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Inequalities for the derivative of a polynomial

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Abstract. Let $P(z)$ be a polynomial of degree n which does not vanish in $|z| < k$, where $k > 0$. For $k \leqslant 1$, it is known that

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^n} \max_{|z|=1} |P(z)|,$$

provided $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z| = 1$, where $Q(z) = z^n \overline{P(1/\bar{z})}$. In this paper we obtain certain refinements of this result. We also present a refinement of a generalization of the theorem of Turán.

Keywords. Maximum modules; inequalities; polynomials.

1. Introduction and statements of results

If $P(z)$ is a polynomial of degree n, then concerning the estimate of $|P'(z)|$ on the unit disk $|z| = 1$, we have

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)|. \tag{1}$$

This inequality is an immediate consequence of Bernstein's theorem on the derivative of a trigonometric polynomial (for reference, see [9]). In (1) equality holds if and only if $P(z)$ has all its zeros at the origin, and so it is natural to seek improvement in (1) under appropriate assumptions on the zeros of $P(z)$. It was conjectured by Erdös and later verified by Lax [7] (see also [2]) that if $P(z)$ does not vanish in $|z| < 1$, then (1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{2}$$

On the other hand, Turán [10] showed that if $P(z)$ has all its zeros in $|z| \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{3}$$

Thus in (2) as well as in (3) equality holds for those polynomials of degree n which have all their zeros on $|z| = 1$.

As an extension of (2) and (3), Malik [8] proved that if $P(z) \neq 0$ in $|z| < k$ where $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k} \max_{|z|=1} |P(z)|, \tag{4}$$

whereas if $P(z)$ has all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{5}$$

As an analogous result to (4) in case $k \leqslant 1$, it was shown by Govil [4] that if $P(z) \neq 0$ in $|z| < k$, $k \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^n} \max_{|z|=1} |P(z)|, \tag{6}$$

provided $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on the circle $|z| = 1$. Here $Q(z) = z^n \overline{P(1/\bar{z})}$.

In this paper we shall first present a refinement of (6) and obtain a bound that depends on the location of all the zeros of $P(z)$. We prove the following theorem.

Theorem 1. *Let*

$$P(z) = \prod_{j=1}^{n} (z - z_j)$$

be a polynomial of degree n which does not vanish in $|z| < k$ *where* $k \leqslant 1$, *and let*

$$Q(z) = z^n \overline{P(1/\bar{z})}.$$

If $|P'(z)|$ *and* $|Q'(z)|$ *become maximum at the same point on the circle* $|z| = 1$, *then*

$$\max_{|z|=1} |P'(z)| \leqslant \frac{1}{1+k^n} \left\{ n - k^n \sum_{j=1}^{n} \frac{|z_j| - k}{|z_j| + k} \right\} \max_{|z|=1} |P(z)|. \tag{7}$$

The result is best possible and equality holds for the polynomial $P(z) = z^n + k^n$, where $k \leqslant 1$.

Next we present the following refinement of Malik's result (5).

Theorem 2. *Let*

$$P(z) = \prod_{j=1}^{n} (z - z_j)$$

be a polynomial of degree n. If $|z_j| \leqslant k_j \leqslant 1, 1 \leqslant j \leqslant n$, *then*

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \left\{ 1 + \frac{1}{1 + \left(\frac{2}{n}\right) \sum_{j=1}^{n} \frac{k_j}{1 - k_j}} \right\} \max_{|z|=1} |P(z)|. \tag{8}$$

The result is best possible and equality holds for the polynomial $P(z) = (z + k)^n$ where $k \leqslant 1$.

It can be easily seen that Theorem 2 includes as special cases Turán's result (3) and Malik's result (5).

Theorem 3. *Let* $P(z)$ *be a polynomial of degree n which does not vanish in* $|z| < k$ *where* $k \leqslant 1$ *and* $Q(z) = z^n \overline{P(1 + \bar{z})}$. *If* $|P'(z)|$ *and* $|Q'(z)|$ *becomes maximum at the same point on* $|z| = 1$, *then*

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^n} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right. . \tag{9}$$

The result is best possible and equality holds for the polynomial $P(z) = z^n + k^n$, where $k \leqslant 1$.

2. Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 1. *If* $P(z)$ *is a polynomial of degree* n, *then on* $|z| = 1$,

$$|P'(z)| + |Q'(z)| \leqslant n \max_{|z|=1} |P(z)|,$$

where

$$Q(z) = z^n \overline{P(1/\bar{z})}.$$

This is a special case of a result due to Govil and Rahman [6] (see also [2]).

We also need the following result which is due to Aziz [1, Theorem 1].

Lemma 2. *If all the zeros of the polynomial*

$$P(z) = \prod_{j=1}^{n} (z - z_j)$$

of degree n *lie in* $|z| \leqslant k$ *where* $k \geqslant 1$, *then*

$$\max_{|z|=1} |P'(z)| \geqslant \frac{2}{1 + k^n} \sum_{j=1}^{n} \frac{k}{k + |z_j|} \max_{|z|=1} |P(z)|.$$

The next result which is due to Gardner and Govil [3] is needed for the proof of Theorem 2.

Lemma 3. *Let*

$$P(z) = \prod_{j=1}^{n} (z - z_j)$$

be a polynomial of degree n. *If* $|z_j| \geqslant k_j \geqslant 1, 1 \leqslant j \leqslant n$, *then for* $|z| = 1$

$$|Q'(z)/P'(z)| \geqslant 1 + \left(\frac{n}{\sum_{j=1}^{n} \frac{1}{k_j - 1}} \right).$$

For the proof of Theorem 3, we need Lemma 4, due to Govil [5].

Lemma 4. *If* $P(z)$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$ *where* $k \geqslant 1$, *then*

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1 + k^n} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=k} |P(z)| \right\}.$$

3. Proofs of the theorems

Proof of Theorem 1. Since all the zeros of $P(z)$ lie in $|z| \geqslant k, k \leqslant 1$, it follows that all the zeros of

$$Q(z) = z^n \overline{P(1/\bar{z})}$$

lie in $|z| \leqslant 1/k, 1/k \geqslant 1$ and $|P(z)| = |Q(z)|$ for $|z| = 1$.

Applying Lemma 2 to the polynomial $Q(z)$, we get

$$\begin{aligned}\max_{|z|=1} |Q'(z)| &\geqslant \frac{2k^n}{1+k^n} \sum_{j=1}^{n} \frac{|z_j|}{k+|z_j|} \max_{|z|=1} |Q(z)| \\ &= \frac{2k^n}{1+k^n} \sum_{j=1}^{n} \frac{|z_j|}{k+|z_j|} \max_{|z|=1} |P(z)|. \end{aligned} \tag{10}$$

Since by hypothesis $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z| = 1$, if

$$\max_{|z|=1} |P'(z)| = |P'(e^{i\alpha})|, \quad 0 \leqslant \alpha < 2\pi, \tag{11}$$

then

$$\max_{|z|=1} |Q'(z)| = |Q'(e^{i\alpha})|. \tag{12}$$

Now by Lemma 1, we have

$$|P'(e^{i\alpha})| + |Q'(e^{i\alpha})| \leqslant n \max_{|z|=1} |P(z)|.$$

This gives with the help of (10), (11) and (12)

$$\begin{aligned} n \max_{|z|=1} |P(z)| &\geqslant \max_{|z|=1} |P'(z)| + \max_{|z|=1} |Q'(z)| \\ &\geqslant \max_{|z|=1} |P'(z)| + \frac{2k^n}{1+k^n} \sum_{j=1}^{n} \frac{|z_j|}{k+|z_j|} \max_{|z|=1} |P(z)|, \end{aligned}$$

which implies

$$\max_{|z|=1} |P'(z)| \leqslant \left\{ n - \frac{2k^n}{1+k^n} \sum_{j=1}^{n} \frac{|z_j|}{k+|z_j|} \right\} \max_{|z|=1} |P(z)|,$$

and after simplification, we get

$$\max_{|z|=1} |P'(z)| \leqslant \frac{1}{1+k^n} \left\{ n - k^n \sum_{j=1}^{n} \frac{|z_j|-k}{|z_j|+k} \right\} \max_{|z|=1} |P(z)|.$$

This proves the desired result.

