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A note on equivariant Euler characteristic

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Abstract. We give a new equivariant cohomological characterization of the equivariant Euler characteristic of a G-simplicial set as defined by Brown. This implies in particular that the equivariant Euler characteristic is a G-homotopy invariant.

Keywords. G-simplicial set; equivariant cohomology; Euler characteristic.

1. Introduction

Serre [5] and Brown [2] defined the Euler characteristic of a discrete group G of finite homological type. Recall that the cohomological dimension cd G is inf n so that the $\mathbb{Z}G$-module $\mathbb{Z}$ with trivial G-action admits projective resolutions of length $\leqslant n$, and that G has finite virtual cohomological dimension vcd G if there exists a finite index subgroup with finite cohomological dimension. A group G is of finite homological type if

(1) vcd $G < \infty$, and
(2) every torsion-free subgroup of finite index has finitely generated rational homology.

If the rational homology of G is finitely generated, then the 'naive' Euler characteristic of G is the integer

$$\tilde{\chi}(G) = \sum(-1)^i \dim_{\mathbb{Q}} H_i(G;\mathbb{Q}),$$

and if G is of finite homological type, then its Euler characteristic is the rational number

$$\chi(G) = \tilde{\chi}(H)/[G\colon H],$$

where H is a torsion-free subgroup of finite index $[G\colon H]$. It is a result of Brown [2] that $\chi(G)$ is independent of the choice of H.

Next, if K is a G-simplicial set, where G is discrete, such that the simplicial set K/G has finitely many non-degenerate cells, and each isotropy subgroup G_x of a simplex x of K is of finite homological type, then the equivariant Euler characteristic of K is

$$\chi_G(K) = \sum(-1)^{\dim x}\chi(G_x),$$

where the sum is over the representatives of non-degenerate simplexes of K/G.

Now suppose that O_G denotes the category of canonical orbits of G whose objects are left coset spaces G/H and whose morphisms are G-maps $\hat{g}\colon G/H \to G/H'$ coming from a subconjugacy relation $g^{-1}Hg \leqslant H'$. Let $\lambda_{\mathbb{Q}}$ denote the contravariant functor from O_G to the category of vector spaces over $\mathbb{Q}$ such that $\lambda_{\mathbb{Q}}(G/H) = \mathrm{Hom}(\mathbb{Q}(G/H),\mathbb{Q})$ where $\mathbb{Q}(G/H)$ is the vector space over $\mathbb{Q}$ with basis G/H, and $\lambda_{\mathbb{Q}}(\hat{g}) = \mathrm{Hom}(\mathbb{Q}(\hat{g}), id)$.

In § 2, we define equivariant cohomology $H_G^*(K;\lambda_{\mathbb{Q}})$ of K which is G-homotopy invariant. In fact, this cohomology is the simplicial analogue of the Bredon cohomology [1]. The purpose of the present paper is to prove the following theorems.

Theorem 1. *If the action of a discrete group G on a G-simplicial set K is such that (i) K/G has only finitely many non-degenerate cells, and (ii) every isotropy subgroup of every simplex of K has finite index in G, and if $\lambda_{\mathbb{Q}}$ is as defined above, then the cohomology groups $H_G^*(K;\lambda_{\mathbb{Q}})$ are finitely generated.*

Theorem 2. *If G acts freely on a simplicial set K satisfying the conditions (i) and (ii) of Theorem 1 then $\chi_G(K) = \chi(G)\sum(-1)^i \dim_{\mathbb{Q}} H_G^i(K;\lambda_{\mathbb{Q}})$, where the summation is from $i = 0$ to $i = \dim K/G$.*

Theorem 3. *If G is of finite cohomological dimension and finite homological type, and K is a G-simplicial set satisfying the conditions of the above theorem, then*

$$\chi_G(K) = \chi(G)\sum(-1)^i \dim_{\mathbb{Q}} H_G^i(K;\lambda_{\mathbb{Q}}),$$

where the summation is from $i = 0$ to $i = \dim K/G$.

