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Degree of approximation of functions in the Hölder metric by Borel's means

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Abstract. After establishing the Fourier character of the series the authors have studied the degree of approximation of functions associated with the same series in the Hölder metric using Borel's mean.

Keywords. Fourier character of the series; Banach space; Hölder metric; Borel's mean.

1. Definitions

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic continuous functions defined on $[-\pi, \pi]$ under sup-norm.

For $0 < \alpha \leqslant 1$ and some positive constant K, the function space H_α is given by

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leqslant K|x-y|^\alpha\}. \tag{1.1}$$

The space H_α is a Banach space [6] with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{x,y}\{\Delta^\alpha f(x,y)\}, \tag{1.2}$$

where

$$\|f\|_c = \sup_{-\pi \leqslant x \leqslant \pi} |f(x)|$$

and

$$\Delta^\alpha f(x,y) = |f(x) - f(y)\|x-y|^{-\alpha}, \quad (x \neq y). \tag{1.3}$$

We shall use the convention that $\Delta^0 f(x,y) = 0$. The metric induced by the norm (1.2) on H_α is called Hölder metric.

Let f be a periodic function of period 2π and integrable in the Lebesgue sense over $[-\pi, \pi]$.

Let the Fourier series associated with f at x be

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x). \tag{1.4}$$

Let us write

$$\Phi_x(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}, \tag{1.5}$$

$$S_n(x) = \sum_{k=0}^{n} A_k(x), \tag{1.6}$$

$$\sigma_n(x) = \frac{1}{n} \sum_{k=0}^{n} S_k(x) \quad (n > 0). \tag{1.7}$$

Given any sequence $\{t_n(x)\}$, its Borel's exponential mean $B_p(t, x)$ is defined by ([2], p. 182)

$$B_p(t, x) = \mathrm{e}^{-p} \sum_{n=0}^{\infty} t_n(x) \frac{p^n}{n!} \quad (p > 0). \tag{1.8}$$

2. Introduction

The convergence and $(C, 1)$ summability problem of the series

$$\sum_{n=1}^{\infty} \frac{S_n(x) - f(x)}{n} \tag{2.1}$$

have been studied by Hardy and Littlewood [3] (also see [7], p. 125). In a recent work Das, Ojha and Ray [1] have investigated the degree of approximation of (2.1) in the Hölder metric by Borel's mean.

In a recent paper Mohanty and Ray [4] have studied the convergence and absolute convergence of the series

$$\sum_{n=1}^{\infty} \frac{\sigma_n(x) - f(x)}{n}. \tag{2.2}$$

As regards convergence they have shown that if

$$\int_0^t \Phi_x(u)\,\mathrm{d}u = o(t),$$

then the series (2.2) converges if and only if

$$\int_{\to 0+}^{\pi} \frac{\Phi_x(u)}{u}\,\mathrm{d}u$$

exists.

3. Main results

Premchandra [5] has studied the degree of approximation problems for Fourier series by Borel's means. The purpose of the present paper is to investigate the degree of approximation problem of the series (2.2) in Hölder metric by Borel's mean. Before we state the main theorem we first proceed to exhibit the Fourier character of the series (2.2).

Fourier character of the series (2.2)

We write

$$\theta_x(u) = \int_u^{\pi} \frac{\Phi_x(v)}{(2 \sin v/2)^2}\,\mathrm{d}v, \tag{3.1}$$

$$\Theta_x(t) = -\int_0^t \theta_x(u)\,\mathrm{d}u. \tag{3.2}$$

For every $f \in \mathrm{Lip}\,\alpha, 0 < \alpha \leqslant 1, \theta \in L(0,\pi)$ and $\Theta_x(t)$ is even, let

$$\Theta_x(t) \sim \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos nt, \tag{3.3}$$

where

$$\begin{aligned} c_0 = c_0(x) &= \frac{2}{\pi}\int_0^\pi \Theta_x(t)\,\mathrm{d}t = \frac{2}{\pi}\int_0^\pi \left(-\int_0^t \theta_x(u)\,\mathrm{d}u\right)\mathrm{d}t \\ &= -\frac{2}{\pi}\int_0^\pi \mathrm{d}t\left(\int_0^t \theta_x(u)\,\mathrm{d}u\right) = -\frac{2}{\pi}\int_0^\pi \theta_x(u)\,\mathrm{d}u\int_u^\pi \mathrm{d}t \\ &= -\frac{2}{\pi}\int_0^\pi (\pi-u)\,\theta_x(u)\,\mathrm{d}u = -\frac{2}{\pi}\int_0^\pi (\pi-u)\,\mathrm{d}u\int_u^\pi \frac{\Phi_x(v)}{(2\sin v/2)^2}\mathrm{d}v \\ &= -\frac{2}{\pi}\int_0^\pi \frac{\Phi_x(v)}{(2\sin v/2)^2}\mathrm{d}v\int_0^v (\pi-u)\,\mathrm{d}u = -\frac{2}{\pi}\int_0^\pi \frac{(\pi v - v^2/2)}{(2\sin v/2)^2}\Phi_x(v)\,\mathrm{d}v \end{aligned} \tag{3.4}$$