Proof of Theorem 2. Since

$$P(z) = \prod_{j=1}^{n} (z - z_j),$$

it follows that

$$Q(z) = z^n\overline{P(1/\bar{z})} = \prod_{j=1}^{n}(1 - z\bar{z}_j)$$

and

$$|P(z)| = |Q(z)| \quad \text{for} \quad |z| = 1.$$

Now by hypothesis $|z_j| \leqslant k_j \leqslant 1$, $j = 1, 2, \ldots, n$. Therefore, $|1/z_j| \geqslant |1/k_j| \geqslant 1$, $j = 1, 2, \ldots, n$ and hence by Lemma 3, for $|z| = 1$

$$|P'(z)/Q'(z)| \geqslant 1 + \frac{n}{\sum_{j=1}^{n} \frac{k_j}{1-k_j}}$$

$$= \frac{\sum_{j=1}^{n}\left(\frac{k_j}{1-k_j} + 1\right)}{\sum_{j=1}^{n} \frac{k_j}{1-k_j}}$$

$$= \frac{\sum_{j=1}^{n} \frac{1}{1-k_j}}{\sum_{j=1}^{n} \frac{k_j}{1-k_j}}$$

which gives, for $|z| = 1$

$$|Q'(z)/P'(z)| \leqslant \frac{\sum_{j=1}^{n} \frac{k_j}{1-k_j}}{\sum_{j=1}^{n} \frac{1}{1-k_j}}$$

$$= \frac{\sum_{j=1}^{n}\left(\frac{1}{1-k_j} - 1\right)}{\sum_{j=1}^{n} \frac{1}{1-k_j}}$$

$$= 1 - \frac{n}{\sum_{j=1}^{n} \frac{1}{1-k_j}}.$$

Hence

$$t|P'(z)| \geqslant |Q'(z)| \quad \text{for } |z| = 1, \tag{13}$$

where

$$t = 1 - \frac{n}{\sum_{j=1}^{n} \frac{1}{1-k_j}}.$$

Also, we have

$$Q'(z) = n\,z^{n-1}\,\overline{P(1/\bar{z})} - z^{n-2}\overline{P'(1/\bar{z})}.$$

Therefore, for $z = e^{i\theta}, 0 \leqslant \theta < 2\pi$

$$Q'(e^{i\theta}) = n\,e^{i(n-1)\theta}\,\overline{P(e^{i\theta})} - e^{i(n-2)\theta}\,\overline{P'(e^{i\theta})}, \tag{14}$$

which gives for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)| \geqslant n|P(z)| - |P'(z)|$$

or

$$|P'(z)| + |Q'(z)| \geqslant n|P(z)| \quad \text{for} \quad |z| = 1. \tag{15}$$

Using (13) in (15), we obtain for $|z| = 1$

$$\begin{aligned}(1+t)\,|P'(z)| &= |P'(z)| + t|P'(z)| \\ &\geqslant |P'(z)| + |Q'(z)| \geqslant n|P(z)|.\end{aligned}$$

Equivalently,

$$|P'(z)| \geqslant \frac{n}{1+t}|P(z)| \quad \text{for} \quad |z| = 1, \tag{16}$$

where t is defined by (13).

