pi_n x$. Therefore the random series $\Sigma \xi_{n|\Delta}\pi_n x$, and consequently $\Sigma \xi_n \pi_n x$, converge in probability. Further, for every $x \in H$ the equality $\Phi x = \Sigma_{n=0}^\infty \xi_n \pi_n x$ holds almost surely on $\Delta$. If $\Phi$ has $r$th moment, then $\mathbb{E}|\xi_n|^r \leqslant \mathbb{E}\|\Phi u_n\|^r < \infty$. Further $\{\Phi u_n\}$ is bounded in $r$-mean, since otherwise we could choose a subsequence $\{u_{n_k}\}$ *of* $\{u_n\}$ such that $\rho_k = \mathbb{E}\|\Phi u_{n_k}\|^r \to +\infty$. Accordingly $\{u_{n_k}/\rho_k^{1/k}\} \to 0$ and $\mathbb{E}\|\Phi(u_{n_k}/\rho_k^{1/k})\|^r = \mathbb{E}\|\Phi u_{n_k}\|^r/\rho_k = 1$ which contradicts the continuity of $\Phi$. □

**Theoren 5.2.** *Let $\{\pi_n\}$ be a sequence of pairwise orthogonal projections on a Hilbert space H and let $\{\xi_n\}$ be a sequence of complex-valued random variables having rth moment with $r \geqslant 2$. Then the following assertions are equivalent:*

1. *For each $x \in H$ the random series $\Sigma \xi_n \pi_n x$, converges in r-mean.*
2. *The sequence $\{\xi_n\}$ is bounded* in *r-mean.*

*Furthermore, in such a case the pointwise sum of the series $\Sigma \xi_n \pi_n$ defines a continuous linear random operator on H having rth moment.*

*Proof.* Assume that the sequence $\{\xi_n\}$ is bounded in $r$-mean. We show that, for each $x \in H$, the series $\Sigma |\xi_n|^2 \|\pi_n\|^2$ in $L_{r/2}(\mathbb{K})$ converges. To this end we note that

$$\sum_{n=1}^{\infty} \| |\xi_n|^2 \|\pi_n x\|^2 \|_{r/2} = \sum_{n=1}^{\infty} \|\pi_n x\|^2 \|\xi_n\|_r^2 \leqslant (\sup \|\xi_n\|_r^2) \sum_{n=1}^{\infty} \|\pi_n x\|^2$$
$$= (\sup \|\xi_n\|_r^2) \|x\|^2.$$

Therefore the series $\Sigma |\xi_n|^2 \|\pi_n\|^2$ converges absolutely in $L_{r/2}(\mathbb{K})$. Furthermore

$$\left\| \sum_{n=1}^{\infty} |\xi_n|^2 \|\pi_n\|^2 \right\|_{r/2} \leqslant (\sup \|\xi_n\|_r^2) \|x\|^2.$$

Given $m$. $n \in \mathbb{N}$ we have $\mathbb{E}\|\Sigma_{k=n}^{n+m} \xi_k \pi_k x\|^r = \mathbb{E}(\Sigma_{k=n}^{n+m} |\xi_k|^2 \|\pi_k x\|^2)^{r/2}$, which converges to zero when $n \to \infty$, for every $m \in \mathbb{N}$. So the series $\Sigma \xi_n \pi_n x$ converges in $\mathscr{L}_r(\mathbb{P}, H)$

and

$$\mathbb{E}\left\| \sum_{n=1}^{\infty} \xi_n \pi_n x \right\|^r \le (\sup \|\xi_n\|_r^r) \|x\|^r,$$

which shows that the linear random operator given by the pointwise sum of the series $\Sigma\xi_n\pi_n$ is continuous.

Conversely assume the series $\Sigma\xi_n\pi_n$ to be pointwise convergent. Then $\{\xi_n\pi_n\}$ gives a sequence of continuous linear operators from $H$ into $L_r(\mathbb{P}, H)$ which is pointwise bounded. Banach–Steinhaus theorem shows that there is a positive number $M$ such that $\sup_{\|x\|=1} \|\xi_n\pi_n x\|_r \le M \ \forall n\in\mathbb{N}$. Since $\sup_{\|x\|=1} \|\xi_n\pi_n x\|_r = \|\xi_n\|_r$, it follows that the sequence $\{\xi_n\}$ is bounded in $r$-mean. □

Unfortunately there exist sequences of complex-valued random variables $\{\xi_n\}$ bounded in $r$-mean with $0 < r < 2$, and sequences $\{\pi_n\}$ of pairwise orthogonal projections for which the random series $\Sigma\xi_n\pi_n x$ does not converge in probability for a suitable $x\in H$. We illustrate this fact in the following.