and for $n \geqslant 1$

$$\begin{aligned} c_n = c_n(x) &= \frac{2}{\pi}\int_0^\pi \Theta_x(t)\cos nt\,\mathrm{d}t \\ &= \frac{2}{\pi}\left[\Theta_x(t)\frac{\sin nt}{n}\right]_0^\pi + \frac{2}{\pi}\int_0^\pi \theta_x(t)\frac{\sin nt}{n}\mathrm{d}t \\ &= \frac{2}{\pi n}\int_0^\pi \theta_x(t)\sin nt\,\mathrm{d}t \\ &= \frac{2}{\pi n}\left[\theta_x(t)\left(\frac{1-\cos nt}{n}\right)\right]_0^\pi + \frac{2}{n\pi}\int_0^\pi \frac{\Phi_x(t)}{(2\sin t/2)^2}\left(\frac{1-\cos nt}{n}\right)\mathrm{d}t \\ &= \frac{2}{\pi n^2}\int_0^\pi \Phi_x(t)\frac{2\sin^2 nt/2}{(2\sin t/2)^2}\mathrm{d}t \\ &= \frac{1}{\pi n^2}\int_0^\pi \Phi_x(t)\left(\frac{\sin nt/2}{\sin t/2}\right)^2\mathrm{d}t = \frac{\sigma_n(x) - f(x)}{n}. \end{aligned} \tag{3.5}$$

Theorem. *Let* $0 \leqslant \beta < \alpha \leqslant 1$ *and* $f \in H_\alpha$. *Then*

$$\| B_p(T) + \tfrac{1}{2}c_0 \|_\beta = O(1)\begin{cases} \dfrac{1}{p^{\alpha-\beta}}, & \alpha - \beta \neq 1 \\ \dfrac{\log p}{p}, & \alpha - \beta = 1 \ (\text{i.e. } \beta = 0, \alpha = 1) \end{cases}, \tag{3.6}$$

where

$$B_p(T) = B_p(T,x) = \mathrm{e}^{-p}\sum_{k=0}^{\infty}\frac{p^k}{k!}T_k(x)$$

$$T_0(x) = 0,\ T_n(x) = \sum_{k=1}^{n}\frac{\sigma_k(x) - f(x)}{k}, n \geqslant 1$$

and c_0 *is defined by* (3.4)

Additional notations

Throughout the present paper we suppose that $0<\delta<\pi/4$ and

$$G(t)=\Theta_x(t)-\Theta_y(t), \tag{3.7}$$

$$Q_p=Q_p(t)=\sin(\tfrac{1}{2}t+p\sin t), \tag{3.8}$$

$$H_p(t)=e^{-p}\sum_{n=0}^{\infty}\frac{p^n}{n!}\sin(n+\tfrac{1}{2})t, \tag{3.9}$$

$$l_p(x)=B_p(T,x)+\tfrac{1}{2}c_0, \tag{3.10}$$

$$\eta=\eta(p)=\pi/p, \tag{3.11}$$

$$P=P(p)=\frac{\pi\log p}{\sqrt{p}}. \tag{3.12}$$

By formal computation we see that

$$\begin{aligned}H_p(t)&=\mathrm{Im}\left\{e^{-p}\sum_{n=0}^{\infty}\frac{p^n}{n!}e^{i(n+\frac{1}{2})t}\right\}\\&=e^{-p(1-\cos t)}\sin(\tfrac{1}{2}t+p\sin t)\\&=e^{-p(1-\cos t)}Q_p(t).\end{aligned} \tag{3.13}$$

4. Lemmas

We use the following lemmas:

Lemma 1. *If* $f\in H_\alpha$, $0<\alpha\leqslant 1$ *and* $0\leqslant\beta<\alpha<1$, *then*

$$\Phi_x(v)-\Phi_y(v)=O(|v|^\alpha), \tag{4.1}$$

$$\Phi_x(v)-\Phi_y(v)=O(|x-y|^\alpha), \tag{4.2}$$

$$\Phi_x(v)-\Phi_y(v)=O(1)|x-y|^\beta|v|^{\alpha-\beta}. \tag{4.3}$$

Proof. Estimates (4.1) and (4.2) follow immediately from the definition of $\Phi_x(v)$ and H_α.
Using (4.2) and (4.1), we get

$$\begin{aligned}\Phi_x(v)-\Phi_y(v)&=[\Phi_x(v)-\Phi_y(v)]^{\beta/\alpha}[\Phi_x(v)-\Phi_y(v)]^{1-\beta/\alpha}\\&=O(1)|x-y|^\beta|v|^{\alpha-\beta}.\end{aligned}$$