Thus $\chi_G(K)$ is a G-homotopy invariant. In particular, if G is free of rank n, then it is of finite homological type because its virtual cohomological dimension is n. Moreover its rational homology is finitely generated because there exists a $K(G,1)$ with one 0-cell and n 1-cells. Therefore $\chi(G) = 1 - n$, and

$$\chi_G(K) = (1-n)\sum(-1)^i \dim_{\mathbb{Q}} H_G^i(K;\lambda_{\mathbb{Q}}),$$

if K is as in Theorem 1.

The plan of the paper is as follows. In § 2, we discuss some basic results with only sketches of proofs, the details of which may be worked out without difficulty. The proofs of the theorems appear in § 3.

2. Equivariant cohomology of a G-simplicial set

A G-simplicial set is a simplicial set K together with an action of G by simplicial maps which commute with the face and degeneracy maps d_i and s_i.

We define the equivariant cohomology $H_G^*(K;\lambda)$ of K with a cofficient system λ (which is a contravariant functor from O_G to the category R-mod of R-modules, R being a commutative ring with (1) as follows. Let $C^n(K;\lambda)$ be the R-module of functions c defined on n-simplexes x of K such that $c(x) \in \lambda(G/G_x)$, where G_x is the isotropy subgroup of x in G. Then define coboundary $\delta: C^n(K;\lambda) \to C^{n+1}(K;\lambda)$ by

$$\delta(c)(x) = \sum_{i=0}^{n+1} (-1)^i \lambda(d_i x \to x) c(d_i x),$$

where $\lambda(d_i x \to x)$ is the homomorphism $\lambda(G/G_{d_i x}) \to \lambda(G/G_x)$ induced from the G-map $G/G_x \to G/G_{d_i x}$ given by the inclusion $G_x \subseteq G_{d_i x}$.

We define an action of G on $C^n(K;\lambda)$ by $(gc)(x) = \lambda(\hat{g})(c(g^{-1}x))$ where $\lambda(\hat{g})$ is the isomorphism $\lambda(G/G_{g^{-1}x}) \to \lambda(G/G_x)$ induced by the conjugacy relation

$g^{-1}G_x g = G_{g^{-1}x}$. Let $C^n_G(K;\lambda)$ be the submodule of G-invariant cochains $(C^n(K;\lambda))^G$. Clearly this makes $C^*_G(K;\lambda)$ a cochain complex, and so we may define

$$H^n_G(K;\lambda) = H_n(C^*_G(K;\lambda)).$$

Note that if the action of G on K is free, then we have

$$H^*_G(K;\lambda) \cong H^*(K/G;\lambda(G/\{e\}))$$

for every coefficient system λ.

Clearly a G-simplicial map $f:K \to L$ induces a cochain map $f^{\#}: C^*_G(L;\lambda) \to C^*_G(K;\lambda)$ defined by $f^{\#}(c)(x) = \lambda(fx \to x)c(fx)$, where $\lambda(fx \to x): \lambda(G/G_{fx}) \to \lambda(G/G_x)$ is the homomorphism induced by the inclusion $G_x \subseteq G_{fx}$. Then $f^{\#}$ induces homomorphism $f^*: H^*_G(L;\lambda) \to H^*_G(K;\lambda)$ satisfying the usual functorial properties.

Lemma 4. *If $f, g : K \to L$ are G-homotopic G-simplicial maps, then*

$$f^* = g^*: H^*_G(L:\lambda) \to H^*_G(K;\lambda).$$

Sketch of Proof. The cochain maps $f^{\#}, g^{\#}: C^*_G(L;\lambda) \to C^*_G(K;\lambda)$ are cochain homotopic by h: $C^n_G(L;\lambda) \to C^{n-1}_G(K;\lambda)$ given by

$$h(c)(x) = \sum_{j=0}^{n-1} (-1)^j \lambda(h_j x \to x)c(h_j x),$$

where $h_j: K_n \to L_{n+1}$ are G-functions constituting a G-homotopy from f to g. ∎