*Examples* 5.3. Let $e_n$ be an orthonormal sequence in a Hilbert space $H$. Consider the sequence $\{\pi_n\}$ of pairwise orthogonal projections on $H$ given by $\pi_n(x) = (x|e_n)e_n$ for every $x\in H$. Also consider the interval $[0, 1]$ endowed with the Lebesgue measure. Given a measurable set $\Delta$, $\chi_\Delta$ stands for the characteristic function of $\Delta$. If $0 < r < 2$, then we consider the sequence $\{\xi_n\}$ of random variables, given by $\xi_{2^k+m} = 2^{k/r}\chi_{[m/2^k,(m+1)/2^k[}$, $m = 0,\ldots,2^k - 1$, $k \geqslant 0$. The sequence $\{\xi_n\}$ is bounded in $r$-mean. Further the sequence $\{\alpha_n\}$ given by $\alpha_{2^k+m} = 2^{-k/r}$, $m = 0,\ldots,2^k - 1, k \geqslant 0$, satisfies that $\Sigma_{n=1}^{\infty}|\alpha_n|^2 < \infty$ and therefore the series $\Sigma_{n=1}^{\infty}\alpha_n e_n$ defines an element $x\in H$. The random series $\Sigma\xi_n\pi_n x$ satisfies

$$\left\| \sum_{n=1}^{2^{k+1}-1} \xi_n\pi_n x \right\| = \sum_{l=0}^{k}\sum_{m=0}^{2^l-1} |2^{l/r}\chi_{[m/2^l,(m+1)/2^l[}2^{-l/r}|^2 = \sum_{l=0}^{k}\chi_{[0,1[} = (k+1)\chi_{[0,1[}$$

and therefore it does not even converge in probability.

The random series $\Sigma\varphi(\lambda_n)\pi_n$ defines (up to equivalence) a continuous linear random operator having $r$th moment whenever $\varphi$ is a random function acting on sp $(K)$ bounded in $r$-mean with $r \ge 2$. We write $\varphi(K)$ to denote it.

COROLLARY 5.4

*Let $\Phi$ be a continuous linear random operator having 2nd moment. Then the following assertions are equivalent:*

1. *For every continuous linear deterministic operator $F$ on $H$ commuting with $K$ we have* $\mathbb{P}[\Phi F \equiv F\Phi] > 0$.
2. *There exists a random function $\varphi$ on* $\mathrm{sp}(K)$ *bounded in mean square such that* $\mathbb{P}[\Phi \equiv \varphi(K)] > 0$.

*Proof.* Assume that assertion 1 holds. By Theorem 5.1 there exists a random function $\varphi$ acting on $\mathrm{sp}(K)$ bounded in mean square such that $\mathbb{P}[\Phi \equiv \varphi(K)] > 0$.

Conversely if 2 is fulfilled, then there exists a measurable set $\Delta$ with $\mathbb{P}[\Delta] = \mathbb{P}[\Phi \equiv \varphi(K)]$ and $\Phi_\Delta \equiv \varphi_\Delta(K)$. Given a continuous linear deterministic operator $F$ commuting with $K$ it is known that $\pi_n F = F\pi_n$ for every $n\in\mathbb{N}$. From this it is easy to check that $\varphi_\Delta(K)Fx = F\varphi_\Delta(K)x$ almost surely for every $x \in H$. Thus $\mathbb{P}[\Phi F \equiv F\Phi] \geq \mathbb{P}[\Delta]$. Therefore assertion 1 follows. □

## References


[1] Beauzamy B, *Introduction to operator theory and invariant subspaces* (Amsterdam: North-Holland) (1988)

[2] Bharucha-Reid A T, *Random Integral Equations* (New York and London: Academic Press) (1972)

[3] Fuglede B, A commutativity theorem for normal operators, *Proc. Nat. Acad. USA* **36** (1950) 35–40

[4] Kleinecke D C, On operator commutators, *Proc. Am. Math. Soc.* **8** (1957) 535–536

[5] Putnam C R, On normal operators in Hilbert space, *Am. J. Math.* **73** (1950) 357–362

[6] Putnam C R, *Commutator properties of Hilbert space operators and related topics*, Ergebnisse series 36, Springer-Verlag, Berlin, Heidelberg, New York (1967)

[7] Rudin R W, *Functional Analysis* (New York: McGraw-Hill) (1973)

[8] Sakai S, *Operator algebras in dynamical systems* (Cambridge: Cambridge University Press) (1991)

[9] Sirokov F V, Proof of a conjecture of Kaplansky, *Usephi. Math. Nauk.* **11** (1956) 167–168

[10] Skorohod A V, *Random Linear Operators* (Holland: D. Reidel Publishing Company) (1984)

[11] Velasco M V and Villena A R, Continuity of random derivations, *Proc. Am. Math. Soc.* **123** (1995) 107–120