Lemma 2. *Let* $0<\delta<\pi/4$. *Then for* $0\leqslant t\leqslant\delta$

$$\sin(p\sin t)-\sin pt=O(pt^3). \tag{4.4}$$

Proof. We have

$$\sin(p\sin t) - \sin pt = 2\cos\left(\frac{p\sin t + pt}{2}\right)\sin\left(\frac{p\sin t - pt}{2}\right)$$
$$= O(1)\left|\sin\frac{1}{2}p(\sin t - t)\right| = O(pt^3).$$

Lemma 3. *Let* $0 < \delta < \pi/4$. *Then for* $c = 2/\pi^2$ *and* $\eta = \pi/p$

$$\text{(i) } e^{-p(1-\cos t)} = O(e^{-pct^2}) \tag{4.5}$$

$$\text{(ii) } e^{-p(1-\cos t)} - e^{-p(1-\cos(t+\eta))} = O(p e^{-pct^2}). \tag{4.6}$$

Proof. (i) Since

$$1 - \cos t \geqslant \frac{2}{\pi^2}t^2 = ct^2 \text{ for all } 0 \leqslant t \leqslant \pi,$$

the estimate (4.5) follows at once. (ii) We have

$$\exp\{-p(1-\cos t)\} - \exp\{-p(1-\cos(t+\eta))\}$$
$$= \exp\{-p(1-\cos t)\}[1 - \exp\{p(\cos(t+\eta) - \cos t)\}]$$
$$= O(1)p\exp\{-p(1-\cos t)\}|\cos(t+\eta) - \cos t|$$
$$= O(pe^{-pct^2}) \quad \text{by (4.5)}.$$

Lemma 4. *If* $f \in H_\alpha$, $0 \leqslant \beta < \alpha \leqslant 1$ *and* $0 < t \leqslant \pi$, *then*

$$\Theta_x(t) = O(1)\begin{cases} t^\alpha, 0 < \alpha < 1 \\ t|\log t|, \alpha = 1 \end{cases}, \tag{4.7}$$

$$G(t) = O(1)\begin{cases} |x-y|^\beta t^{\alpha-\beta}, \alpha - \beta \neq 1 \\ t|\log t|, \alpha - \beta = 1 \text{ (i.e. } \beta = 0, \alpha = 1) \end{cases}, \tag{4.8}$$

$$G(t+\eta) - G(t) = O(1)\begin{cases} |x-y|^\beta \eta t^{\alpha-\beta-1}, \alpha - \beta \neq 1 \\ \eta|\log t|, \alpha - \beta = 1 \text{ (i.e. } \beta = 0, \alpha = 1) \end{cases} \tag{4.9}$$

Proof of (4.7). Proof follows at once from the definition of $\Theta_x(t)$.

Proof of (4.8). Since $f \in H_\alpha \Rightarrow \Phi_x(v) = O(|v|^\alpha)$ from (3.1) and (3.2), we get

$$\Theta_x(t) = -\int_0^t du\left(\int_u^\pi \frac{\Phi_x(v)}{(2\sin v/2)^2}du\right)$$
$$= O(1)|x-y|^\beta \int_0^t du \int_u^\pi v^{\alpha-\beta-2}dv$$
$$= O(1)\begin{cases} |x-y|^\beta \int_0^t u^{\alpha-\beta-1}du, \quad \alpha-\beta \neq 1 \\ \int_0^t |\log u|\,du, \ \alpha - \beta = 1 \text{ (i.e. } \beta = 0, \alpha = 1) \end{cases}$$

$$= O(1)\begin{cases} |x-y|^{\beta}\, t^{\alpha-\beta}, & \alpha-\beta \neq 1 \\ t\,|\log t|, & \alpha-\beta = 1 \text{ (i.e. } \beta = 0, \alpha = 1) \end{cases}.$$

Proof of (4.9). Using Lemma 1, we get

$$G'(t) = -\int_t^{\pi} \frac{\Phi_x(v) - \Phi_y(v)}{(2\sin v/2)^2}\,\mathrm{d}v$$

$$= O(1)|x-y|^{\beta}\int_t^{\pi} v^{\alpha-\beta-2}\,\mathrm{d}v$$

$$= O(1)\begin{cases} |x-y|^{\beta}\, t^{\alpha-\beta-1}, & \alpha-\beta \neq 1 \\ |\log t|, & \alpha-\beta = 1 \text{ (i.e. } \beta = 0, \alpha = 1) \end{cases}.$$

By mean value theorem for some t_1 with $t < t_1 < t + \eta$

$$G(t+\eta) - G(t) = \eta\, G'(t_1)$$

$$= O(1)\eta\begin{cases} |x-y|^{\beta}\, t^{\alpha-\beta-1}, & \alpha-\beta \neq 1 \\ |\log t|, & \alpha-\beta = 1 \text{ (i.e. } \beta = 0, \alpha = 1) \end{cases}.$$