Alternatively, the cochain complex $C^*_G(K;\lambda)$ may be defined as follows. Consider for each $n = \geqslant 0$ a coefficient system $\underline{C}_n(K): O_G \to R$-mod by setting $\underline{C}_n(K)(G/H) = C_n(K^H;R)$ which is the free R-module generated by the n-simplexes of K^H, and, for a G-map $\hat{g}: G/H \to G/H'$, $g^{-1}Hg \subseteq H'$, setting $\underline{C}_n(K)(\hat{g}) = g_*$ which is the chain map induced by the left translation $g: K^{H'} \to K^H$. This gives a chain complex $\underline{C}_*(K)$ in the abelian category of coefficient systems, and if λ is a coefficient system, then $\mathrm{Hom}(\underline{C}_*(K),\lambda)$, which is the R-module of natural transformations $\underline{C}_*(K) \to \lambda$, becomes a cochain complex.

Lemma 5. *There is an isomorphism of cochain complexes*

$$\alpha: C^*_G(K;\lambda) \to \mathrm{Hom}(\underline{C}_*(K), \lambda).$$

Sketch of Proof. Define α by $\alpha(c)(G/H)(x) = \lambda(G_x \to H)(c(x))$, where $x \in K^H_n$ and $\lambda(G_x \to H): \lambda(G/G_x) \to \lambda(G/H)$ is the homomorphism induced by the inclusion $H \subseteq G_x$. Next, define the inverse α' of α by $\alpha'(T)(x) = T(G/G_x)(x)$. ∎

Note that $\underline{C}_*(K)$ is projective in the abelian category of coefficient systems which has sufficiently many injectives, and if λ^* is an injective resolution of λ, then we have a double complex $\mathrm{Hom}(\underline{C}_*(K), \lambda^*)$. The homological algebra applied to this double complex yields a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p(\underline{H}_q(K), \lambda) \Rightarrow H^{p+q}_G(K;\lambda),$$

where $\underline{H}_q(K): O_G \to R$-mod is the cofficient system given by

$$\underline{H}_q(K)(G/H) = H_q(K^H;R) \quad \text{and} \quad \underline{H}_q(K)(\hat{g}) = H_q(g).$$

Lemma 6. *If* $f\colon K \to L$ *is a G-simplicial map such that each* $f^H = f|K^H\colon K^H \to L^H$, $H \subseteq G$, *induces isomorphism in the classical homology with R coefficients, then*

$$f^*\colon H^*_G(L;\lambda) \to H^*_G(K;\lambda)$$

is an isomorphism for every coefficient system λ.

Sketch of Proof. We have an isomorphism $f_*\colon \underline{H}_q(K) \to \underline{H}_q(L)$ given by $f_*(G/H) = f^H_*$. This extends to an isomorphism f^* between the spectral sequences. ■

For a G-simplicial set K, let RK denote the G-simplicial R-module with the set of n-simplexes $(RK)_n = RK_n$ which is the free R-module with basis K_n, and the face and degeneracy maps as the linear extensions of the corresponding maps of K. The G-action on RK is also defined similarly.

Lemma 7. *There is an isomorphism* $H^*_G(K;\lambda) \cong H^*_G(RK;\lambda)$.

Sketch of Proof. We have a cochain isomorphism $\theta\colon \mathrm{Hom}(\underline{C}_*(K),\lambda) \to \mathrm{Hom}(\underline{C}_*(RK),\lambda)$ given by $\theta(T)(G/H)(\sum n_i x_i) = T(G/H)(\sum n_i x_i)$. ■

Let NRK denote the G-pre-simplicial module (degeneracy not considered) where the set of n-simplexes is $\{x \in RK_n : d_i x = 0, 0 \leqslant i < n\}$, and the nth face operator is d_n.

Lemma 8. *There is an isomorphism* $H^*_G(K;\lambda) \cong H^*_G(NRK;\lambda)$.

