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On a general theorem concerning some absolute summability methods

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Abstract. In this note a general theorem covering several absolute summability methods e.g, $|\bar{N}, p_n|_k, |R, p_n|_k$, is proved.

Keywords. Summability methods

1. Introduction

Let Σa_n be a given infinite series with the sequence of partial sums (s_n). By u_n we denote the nth $(C, 1)$ mean of the sequence (s_n). The series Σa_n is said to be summable $|C, 1|_k$, $k \geqslant 1$, if (see [3])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k < \infty.$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty \quad (P_{-i} = p_{-i} = 0, \quad i \geqslant 1).$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (t_n) of the Riesz means of the sequence(s_n) generated by the sequence of coefficients (p_n) (see [4]). The series Σa_n is said to be summable $|R, p_n|_k, k \geqslant 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty.$$

The series Σa_n is said to be summable $|\bar{N}, p_n|_k, k \geqslant 1$, (see [1]), if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty.$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), then both of $|R, p_n|_k$ and $|\bar{N}, p_n|_k$ is the same as $|C, 1|_k$ (resp. $|R, p_n|$, $|\bar{N}, p_n|$) summability.

We assume that (f_n), (g_n) and (h_n) are positive sequences.

2. Main theorem

Theorem. *Let* $Y_n = (1/F_{n-1} H_n) \Sigma_{v=1}^{n} F_{v-1} x_v \xi_v$ *such that* $F_n = \Sigma_{v=1}^{n} f_v \to \infty$,

$X_n = (1/G_n)\,\Sigma_{v=1}^{n} g_v x_v$. *Suppose that*

$$F_n \Delta g_n^{-1} = O(f_n g_n^{-1}),$$

$$\sum_{n=v+1}^{\infty} \frac{1}{F_{n-1} H_n^k} = O\left(\frac{1}{f_v H_v^k}\right),$$

and

$$\sum_{n=1}^{\infty} \left| \Delta \left| \frac{1}{F_{n-1}} \sum_{v=1}^{n-1} \frac{F_v G_v}{g_{v+1} H_v} \Delta \varepsilon_v \right|^k \right| < \infty.$$

Then the necessary and sufficient conditions that $\Sigma |Y_n|^k < \infty$ whenever $\Sigma |X_n|^k < \infty$ are

(i) $\varepsilon_n = O\left\{\frac{g_n H_n}{G_n}\right\}$,

(ii) $\Delta \varepsilon_n = O\left\{\frac{g_{n+1} H_n f_n}{G_n F_n}\right\}$.

Remark. It may be mentioned that on putting $f_n = q_n$, $g_n = P_{n-1}$, we obtain

1. $|\bar{N}, q_n|_k$, $|R, q_n|_k$ summability of $\Sigma a_n \varepsilon_n$ by putting $H_n = (Q_n/q_n)^{1/k}$, $H_n = n^{1/k-1}(Q_n/q_n)$ respectively, and
2. $|\bar{N}, p_n|_k$, $|R, p_n|_k$ summability of Σa_n by putting $G_n = (P_n/p_n)^{1/k} P_{n-1}$, $G_n = n^{1/k-1}$ $(P_n P_{n-1}/p_n)$ respectively.

3. Lemmas

We need the following lemmas for the proof of our theorem.

Lemma 1. *(Bor [2]). Let $k \geqslant 1$ and let $A = (a_{nv})$ be an infinite matrix. In order that $A \varepsilon (l^k; l^k)$ it is necessary that*

$$a_{nv} = O(1) \quad (\textit{all } n, v).$$

Lemma 2. *Let $\Sigma_{v=1}^{n} x_v = X_n$, $x_v \geqslant 0$, $\Sigma_{v=1}^{n} x_v \varepsilon_v = O(X_n)$, $\Sigma |\Delta \varepsilon_n| < \infty$. Then $\varepsilon_n = O(1)$.*