5. Proof of the theorem

Using (3.5) and writing

$$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin t/2}$$

we have

$$T_n(x) = \sum_{k=1}^{n} \frac{\sigma_k(x) - f(x)}{k} = \tfrac{1}{2}c_0 + \sum_{k=1}^{n} c_k - \tfrac{1}{2}c_0$$

$$= \frac{2}{\pi}\int_0^{\pi} \Theta_x(t)\, D_n(t)\,\mathrm{d}t - \tfrac{1}{2}c_0$$

and hence

$$B_p(T,x) = \mathrm{e}^{-p}\sum_{n=0}^{\infty} \frac{p^n}{n!} T_n(x)$$

$$= \mathrm{e}^{-p}\sum_{n=0}^{\infty} \left\{\frac{2}{\pi}\int_0^{\pi} \Theta_x(t)\, D_n(t)\,\mathrm{d}t - \tfrac{1}{2}c_0\right\}\frac{p^n}{n!}$$

$$= \frac{2}{\pi}\mathrm{e}^{-p}\int_0^{\pi} \Theta_x(t)\left(\sum_{n=0}^{\infty} \frac{p^n}{n!} D_n(t)\right)\mathrm{d}t - \tfrac{1}{2}c_0$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\Theta_x(t)}{2\sin t/2}\left\{\mathrm{e}^{-p}\sum_{n=0}^{\infty} \frac{p^n}{n!}\sin\left(n+\frac{1}{2}\right)t\right\}\mathrm{d}t - \tfrac{1}{2}c_0$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\Theta_x(t)}{2\sin t/2} H_p(t)\,\mathrm{d}t - \tfrac{1}{2}c_0$$

from which it follows that

$$I_p(x) = B_p(T, x) + \frac{1}{2}c_0 = \frac{2}{\pi}\int_0^{\pi} \frac{\Theta_x(t)}{2\sin t/2} H_p(t)\,dt. \tag{5.1}$$

Now by (5.1), (3.7) and (3.13) we get

$$\begin{aligned} I_p(x) - I_p(y) &= \frac{2}{\pi}\int_0^{\pi} \frac{\Theta_x(t) - \Theta_y(t)}{2\sin t/2} H_p(t)\,dt \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{G(t)}{2\sin t/2} e^{-p(1-\cos t)} \sin\left(\frac{1}{2}t + p\sin t\right) dt \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{G(t)}{2\sin t/2} e^{-p(1-\cos t)} Q_p(t)\,dt \\ &= \frac{2}{\pi}\left[\int_0^{\eta} + \int_{\eta}^{\delta} + \int_{\delta}^{\pi}\right] \frac{G(t)e^{-p(1-\cos t)}}{2\sin t/2} Q_p(t)\,dt \\ &= I_1 + I_2 + I_3, \quad \text{say}. \end{aligned} \tag{5.2}$$

For the proof of our theorem we are required to estimate

(i) $\sup_{x,y} \Delta^{\beta} l_p(x, y) = \sup_{x,y} \frac{|l_p(x) - l_p(y)|}{|x-y|^{\beta}}, (x \neq y)$ and

(ii) $\|l_p\|_c$.

In what follows, we first consider $\sup_{x,y} \Delta^{\beta} l_p(x, y)$ in the case when

$$\alpha - \beta \neq 1 \ (\text{i.e.} \quad 0 < \beta < \alpha \leqslant 1 \text{ or } \beta = 0, \quad 0 < \alpha < 1).$$

By Lemma 4

$$\begin{aligned} I_1 &= \frac{2}{\pi}\int_0^{\eta} \frac{G(t)e^{-p(1-\cos t)}}{2\sin t/2} Q_p(t)\,dt \\ &= O(1)|x-y|^{\beta}\int_0^{\eta} t^{\alpha-\beta}(1+p)\,dt \\ &= O(1)\frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, \quad (\alpha - \beta \neq 1). \end{aligned} \tag{5.3}$$

By Lemmas 3 and 4

$$\begin{aligned} I_3 &= \frac{2}{\pi}\int_{\delta}^{\pi} \frac{G(t)}{2\sin t/2} e^{-p(1-\cos t)} Q_p(t)\,dt \\ &= O(1)|x-y|^{\beta}\int_{\delta}^{\pi} t^{\alpha-\beta-1} e^{-pct^2}\,dt, (\alpha - \beta \neq 1) \\ &= O(1)\,|x-y|^{\beta}\, e^{-pc\delta^2}\int_{\delta}^{\pi} t^{\alpha-\beta-1}\,dt \\ &= O(1)\frac{|x-y|^{\beta}}{p^{\Delta}}, \text{positive } \Delta \text{ however large.} \end{aligned} \tag{5.4}$$