Sketch of Proof. Consider the inclusion map $i\colon \underline{C}_*(NRK) \to \underline{C}_*(RK)$. By May [4, (22.3)], $i(G/H)\colon C_*(NRK^H;R) \to C_*(RK;R)$ induces isomorphism on homology for each $H \subseteq G$. The proof then follows from Lemmas 6 and 7. ■

It may be noted in passing that if X is a G-space and SX the associated singular G-simplicial set, then the cohomology $H^*_G(SX;\lambda)$ is isomorphic to the equivariant singular cohomology of X with coefficient system λ (see Illman [3]), for every λ.

3. Proofs of theorems

Proof of Theorem 1. In view of Lemma 8, it is sufficient to prove that the vector space $C^n_G(NRK;\lambda_{\mathbb{Q}})$ is finitely generated. Let $x_1, \ldots, x_k$ denote the representatives of the orbit classes of the non-degenerate n-simplexes which lie in NRK. Suppose that for $1 \leqslant l \leqslant k$, the isotropy group G_{x_l} has index m_l in G. Fix a coset representation

$$G/G_{x_l} = \{a_{l_1} G_{x_l}, \ldots, a_{l_{m_l}} G_{x_l}\},\ 1 \leqslant l \leqslant k,\ a_{l_i} \in G.$$

Then define cochains c_{ij} by

$$c_{ij}(x_l) = \begin{cases} 0 & j \neq l \\ (a_{l_i} G_{x_l})^* & j = l,\ 1 \leqslant i \leqslant m_l \\ 0 & j = l, i > m_l \end{cases}$$

where $(a_{l_i} G_{x_l})^*$ are basis dual to $a_{l_i} G_{x_l}$. There is a unique way to define c_{ij} on the orbit of x_l so that $c_{ij} \in C^n_G(NRK;\lambda_{\mathbb{Q}})$. It is also clear that the set $\{c_{ij}\}$ is a linearly independent set, and that any invariant cochain can be written in terms of the c_{ij}'s. This proves the theorem.

Proof of Theorem 2. The group G is necessarily finite. Therefore $\chi_G(K)$ is defined, and, by Theorem 1, the groups $H^*_G(K;\lambda_{\mathbb{Q}})$ are finitely generated. Also, as the action is free, we have

$$\chi_G(K) = \sum_i (-1)^i N_i = \sum_i (-1)^i \dim_{\mathbb{Q}} H^i(K/G;\mathbb{Q}),$$

where N_i denotes the number of non-degenerate i-simplexes of K modulo the action. Consequently, $\chi_G(K) = \chi(K/G)$, the Euler characteristic of K/G. On the other hand the nature of the action implies $H^*_G(K;\lambda_{\mathbb{Q}}) \cong H^*(K/G;\mathbb{Q}(G))$ and, as

$$\dim_{\mathbb{Q}} H^*(K/G;\, \mathbb{Q}(G)) = |G| \dim_{\mathbb{Q}} H^*(K/G;\, \mathbb{Q}),$$

the theorem follows.

Proof of Theorem 3. Since G has finite cohomological dimension, it is torsion free. Also, since G is of finite homological type, the isotropy subgroups G_x also have finite homological type, by a result of Brown [2, IX (6.3)]. Therefore $\chi(G_x)$ is defined, and

$$\begin{aligned}\chi_G(K) &= \sum (-1)^{\dim x} \chi(G_x) = \sum (-1)^{\dim x} \chi(G)\,[G : G_x] \\ &= \chi(G) \sum_{i=0}^{\dim(K/G)} (-1)^i \dim_{\mathbb{Q}} C^i_G(K;\lambda_{\mathbb{Q}}), \\ &= \chi(G) \sum_{i=0}^{\dim(K/G)} (-1)^i \dim_{\mathbb{Q}} H^i_G(K;\lambda_{\mathbb{Q}}).\end{aligned}$$

The last step follows since we are dealing with vector spaces. This completes the proof.

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