Since

$$\begin{aligned}\frac{1}{1+t} &= \frac{1}{2 - \left\{n \Big/ \sum_{j=1}^{n} \frac{1}{1-k_j}\right\}} = \sum_{j=1}^{n}\frac{1}{1-k_j} \Big/ \sum_{j=1}^{n}\left[\frac{2}{1-k_j} - 1\right] \\ &= \frac{1}{2}\sum_{j=1}^{n}\frac{1+k_j+1-k_j}{1-k_j} \Big/ \sum_{j=1}^{n}\frac{1+k_j}{1-k_j} \\ &= \frac{1}{2}\left[1 + n \Big/ \sum_{j=1}^{n}\frac{1+k_j}{1-k_j}\right] = \frac{1}{2}\left[1 + \left(n \Big/ \frac{\Sigma 1 - k_j + 2k_j}{1-k_j}\right)\right] \\ &= \frac{1}{2}\left[1 + n \Big/ \left(n + 2\sum_{j=1}^{n}\frac{k_j}{1-k_j}\right)\right] \\ &= \frac{1}{2}\left[1 + \frac{1}{1 + \frac{2}{n}\sum_{j=1}^{n}\frac{k_j}{1-k_j}}\right],\end{aligned}$$

from (16) we get

$$\max_{|z|=1}|P'(z)| \geqslant \frac{n}{2}\left[1 + \frac{1}{1 + \frac{2}{n}\sum_{j=1}^{n}\frac{k_j}{1-k_j}}\right]\max_{|z|=1}|P(z)|,$$

and this completes the proof of Theorem 2.

Proof of Theorem 3. Since all the zeros of $P(z)$ lie in $|z| \geqslant k$ where $k \leqslant 1$, all the zeros of $Q(z) = z^n\overline{P(1/\bar{z})}$ lie in $|z| \leqslant (1/k)$ where $(1/k) \geqslant 1$ and $|Q(z)| = |P(z)|$ for $|z| = 1$. Applying Lemma 4 to the polynomial $Q(z)$, we get

$$\max_{|z|=1} |Q'(z)| \geqslant \frac{nk^n}{1+k^n}\left\{\max_{|z|=1}|Q(z)| + \min_{|z|=(1/k)} |Q(z)|\right\}$$
$$= \frac{nk^n}{1+k^n}\left\{\max_{|z|=1}|P(z)| + \min_{|z|=(1/k)} |Q(z)|\right\}. \tag{17}$$

Now

$$\min_{|z|=(1/k)} |Q(z)| = \min_{|z|=1} |Q(z/k)|$$
$$= \min_{|z|=1} \left|\frac{z^n}{k^n}\overline{P(k/\bar{z})}\right|$$
$$= \frac{1}{k^n}\min_{|z|=k} |P(z)|.$$

Using this in (17), we obtain

$$\max_{|z|=1} |Q'(z)| \geqslant \frac{nk^n}{1+k^n}\left\{\max_{|z|=1}|P(z)| + \frac{1}{k^n}\min_{|z|=k} |P(z)|\right\}. \tag{18}$$

Since by hypothesis $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z| = 1$, if we choose α such that

$$\max_{|z|=1} |P'(z)| = |P'(e^{i\alpha})| \quad \text{where} \quad 0 \leqslant \alpha < 2\pi,$$

then

$$\max_{|z|=1} |Q'(z)| = |Q'(e^{i\alpha})|.$$

Hence from (18), we have

$$|Q'(e^{i\alpha})| \geqslant \frac{nk^n}{1+k^n}\left\{\max_{|z|=1}|P(z)| + \frac{1}{k^n}\min_{|z|=1} |P(z)|\right\}.$$

This in conjunction with Lemma 1 gives

$$n\max_{|z|=1}|P(z)| \geqslant |P'(e^{i\alpha})| + |Q'(e^{i\alpha})|$$
$$\geqslant |P'(e^{i\alpha})| + \frac{nk^n}{1+k^n}\left\{\max_{|z|=1}|P(z)| + \frac{1}{k^n}\min_{|z|=1} |P(z)|\right\}.$$

This implies

$$|P'(e^{i\alpha})| \leqslant \frac{n}{1+k^n}\left\{\max_{|z|=1}|P(z)| - \min_{|z|=k} |P(z)|\right\},$$

which is equivalent to the desired result.

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