Now we write

$$\pi I_2 = \int_\eta^\delta \frac{G(t)}{\sin t/2} e^{-p(1-\cos t)} \sin\left(\frac{1}{2}t + p\sin t\right) dt$$

$$= \int_\eta^\delta G(t) e^{-p(1-\cos t)} \cos(p\sin t)\, dt$$

$$+ \int_\eta^\delta G(t)\cot(t/2)\, e^{-p(1-\cos t)} \sin(p\sin t)\, dt$$

$$= I_{2,1} + I_{2,2}, \quad \text{say.} \tag{5.5}$$

Using Lemmas 3 and 4 we have for $\alpha - \beta \neq 1$

$$I_{2,1} = \int_\eta^\delta G(t) e^{-p(1-\cos t)} \cos(p\sin t)\, dt$$

$$= O(1)|x-y|^\beta \int_\eta^\delta t^{\alpha-\beta} e^{-pct^2} dt$$

$$= O(1)\frac{|x-y|^\beta}{p} \int_\eta^\delta t^{\alpha-\beta-1} \frac{d}{dt}(-e^{-pct^2})\, dt$$

$$= O(1)|x-y|^\beta p^{-1} |[t^{\alpha-\beta-1} e^{-pct^2}]_\delta^\eta|$$

$$+ O(1)|x-y|^\beta p^{-1} \int_\eta^\delta t^{\alpha-\beta-2} e^{-pct^2} dt$$

$$= O(1)\frac{|x-y|^\beta}{p^{\alpha-\beta}} + O(1)\frac{|x-y|^\beta}{p} \int_\eta^\delta t^{\alpha-\beta-2} dt$$

$$= O(1)\frac{|x-y|^\beta}{p^{\alpha-\beta}} \qquad (\alpha - \beta \neq 1). \tag{5.6}$$

We write

$$I_{2,2} = 2\int_\eta^\delta \frac{G(t)}{2\tan t/2} e^{-p(1-\cos t)} \sin(p\sin t)\, dt$$

$$= 2\int_\eta^\delta \frac{G(t)}{t} e^{-p(1-\cos t)} \sin(p\sin t)\, dt$$

$$+ 2\int_\eta^\delta G(t)\left\{\frac{1}{2\tan(t/2)} - \frac{1}{t}\right\} e^{-p(1-\cos t)} \sin(p\sin t)\, dt$$

$$= J_1 + J_2, \quad \text{say.} \tag{5.7}$$

By Lemmas 3 and 4 we have for $\alpha - \beta \neq 1$

$$J_2 = O(1)|x-y|^\beta \int_\eta^\delta t^{\alpha-\beta+1} e^{-pct^2} dt$$

$$= O(1)\frac{|x-y|^\beta}{p} \int_\eta^\delta t^{\alpha-\beta} \frac{d}{dt}(-e^{-pct^2})\, dt$$

$$= O(1)\frac{|x-y|^{\beta}}{p}\left\{|[t^{\alpha-\beta}\mathrm{e}^{-pct^2}]_{\delta}^{\eta}| + (\alpha-\beta)\int_{\eta}^{\delta} t^{\alpha-\beta-1}\mathrm{e}^{-pct^2}\,\mathrm{d}t\right\}$$

$$= O(1)\frac{|x-y|^{\beta}}{p^{1+\alpha-\beta}} + O(1)\frac{|x-y|^{\beta}}{p}\int_{\eta}^{\delta} t^{\alpha-\beta-1}\,\mathrm{d}t$$

$$= O(1)\frac{|x-y|^{\beta}}{p}. \tag{5.8}$$

Now we write

$$J_1 = 2\int_{\eta}^{\delta}\frac{G(t)}{t}\,\mathrm{e}^{-p(1-\cos t)}\sin(p\sin t)\,\mathrm{d}t$$

$$= 2\int_{\eta}^{\delta}\frac{G(t)}{t}\,\mathrm{e}^{-p(1-\cos t)}\sin pt\,\mathrm{d}t$$

$$+ 2\int_{\eta}^{\delta}\frac{G(t)}{t}\,\mathrm{e}^{-p(1-\cos t)}\{\sin(p\sin t) - \sin pt\}\mathrm{d}t$$

$$= J_{1,1} + J_{1,2}, \qquad \text{say}. \tag{5.9}$$

By Lemmas 2, 3 and 4 we have for $\alpha - \beta \neq 1$

$$J_{1,2} = 2\int_{\eta}^{\delta}\frac{G(t)}{t}\,\mathrm{e}^{-p(1-\cos t)}\{\sin(p\sin t) - \sin pt\}\mathrm{d}t$$

$$= O(1)|x-y|^{\beta}\int_{\eta}^{\delta} t^{\alpha-\beta-1}\,pt^3\,\mathrm{e}^{-pct^2}\,\mathrm{d}t$$

$$= O(1)|x-y|^{\beta}\int_{\eta}^{\delta} t^{\alpha-\beta+1}\frac{\mathrm{d}}{\mathrm{d}t}(-\mathrm{e}^{-pct^2})\,\mathrm{d}t$$

$$= O(1)\,|x-y|^{\beta}\,|[t^{\alpha-\beta+1}\,\mathrm{e}^{-pct^2}]_{\delta}^{\eta}|$$

$$+ O(1)|x-y|^{\beta}\int_{\eta}^{\delta} t^{\alpha-\beta}\mathrm{e}^{-pct^2}\,\mathrm{d}t$$

$$= O(1)\frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, \tag{5.10}$$

using (5.6) for the estimation of the last integral.

