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Bredon cohomology of cyclic geometric realization of G-cyclic sets

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Abstract. We define equivariant cyclic and Hochschild cohomology modules of a cyclic object X in the category of G-sets and relate them with the Bredon cohomologies of the cyclic geometric realization $|X|^{cy}$.

Keywords. Cyclic cohomology; Hochschild cohomology; cyclic set; equivariant cohomology.

1. Introduction

Let G be a discrete group. A G-cyclic set is a cyclic object in the category $G\mathscr{S}$ of G-sets. More precisely, a G-cyclic set is a cyclic set $X = \{X_n; d_i, s_i, t_n\}$, where X_n is a G-set for each n and the maps d_i, s_i, t_n are G-equivariant. This may also be considered as a covariant functor $X\colon \Delta C^{op} \to G\mathscr{S}$, where ΔC is the cyclic category. Similarly, by a G-cyclic module, we shall mean a cyclic object in the category of G-modules over a commutative ring k with 1. If X is a G-cyclic set then the cyclic module $k[X]$, associated to the cyclic set X becomes in a natural way a G-cyclic module. For a cyclic set X, the geometric realization $|X|$ is an S^1-space and let $|X|^{cy} = ES^1 \times_{S^1} |X|$ be the cyclic geometric realization of X. Then a well-known result in cyclic cohomology states that

$$H^*(k[X]) \cong H^*(|X|; k),\ HC^*(k[X]) \cong H^*(|X|^{cy}; k) =: H^*_{S^1}(|X|; k).$$

We refer [3, 5, 6], for various results of this type. The aim of this paper is to prove a similar result for G-cyclic set.

Let O_G be the category of canonical orbits [1], the objects of O_G are homogeneous spaces G/H, and a morphism $\hat{g}\colon G/H \to G/K$ is a G-map corresponding to a subconjugacy relation $g^{-1}Hg \subseteq K$. Let k be a commutative ring with 1. An O_G-module is a contravariant functor $O_G \to k$-mod. The collection of O_G-modules along with natural transformations between them as morphisms form an abelian category Vec_G. For any k-module E, we denote by $\bar{E}$ the constant functor $O_G \to k$-mod at E, explicitly, $\bar{E}(G/H) = E$ for all G/H and $\bar{E}(\hat{g}) = id$. Then $\bar{0}$ is the zero object of Vec_G. We shall also consider O_G-groups and O_G-cyclic sets, defined similarly. If X is a G-cyclic set, then the geometric realization $|X|$ of the underlying simplicial set is an S^1-space and also a G-space, and the two actions on $|X|$ commute. As a consequence, the cyclic geometric realization $|X|^{cy} = ES^1 \times_{S^1} |X|$ is a G-space. Let $H^*_G(|X|^{cy}; \lambda)$ denote the Bredon–Illman cohomology groups of $|X|^{cy}$ with coefficients in the O_G-module λ, [1, 4]. One should expect to have an interpretation of the cohomology groups $H^*_G(|X|; \lambda)$ and $H^*_G(|X|^{cy}; \lambda)$ in terms of certain Hochschild and cyclic cohomology groups of $k[X]$. For a G-cyclic set X, we define equivariant Hochschild and cyclic cohomology modules of $k[X]$ and prove that they are isomorphic to $H_G^*(|X|;\lambda)$ and $H_G^*(|X|^{cy};\lambda)$ respectively. We also deduce Connes' exact sequence for these equivariant Hochschild and cyclic cohomology modules.

2 Equivariant Hochschild and cyclic cohomology

Let X be a G-cyclic set, $k[X]$ the associated G-cyclic module and $\lambda: O_G \to k$-mod an O_G-module. Note that for any subgroup H of G, the H-fixed point set X^H is a cyclic set. Let $\underline{k[X]}_n: O_G \to k$-mod be the functor defined as follows. For an object G/H of O_G $\underline{k[X]}_n(G/H) = k[X^H]_n$, and for a morphism $\hat{g}: G/H \to G/K$ of O_G, $\underline{k[X]}_n(\hat{g}): k[X^K]_n \to k[X^H]_n$ is induced by $g: X^K \to X^H$, $x \mapsto gx$. Let $\partial_n: \underline{k[X]}_n \to \underline{k[X]}_{n-1}$ be the natural transformation defined by $\partial_n(G/H): k[X^H]_n \to k[X^H]_{n-1}$, $\partial_n(G/H) = \Sigma(-1)^i d_i$. Clearly $\partial_{n-1} \circ \partial_n = \bar{0}$, and thus we have a chain complex $\underline{k[X]}_*$ in Vec_G. Let $k[X]_G^n = \mathrm{Hom}(\underline{k[X]}_n, \lambda)$ where Hom is in the category Vec_G, and $\delta^n: k[X]_G^n \to k[X]_G^{n+1}$ be induced by ∂_{n+1}. Then $k[X]_G^*$ is a cochain complex in k-mod. We define equivariant Hochschild cohomology modules by $H_G^n(k[X]; \lambda) =: H^n(k[X]_G^*)$.