Proof. Abel's transformation gives

$$\sum_{v=1}^{n-1} X_v(\Delta \varepsilon_v) + X_n \varepsilon_n = O(X_n)$$

$$X_n |\varepsilon_n| = O(X_n) + O(X_n) \sum_{v=1}^{n-1} |\Delta \varepsilon_n|$$

$$= O(X_n) + O(X_n)$$

$$= O(X_n),$$

Then

$$\varepsilon_n = O(1).$$

4. Proof of the theorem

Sufficiency. We have via Abel's transformation

$$
\begin{aligned}
Y_n &= \frac{1}{F_{n-1}H_n}\sum_{v=1}^{n-1} G_v X_v \Delta(F_{v-1} g_v^{-1}\varepsilon_v) + \frac{G_n X_n}{g_n H_n}\varepsilon_n \\
&= \frac{1}{F_{n-1}H_n}\sum_{v=1}^{n-1}\{-G_v X_v f_v g_v^{-1}\varepsilon_v + G_v X_v\, F_v \Delta g_v^{-1}\varepsilon_v \\
&\quad + G_v X_v\, F_v\, g_{v+1}^{-1}\Delta\varepsilon_v\} + \frac{G_n X_n}{g_n H_n}\varepsilon_n \qquad (2)\\
&= Y_{n,1} + Y_{n,2} + Y_{n,3} + Y_{n,4}, \text{ say.}
\end{aligned}
$$

By Minkowski's inequality,

$$\sum_{n=1}^{m} |Y_n|^k = O(1)\sum_{n=1}^{m}\sum_{r=1}^{4}|Y_{n,r}|^k.$$

Applying Hölder's inequality

$$
\begin{aligned}
\sum_{n=2}^{m+1}|Y_{n,1}|^k &= \sum_{n=2}^{m+1}\left|\frac{1}{F_{n-1}H_n}\sum_{v=1}^{n-1} G_v X_v f_v g_v^{-1}\varepsilon_v\right|^k \\
&\leqslant \sum_{n=2}^{m+1}\frac{1}{F_{n-1}H_n^k}\sum_{v=1}^{n-1} G_v^k X_v^k f_v g_v^{-k}|\varepsilon_v|^k\left\{\frac{1}{F_{n-1}}\sum_{v=1}^{n-1} f_v\right\}^{k-1} \\
&\leqslant O(1)\sum_{v=1}^{m} G_v^k X_v^k f_v g_v^{-k}|\varepsilon_v|^k \sum_{n=v+1}^{m+1}\frac{1}{F_{n-1}H_n^k} \\
&\leqslant O(1)\sum_{v=1}^{m}\frac{G_v^k X_v^k}{H_v^k g_v^k}|\varepsilon_v|^k,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=2}^{m+1}|Y_{n,2}|^k &= \sum_{n=2}^{m+1}\left|\frac{1}{F_{n-1}H_n}\sum_{v=1}^{n-1} G_v X_v\left(\frac{F_v}{f_v}\right) f_v \Delta g_v^{-1}\varepsilon_v\right|^k \\
&\leqslant \sum_{n=2}^{m+1}\frac{1}{F_{n-1}H_n^k}\sum_{v=1}^{n-1} G_v^k X_v^k\left(\frac{F_v}{f_v}\right)^k f_v|\Delta g_v^{-1}|^k|\varepsilon_v|^k\left\{\frac{1}{F_{n-1}}\sum_{v=1}^{n-1} f_v\right\}^{k-1} \\
&\leqslant O(1)\sum_{v=1}^{m} G_v^k X_v^k\left(\frac{F_v}{f_v}\right)^k f_v|\Delta g_v^{-1}|^k|\varepsilon_v|^k\sum_{n=v+1}^{m+1}\frac{1}{F_{n-1}H_n^k} \\
&\leqslant O(1)\sum_{v=1}^{m}\frac{G_v^k X_v^k}{H_v^k g_v^k}|\varepsilon_v|^k,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=2}^{m+1}|Y_{n,3}|^k &= \sum_{n=2}^{m+1}\left|\frac{1}{F_{n-1}H_n}\sum_{v=1}^{n-1} G_v^k X_v^k\left(\frac{F_v}{f_v}\right) f_v g_{v+1}^{-1}\Delta\varepsilon_v\right|^k \\
&\leqslant \sum_{n=2}^{m+1}\frac{1}{F_{n-1}H_n^k}\sum_{v=1}^{n-1} G_v^k X_v^k\left(\frac{F_v}{f_v}\right)^k f_v g_{v+1}^{-k}|\Delta\varepsilon_v|^k\left\{\frac{1}{F_{n-1}}\sum_{v=1}^{n-1} f_v\right\}^{k-1} \\
&\leqslant O(1)\sum_{v=1}^{m} G_v^k X_v^k\left(\frac{F_v}{f_v}\right)^k f_v g_{v+1}^{-k}|\Delta\varepsilon_v|^k\sum_{n=v+1}^{m+1}\frac{1}{F_{n-1}H_n^k}
\end{aligned}
$$