Now we write

$$J_{1,1} = 2\int_{\eta}^{\delta}\frac{G(t)}{t}\mathrm{e}^{-p(1-\cos t)}\sin pt\,\mathrm{d}t$$

$$= \left[\int_{\eta}^{\delta} + \int_{2\eta}^{\delta+\eta} + \int_{\eta}^{2\eta} - \int_{\delta}^{\delta+\eta}\right]\frac{G(t)}{t}\mathrm{e}^{-p(1-\cos t)}\sin pt\,\mathrm{d}t$$

$$= \int_{\eta}^{\delta}\left[\frac{G(t)}{t}\mathrm{e}^{-p(1-\cos t)} - \frac{G(t+\eta)}{t+\eta}\,\mathrm{e}^{-p(1-\cos(t+\eta))}\right]\sin pt\,\mathrm{d}t$$

$$+ \left[\int_{\eta}^{2\eta} - \int_{\delta}^{\delta+\eta}\right]\frac{G(t)}{t}\mathrm{e}^{-p(1-\cos t)}\sin pt\,\mathrm{d}t$$

$$= \int_\eta^\delta \left\{ \frac{G(t) - G(t+\eta)}{t} \right\} e^{-p(1-\cos t)} \sin pt \, dt$$

$$+ \int_\eta^\delta G(t+\eta) \left\{ \frac{1}{t} - \frac{1}{t+\eta} \right\} e^{-p(1-\cos t)} \sin pt \, dt$$

$$+ \int_\eta^\delta \frac{G(t+\eta)}{t+\eta} \{ e^{-p(1-\cos t)} - e^{-p(1-\cos(t+\eta))} \} \sin pt \, dt$$

$$+ \left(\int_\eta^{2\eta} - \int_\delta^{\delta+\eta} \right) \frac{G(t)}{t} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= R_1 + R_2 + R_3 + R_4 - R_5, \quad \text{say}. \tag{5.11}$$

We write

$$R_1 = \left(\int_\eta^P + \int_P^\delta \right) \left\{ \frac{G(t) - G(t+\eta)}{t} \right\} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= R_{1,1} + R_{1,2}, \quad \text{say}. \tag{5.12}$$

Using (4.9) of Lemma 4, we get

$$R_{1,1} = \int_\eta^P \frac{G(t) - G(t+\eta)}{t} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= O(1) |x-y|^\beta \eta \int_\eta^P t^{\alpha-\beta-2} \, dt$$

$$= O(1) \frac{|x-y|^\beta}{p^{\alpha-\beta}} (\alpha - \beta \neq 1). \tag{5.13}$$

Again using (4.9) and Lemma 3

$$R_{1,2} = \int_P^\delta \frac{G(t) - G(t+\eta)}{t} e^{-p(1-\cos t)} \sin pt \, dt$$

$$= O(1) |x-y|^\beta \eta \int_P^\delta t^{\alpha-\beta-2} \, e^{-pct^2} dt$$

$$= O(1) \frac{|x-y|^\beta}{p} e^{-pc\delta^2} \int_P^\delta t^{\alpha-\beta-2} \, dt$$

$$= O(1) \frac{|x-y|^\beta}{p^\Delta}, \text{ positive } \Delta \text{ however large}. \tag{5.14}$$

From (5.12), (5.13) and (5.14), we have

$$R_1 = O(1) \frac{|x-y|^\beta}{p^{\alpha-\beta}}, \quad (\alpha - \beta \neq 1). \tag{5.15}$$

Using Lemmas 3 and 4 and adopting the arguments similar to those used in estimating $I_{2,1}$, we get

$$R_3 = \int_\eta^\delta \frac{G(t+\eta)}{t+\eta}\{\mathrm{e}^{-p(1-\cos t)} - \mathrm{e}^{-p(1-\cos(t+\eta))}\}\sin pt\,\mathrm{d}t$$

$$= O(1)|x-y|^\beta \int_\eta^\delta t^{\alpha-\beta}\mathrm{e}^{-pct^2}\mathrm{d}t$$

$$= O(1)\frac{|x-y|^\beta}{p^{\alpha-\beta}}, \quad (\alpha-\beta \neq 1). \tag{5.16}$$

