Proc. Indian Acad. Sci. (Math. Sci.), Vol. 107, No. 2, May 1997, pp. 197–199.

On self-reciprocal polynomials

ABDUL AZIZ and B A ZARGAR

Post-graduate Department of Mathematics, University of Kashmir, Hazratbal 190006, Kashmir, India

MS received 12 February 1996; revised 25 July 1996

Abstract. In this paper we establish a sharp result concerning integral mean estimates for self-reciprocal polynomials.

Keywords. Integral mean estimates; self-reciprocal polynomials; inequalities.

Let $P(z)$ be a polynomial of degree at most n. Then $P(z)$ is said to be self-reciprocal if it satisfies the condition

$$z^n P(1/z) = P(z) \quad \text{for all} \quad z \in C. \tag{1}$$

Polynomials $P(z)$ satisfying (1) were studied in the past by other authors (see for example [1, 2, 4–7]). Govil, Jain and Labelle [6] have proved that if $P(z)$ is a polynomial satisfying (1) and if in addition $P(z)$ has all its zeros either in the left half plane or in the right half plane, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{\sqrt{2}} \max_{|z|=1} |P(z)| \tag{2}$$

and

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{3}$$

It was shown by Dewan and Govil [3] and independently by the first author [1, Theorem 5] that inequality (3) holds for all polynomials $P(z)$ satisfying (1). Here we present the following interesting result which is L^2 analogue of (2) and (3) for all polynomials satisfying (1).

Theorem. *If $P(z)$ is a self-reciprocal polynomial of degree at most n*, then

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |P'(e^{i\theta})|^2 d\theta\right\}^{1/2} \leqslant \frac{n}{\sqrt{2}}\left\{\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^2 d\theta\right\}^{1/2} \tag{4}$$

and

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |P'(e^{i\theta})|^2 d\theta\right\}^{1/2} \geqslant \frac{n}{2}\left\{\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^2 d\theta\right\}^{1/2}. \tag{5}$$

Both the estimates are sharp. Equality in (4) *holds for $P(z) = c(z^n + 1)$ for all $n \geqslant 1$ and in* (5) *for $P(z) = cz^{n/2}$ where n is an even positive integer.*

Proof of the Theorem. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial of degree at most n and let

$$Q(z) = z^n P(1/z) = a_n + a_{n-1}z + \cdots + a_1 z^{n-1} + a_0 z^n.$$

Then we have

$$P'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + 2a_2 z + a_1$$

and

$$Q'(z) = na_0 z^{n-1} + (n-1)a_1 z^{n-2} + \cdots + 2a_{n-2}z + a_{n-1}.$$

Now

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\{|P'(e^{i\theta})|^2 + |Q'(e^{i\theta})|^2\}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}|P'(e^{i\theta})|^2\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}|Q'(e^{i\theta})|^2\,d\theta \\
&= \sum_{j=0}^{n} j^2|a_j|^2 + \sum_{j=0}^{n}(n-j)^2|a_j|^2 = \sum_{j=0}^{n}(j^2 + (n-j)^2)|a_j|^2 \\
&\leqslant \sum_{j=0}^{n} n^2|a_j|^2 = \frac{n^2}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^2\,d\theta. \qquad (6)
\end{aligned}$$

Now suppose $P(z)$ is a self-reciprocal polynomial of degree at most n. Then we have $z^n P(1/z) = P(z)$ for all $z \in C$, or equivalently $Q(z) = P(z)$ for all $z \in C$. This implies $Q'(z) = P'(z)$ for all $z \in C$ and hence in particular

$$|Q'(e^{i\theta})|^2 = |P'(e^{i\theta})|^2 \quad \text{for} \quad 0 \leqslant \theta < 2\pi.$$

Using this in (6), we get

$$\frac{1}{2\pi}\int_0^{2\pi}|P'(e^{i\theta})|^2 \leqslant \frac{n^2}{2}\cdot\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^2\,d\theta,$$

from which (4) follows immediately.