$$\leqslant O(1)\sum_{v=1}^{m}\frac{G_v^k X_v^k F_v^k}{H_v^k g_{v+1}^k f_v^k}|\Delta\varepsilon_v|^k,$$

$$\sum_{n=1}^{m}|Y_{n,4}|^k=\sum_{n=1}^{m}\left|\frac{G_n X_n}{g_n H_n}\varepsilon_n\right|^k$$

$$\leqslant O(1)\sum_{n=1}^{m}\frac{G_n^k X_n^k}{g_n^k H_n^k}|\varepsilon_n|^k.$$

The result follows by the hypothesis.

Necessity of (i). By (1), we are able to write the matrix transforming (X_n) into (Y_n). Since $\Sigma|X_n|^k<\infty\Rightarrow\Sigma|Y_n|^k<\infty$, the matrix $\varepsilon(l^k;l^k)$. By lemma 1, a necessary condition for this implication is that the elements (in particular the diagonal elements) of this matrix should be bounded. Hence (i).

Necessity of (ii). From (2), we have

$$|Y_{n,3}|\leqslant|Y_{n,1}|+|Y_{n,2}|+|Y_{n,4}|+|Y_n|.$$

By Minkowski's inequality,

$$\sum_{n=1}^{m}|Y_{n,3}|^k\leqslant O(1)\sum_{n=1}^{m}\{|Y_{n,1}|^k+|Y_{n,2}|^k+|Y_{n,4}|^k+|Y_n|^k\}.$$

Suppose that $\Sigma|X_n|^k<\infty\Rightarrow\Sigma|Y_n|^k<\infty$. Then we have via the proof of sufficiency using (i),

$$\sum_{n=1}^{m}|Y_{n,3}|^k\leqslant O(1)\sum_{n=1}^{m}|X_n|^k,$$

$$\sum_{n=1}^{m}\frac{1}{H_n^k}\left|\frac{1}{F_{n-1}}\sum_{v=1}^{n-1}G_v X_v F_v g_{v+1}^{-1}\Delta\varepsilon_v\right|^k=O(1)\sum_{n=1}^{m}|X_n|^k.$$

Now, put $X_n=1/H_n$, we obtain

$$\sum_{n=1}^{m}\frac{1}{H_n^k}\left|\frac{1}{F_{n-1}}\sum_{v=1}^{n-1}f_v\left\{\frac{F_v G_v}{f_v g_{v+1}H_v}\Delta\varepsilon_v\right\}\right|^k=O(1)\sum_{n=1}^{m}1/H_n^k.$$

This, by Lemma 2, implies

$$\frac{1}{F_{n-1}}\sum_{v=1}^{n-1}f_v\left|\frac{F_v G_v}{f_v g_{v+1}H_v}\Delta\varepsilon_v\right|^k=O(1).$$

But

$$\sum_{v=1}^{n-1}f_v=F_{n-1}\to\infty,$$

Then, we should have

$$\frac{F_v G_v}{f_v g_{v+1}H_v}\Delta\varepsilon_v=O(1).$$

This completes the proof of the theorem.

5. Applications

With the aid of the remark, the following results are consequences of the theorem.