By Lemma 4, we have

$$R_4 = \int_\eta^{2\eta} \frac{G(t)}{t}\mathrm{e}^{-p(1-\cos t)}\sin pt\,\mathrm{d}t$$

$$= O(1)|x-y|^\beta \int_\eta^{2\eta} t^{\alpha-\beta-1}\mathrm{d}t$$

$$= O(1)\frac{|x-y|^\beta}{p^{\alpha-\beta}}, \quad (\alpha-\beta \neq 1). \tag{5.17}$$

Again by Lemmas 3 and 4

$$R_5 = \int_\delta^{\delta+\eta} \frac{G(t)}{t}\mathrm{e}^{-p(1-\cos t)}\sin pt\,\mathrm{d}t$$

$$= O(1)|x-y|^\beta \int_\delta^{\delta+\eta} t^{\alpha-\beta-1}\,\mathrm{e}^{-pct^2}\mathrm{d}t, \quad (\alpha-\beta\neq 1)$$

$$= O(1)|x-y|^\beta\,\mathrm{e}^{-pc\delta^2}\int_\delta^{\delta+\eta} t^{\alpha-\beta-1}\,\mathrm{d}t$$

$$= O(1)\frac{|x-y|^\beta}{p^\Delta}, \quad \text{positive } \Delta \text{ however large}. \tag{5.18}$$

Now we write

$$2R_2 = 2\int_\eta^\delta G(t+\eta)\left\{\frac{1}{t} - \frac{1}{t+\eta}\right\}\mathrm{e}^{-p(1-\cos t)}\sin pt\,\mathrm{d}t$$

$$= 2\eta\int_\eta^\delta \frac{G(t+\eta)}{t(t+\eta)}\mathrm{e}^{-p(1-\cos t)}\sin pt\,\mathrm{d}t$$

$$= \eta\left[\int_\eta^\delta + \int_{2\eta}^{\delta+\eta} + \int_\eta^{2\eta} - \int_\delta^{\delta+\eta}\right]\frac{G(t+\eta)}{t(t+\eta)}\mathrm{e}^{-p(1-\cos t)}\sin pt\,\mathrm{d}t$$

$$= \eta\int_\eta^\delta\left\{\frac{G(t+\eta)}{t(t+\eta)}\mathrm{e}^{-p(1-\cos t)} - \frac{G(t+\eta)}{(t+\eta)(t+2\eta)}\mathrm{e}^{-p(1-\cos(t+\eta))}\right\}\sin pt\,\mathrm{d}t$$

$$+\eta\left[\int_\eta^{2\eta} - \int_\delta^{\delta+\eta}\right]\frac{G(t+\eta)}{t(t+\eta)}\mathrm{e}^{-p(1-\cos t)}\sin pt\,\mathrm{d}t$$

$$= \eta \int_{\eta}^{\delta} \frac{G(t+\eta) - G(t+2\eta)}{t(t+\eta)} \mathrm{e}^{-p(1-\cos t)} \sin pt \, \mathrm{d}t$$

$$+ \eta \int_{\eta}^{\delta} \frac{G(t+2\eta)}{t+\eta} \left\{ \frac{1}{t} - \frac{1}{t+2\eta} \right\} \mathrm{e}^{-p(1-\cos t)} \sin pt \, \mathrm{d}t$$

$$+ \eta \int_{\eta}^{\delta} \frac{G(t+2\eta)}{(t+\eta)(t+2\eta)} \{ \mathrm{e}^{-p(1-\cos t)} - \mathrm{e}^{-p(1-\cos(t+\eta))} \} \sin pt \, \mathrm{d}t$$

$$+ \eta \left\{ \int_{\eta}^{2\eta} - \int_{\delta}^{\delta+\eta} \right\} \frac{G(t+\eta)}{t(t+\eta)} \mathrm{e}^{-p(1-\cos t)} \sin pt \, \mathrm{d}t$$

$$= L_1 + L_2 + L_3 + L_4 - L_5, \text{ say}. \tag{5.19}$$

By (4.9) of Lemma 4, we get

$$L_1 = \eta \int_{\eta}^{\delta} \frac{G(t+\eta) - G(t+2\eta)}{t(t+\eta)} \mathrm{e}^{-p(1-\cos t)} \sin pt \, \mathrm{d}t$$

$$= O(1) |x-y|^{\beta} \eta^2 \int_{\eta}^{\delta} t^{\alpha-\beta-3} \, \mathrm{d}t$$

$$= O(1) \frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, \quad (\alpha - \beta \neq 1). \tag{5.20}$$