To prove (5), we take $A = zP'(z)$ and $B = nP(z) - zP'(z)$ in the identity

$$2|A|^2 + 2|B|^2 = |A+B|^2 + |A-B|^2,$$

we get

$$2|zP'(z)|^2 + 2|nP(z) - zP'(z)|^2 = n^2|P(z)|^2 + |2zP'(z) - nP(z)|^2.$$

This in particular for $z = e^{i\theta}$, $0 \leqslant \theta < 2\pi$, implies that

$$|P'(e^{i\theta})|^2 + |nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})|^2 \geqslant \frac{n^2}{2}|P(e^{i\theta})|^2,$$

which gives

$$\begin{aligned}
&\int_0^{2\pi}|P'(e^{i\theta})|^2\,d\theta + \int_0^{2\pi}|nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})|^2\,d\theta \\
&\quad \geqslant \frac{n^2}{2}\int_0^{2\pi}|P(e^{i\theta})|^2\,d\theta. \qquad (7)
\end{aligned}$$

Now by hypothesis, $P(z) = z^n P(1/z)$ for all $z \in C$ and therefore, we have

$$zP'(z) = nz^n P(1/z) - z^{n-1} P'(1/z).$$

This in particular for $z = \mathrm{e}^{-i\theta}$, $0 \leqslant \theta < 2\pi$, gives

$$\begin{aligned} |P'(\mathrm{e}^{-i\theta})| &= |\mathrm{e}^{-i\theta} P'(\mathrm{e}^{-i\theta})| = |n\, \mathrm{e}^{-in\theta} P(\mathrm{e}^{i\theta}) - \mathrm{e}^{-i(n-1)\theta} P'(\mathrm{e}^{i\theta})| \\ &= |nP(\mathrm{e}^{i\theta}) - \mathrm{e}^{i\theta} P'(\mathrm{e}^{i\theta})|. \end{aligned} \tag{8}$$

Using (8) in (7), we obtain

$$\int_0^{2\pi} |P'(\mathrm{e}^{i\theta})|^2 \,\mathrm{d}\theta + \int_0^{2\pi} |P'(\mathrm{e}^{-i\theta})|^2 \,\mathrm{d}\theta \geqslant \frac{n^2}{2} \int_0^{2\pi} |P(\mathrm{e}^{i\theta})|^2 \,\mathrm{d}\theta. \tag{9}$$

Since it can be easily verified that

$$\int_0^{2\pi} |P'(\mathrm{e}^{-i\theta})|^2 \mathrm{d}\theta = \int_0^{2\pi} |P'(\mathrm{e}^{i\theta})|^2 \,\mathrm{d}\theta,$$

from (9) it follows that

$$\int_0^{2\pi} |P'(\mathrm{e}^{i\theta})|^2 \,\mathrm{d}\theta \geqslant \frac{n^2}{4} \int_0^{2\pi} |P(\mathrm{e}^{i\theta})|^2 \,\mathrm{d}\theta$$

which is equivalent to (5) and this completes the proof of the theorem.

References

[1] Abdul Aziz, Inequalities for the derivative of a polynomial, *Proc. Am. Math. Soc.* **89** (1983) 259–266

[2] Abdul Aziz and Mohammad Q G, Simple proof of a theorem of Erdös and Lax, *Proc. Am. Math. Soc.* **80** (1980) 119–122

[3] Dewan K K, Extremal properties and coefficient estimates for polynomials with restricted zeros and on location of the zeros of polynomials, Ph.D. Thesis, Indian Institute of Technology, Delhi (1980)

[4] Frappier C and Rahman Q I, On an inequality of S Bernstein, *Canad. J. Math.* **34** (1982) 932–944

[5] Frappier C, Rahman Q I and St. Ruscheweyh, New inequalities for polynomials, *Trans. Am. Math. Soc.* **288** (1985) 69–99

[6] Govil N K, Jain V K and Labelle G, Inequalities for polynomials satisfying $P(z) \equiv z^n P(1/z)$, *Proc. Am. Math. Soc.* **57** (1976) 238–242

[7] Jain V K, Inequalities for polynomials satisfying $P(z) \equiv z^n P(1/z)$–II, *J. Indian Math. Soc.* **59** (1993) 167–170