Theorem 2. *Let $p_nQ_n = O(P_nq_n)$. Then the necessary and sufficient conditions that $\Sigma a_n\varepsilon_n$ is summable $|N,q_n|_k$, whenever Σa_n is summable $|\bar{N},p_n|_k$, $k \geqslant 1$, are*

$$\varepsilon_n = O\left\{\left(\frac{p_nQ_n}{P_nq_n}\right)^{1/k}\right\},\quad \Delta\varepsilon_n = O\left\{\left(\frac{p_n}{P_{n-1}}\right)\left(\frac{P_nq_n}{p_nQ_n}\right)^{1-1/k}\right\},$$

provided that

$$\sum_{n=1}^{m}\left|\Delta\left|\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_{v-1}Q_v}{P_v}\right)\left(\frac{P_vq_v}{p_vQ_v}\right)^{1/k}\Delta\varepsilon_v\right|^k\right| < \infty.$$

Theorem 3. *Let $p_nQ_n = O(P_nq_n)$ and*

$$\sum_{n=v}^{m}\frac{n^{k-1}q_n^k}{Q_n^kQ_{n-1}} = O\left(\frac{v^{k-1}q_v^{k-1}}{Q_v^k}\right). \tag{3}$$

Then the necessary and sufficient conditions that $\Sigma a_n\varepsilon_n$ is summable $|R,q_n|_k$ whenever Σa_n is summable $|R,p_n|_k$, $k \geqslant 1$, are

$$\varepsilon_n = O\left(\frac{p_nQ_n}{P_nq_n}\right),\quad \Delta\varepsilon_n = O\left(\frac{p_n}{P_{n-1}}\right),$$

provided that

$$\sum_{n=1}^{m}\left|\Delta\left|\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}\frac{P_{v-1}q_v}{p_v}\Delta\varepsilon_v\right|^k\right| < \infty.$$

Theorem 4. *Let $p_nQ_n = O(P_nq_n)$. Then the necessary and sufficient conditions that $\Sigma a_n\varepsilon_n$* is summable *$|\bar{N},q_n|_k$ whenever Σa_n is summable $|R,p_n|_k$, $k \geqslant 1$, are*

$$\varepsilon_n = O\left\{\left(\frac{np_n}{P_n}\right)\left(\frac{Q_n}{nq_n}\right)^{1/k}\right\},\quad \Delta\varepsilon_n = O\left\{\left(\frac{p_n}{P_{n-1}}\right)\left(\frac{nq_n}{Q_n}\right)^{1-1/k}\right\},$$

provided that

$$\sum_{n=1}^{m}\left|\Delta\left|\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}\frac{P_{v-1}Q_v}{vp_v}\left(\frac{vq_v}{Q_v}\right)^{1/k}\Delta\varepsilon_n\right|^k\right| < \infty.$$

Theorem 5. *Let $p_nQ_n = O(P_nq_n)$ and (3) is satisfied. Then the necessary and sufficient conditions that $\Sigma a_n\varepsilon_n$* is summable *$|R,q_n|_k$ whenever Σa_n is summable $|\bar{N},p_n|_k$, $k \geqslant 1$, are*

$$\varepsilon_n = O\left\{\left(\frac{Q_n}{nq_n}\right)\left(\frac{np_n}{P_n}\right)^{1/k}\right\},\quad \Delta\varepsilon_n = O\left\{\left(\frac{p_n}{P_{n-1}}\right)\left(\frac{P_n}{np_n}\right)^{1-1/k}\right\},$$

provided that

$$\sum_{n=1}^{m}\left|\Delta\left|\frac{1}{Q_{n-1}}\sum_{v=1}^{n-1}\left(\frac{vq_vP_{v-1}}{P_v}\right)\left(\frac{P_v}{vp_v}\right)^{1/k}\right|^k\right| < \infty.$$

Theorem 6. (Bor and Thorpe [1]). *Let* $p_n Q_n = O(P_n q_n)$ *and* $P_n q_n = O(p_n Q_n)$. *Then* Σa_n *is summable* $|\bar{N}, q_n|_k$ *iff it is summable* $|\bar{N}, p_n|_k$, $k \geqslant 1$.
The proof follows from theorem 2 by putting $\varepsilon_n = 1$.

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