By Lemma 4 ($\alpha - \beta \neq 1$), we have

$$L_2 = \eta \int_{\eta}^{\delta} \frac{G(t+\eta)}{t+\eta} \left\{ \frac{1}{t} - \frac{1}{t+2\eta} \right\} \mathrm{e}^{-p(1-\cos t)} \sin pt \, \mathrm{d}t$$

$$= O(1) |x-y|^{\beta} \eta^2 \int_{\eta}^{\delta} t^{\alpha-\beta-3} \mathrm{d}t$$

$$= O(1) \frac{|x-y|^{\beta}}{p^{\alpha-\beta}}. \tag{5.21}$$

Usinig Lemmas 3, 4 and integrating by parts, we have ($\alpha - \beta \neq 1$)

$$L_3 = \eta \int_{\eta}^{\delta} \frac{G(t+2\eta)}{(t+\eta)(t+2\eta)} \{ \mathrm{e}^{-p(1-\cos t)} - \mathrm{e}^{-p(1-\cos(t+\eta))} \} \sin pt \, \mathrm{d}t$$

$$= O(1) |x-y|^{\beta} \eta \int_{\eta}^{\delta} t^{\alpha-\beta-2} (t \mathrm{e}^{-pct^2}) \mathrm{d}t$$

$$= O(1) |x-y|^{\beta} \eta^2 \int_{\eta}^{\delta} t^{\alpha-\beta-2} \frac{\mathrm{d}}{\mathrm{d}t} (-\mathrm{e}^{-pct^2}) \mathrm{d}t$$

$$= O(1) \frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, \quad (\alpha - \beta \neq 1). \tag{5.22}$$

By Lemma 4, ($\alpha - \beta \neq 1$) we have

$$L_4 = \eta \int_{\eta}^{2\eta} \frac{G(t+\eta)}{t(t+\eta)} \mathrm{e}^{-p(1-\cos t)} \sin pt \, \mathrm{d}t$$

$$= O(1)|x-y|^{\beta}\eta\int_{\eta}^{2\eta} t^{\alpha-\beta-2}\,dt$$

$$= O(1)\frac{|x-y|^{\beta}}{p^{\alpha-\beta}}, (\alpha-\beta\neq 1). \tag{5.23}$$

Lastly by Lemma 4 $(\alpha-\beta\neq 1)$, we get

$$L_5 = \eta\int_{\delta}^{\delta+\eta}\frac{G(t+\eta)}{t(t+\eta)}e^{-p(1-\cos t)}\sin pt\,dt$$

$$= O(1)|x-y|^{\beta}\eta\int_{\delta}^{\delta+\eta} t^{\alpha-\beta-2}\,dt$$

$$= O(1)\frac{|x-y|^{\beta}}{p},\quad (\alpha-\beta\neq 1). \tag{5.24}$$

Collecting the results from (5.19) to (5.24), we get

$$R_2 = O(1)\frac{|x-y|^{\beta}}{p^{\alpha-\beta}},\quad (\alpha-\beta\neq 1). \tag{5.25}$$

From (5.11), (5.15), (5.16), (5.17), (5.18) and (5.25) we get

$$J_{1,1} = O(1)\frac{|x-y|^{\beta}}{p^{\alpha-\beta}},\quad (\alpha-\beta\neq 1). \tag{5.26}$$

Collecting the results from (5.5) to (5.10) and using (5.26) we get

$$I_2 = O(1)\frac{|x-y|^{\beta}}{p^{\alpha-\beta}},\quad (\alpha-\beta\neq 1). \tag{5.27}$$

Using the estimates of $I_k\,(k=1,2,3)$ in (5.2) (note that estimate of I_2 dominates over that of I_1 and I_3), we obtain for $\alpha-\beta\neq 1$

$$\sup_{\substack{x,y\\ x\neq y}}(\Delta^{\beta} l_p(x,y)) = \sup_{x,y}\frac{|l_p(x)-l_p(y)|}{|x-y|^{\beta}} = O(1)\frac{1}{p^{\alpha-\beta}}. \tag{5.28}$$

In the case where $\alpha-\beta=1$, (i.e., $\beta=0$, $\alpha=1$) proceeding in the lines similar to those used above in the case $\alpha-\beta\neq 1$, we can prove that

$$\sup_{\substack{x,y\\ x\neq y}}(\Delta^{\beta} l_p(x,y)) = O(1)\frac{\log p}{p},\ (\alpha-\beta=1). \tag{5.29}$$

As $f\in H_{\alpha}\Rightarrow \Phi_x(t)=O(|t|^{\alpha})$ and so proceeding as above (in fact cases $\beta=0$ of (5.28) and (5.29)), we get

$$\|l_p(x)\|_c = O(1)\begin{cases}\dfrac{1}{p^{\alpha}}, & 0<\alpha<1\\[2ex] \dfrac{\log p}{p}, & \alpha=1\end{cases} \tag{5.30}$$

From (5.28), (5.29) and (5.30), we have

$$\|l_p(x)\|_\beta = \|l_p(x)\|_c + \sup_{\substack{x,y \\ x\neq y}} (\Delta^\beta l_p(x,y))$$

$$= O(1)\begin{cases} \dfrac{1}{p^{\alpha-\beta}}, & \alpha-\beta\neq 1 \\ \dfrac{\log p}{p}, & \alpha-\beta\equiv 1, \quad (\text{i.e.}, \beta=0, \alpha=1) \end{cases} .$$

This completes the proof of the theorem.

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