For any cyclic set Y, let $C(k[Y])$ denote the first quadrant cyclic bicomplex (cf. [6], p. 76) associated to $k[Y]$, $C(k[Y])_{p,q} = C(k[Y_q])$, $p \geqslant 0$, and Tot $C(k[Y])$ be the corresponding total complex. Given a G-cyclic set X, define O_G-modules $\underline{\mathrm{Tot}}\ C(k[X])_n: O_G \to k$-mod by $\underline{\mathrm{Tot}}\ C(k[X])_n(G/H) = \mathrm{Tot}\ C(k[X^H])_n$ for every object G/H of O_G, and for a morphism $\hat{g}: G/H \to G/K$, $\underline{\mathrm{Tot}}\ C(k[X])_n(\hat{g})$ is induced by $g: X^K \to X^H$. Note that the map $g: X^K \to X^H$ of cyclic sets induces a map of bicomplexes $C(k[X^K]) \to C(k[X^H])$. As before, let $\partial_n: \underline{\mathrm{Tot}}\ C(k[X])_n \to \underline{\mathrm{Tot}}\ C(k[X])_{n-1}$ be the natural transformation defined by $\partial_n(G/H)$: $\mathrm{Tot}\ C(k[X^H])_n \to \mathrm{Tot}\ C(k[X^H])_{n-1}$, where $\partial_n(G/H)$ is the boundary map of Tot $C(k[X^H])$. Set $\mathrm{Tot}\ C(k[X])_G^n = \mathrm{Hom}\ (\underline{\mathrm{Tot}}\ C(k[X])_n; \lambda)$. Then $\mathrm{Tot}\ C(k[X])_G^*$ become a cochain complex in k-mod with the coboundary induced by the natural transformation ∂_n. We define equivariant cyclic cohomology modules of $k[X]$ with coefficient λ by $HC_G^n(k[X]; \lambda) =: H^n(\mathrm{Tot}\ C(k[X])_G^*)$.

Remark 2.1. For any bicomplex $C_{..}$, let $C_{..}^{(2)}$ denote the bicomplex consisting of the first two columns of $C_{..}$. Then for a G-cyclic set X, we can define as before, a chain complex $\underline{\mathrm{Tot}}\ C(k[X])_*^{(2)}$ in Vec_G, and the corresponding cochain complex in k-mod. Now there is a quasi-isomorphism between $\underline{k[X]}_*$ and $\underline{\mathrm{Tot}}\ C(k[X])_*^{(2)}$ and thus the corresponding homology functors

$$H_n(\underline{k[X]}_*),\ H_n(\underline{\mathrm{Tot}}\ C(k[X]_*^{(2)}): O_G \to k\text{-mod}$$

are isomorphic. Also note that there are spectral sequences with E^2 terms

$$E^2_{p,q} = \begin{cases} \mathrm{Ext}^p\ (H_q(\underline{k[X]}_*), \lambda) \Rightarrow H_G^{p+q}(k[X]; \lambda) \\ \mathrm{Ext}^p(H_q(\underline{\mathrm{Tot}}\ C(k[X])_*^{(2)}), \lambda) \Rightarrow H^{p+q}(\mathrm{Hom}(\underline{\mathrm{Tot}}\ C(k[X])_*^{(2)}; \lambda)). \end{cases}$$

It now follows that

$$H_G^n(k[X]; \lambda) \cong H^n(\mathrm{Hom}(\underline{\mathrm{Tot}}\ C(k[X])_*^{(2)}; \lambda)).$$

Theorem 2.2. (*Connes' periodicity exact sequence*). *If* $\lambda \in \mathrm{Vec}_G$ *then for any G-cyclic set X there exists a long exact sequence*

$$\cdots \to H^n_G(k[X];\lambda) \xrightarrow{B} HC^{n-1}_G(k[X];\lambda) \xrightarrow{S} HC^{n+1}_G(k[X];\lambda)$$

$$\xrightarrow{I} H^{n+1}_G(k[X];\lambda) \xrightarrow{B} \cdots.$$

Proof. For any cyclic set Y, let $C(k[Y])\,[2,0]$ be the bicomplex $(C(k[Y])\,[2,0])_{p,q} = C(k[Y])_{p-2,q}$. Then for a G-cyclic set X, we have a chain complex $\underline{\mathrm{Tot}}\ C(k[X])\,[2,0]_*$ in Vec_G defined as before. Note that there is a split short exact sequence of complexes in Vec_G

$$0 \to \underline{\mathrm{Tot}}\ C(k[X])^{(2)}_* \to \underline{\mathrm{Tot}}\ C(k[X])_* \to \underline{\mathrm{Tot}}\ C(k[X])\,[2,0]_* \to 0.$$

Applying $\mathrm{Hom}(-,\lambda)$ we get a short exact sequence of cochain complexes of k-modules. By the Remark 2.1, the associated long exact sequence yields the result. ∎

Our main result is the following theorem.

Theorem 2.3. *Let X be a G-cyclic set and $k[X]$ the associated G-cyclic module and* $\lambda \in \mathrm{Vec}_G$, then

(1) $H^n_G(k[X];\lambda) \cong H^n_G(|X|;\lambda)$

(2) $HC^n_G(k[X];\lambda) \cong H^n_G(|X|^{\mathrm{cy}};\lambda)$.

Before we prove the theorem we need to set up some more notations.

For any small category $\mathscr{C}$, let colim: Funct $(\mathscr{C}, k\text{-mod}) \to k$-mod be the covariant right exact functor which assigns to a functor $F:\mathscr{C} \to k$-mod its colimit colim F.

Let $\mathrm{Tor}^{\mathscr{C}}_i(k,-)$: Funct $(\mathscr{C}, k\text{-mod}) \to k$-mod, $i \geq 0$, denote the left derived functors of colim. Then $\mathrm{Tor}^{C}_0(k,-) = \mathrm{colim}$. For a G-cyclic set X we have the following contravariant functors from O_G to k-mod.

(i) $\underline{\mathrm{Tor}}^{\Delta C^{\mathrm{op}}}_n(k, k[X])$,

(ii) $\underline{\mathrm{Tor}}^{\Delta C^{\mathrm{op}}_X}_n(k,k)$,

(iii) $\underline{HC}_n(k[X])$,

(iv) $\underline{H}_n(\|X\|^{\mathrm{cy}};k)$,

(v) $\underline{H}_n(|X|^{\mathrm{cy}};k)$.

To describe the functor (i), note that for every subgroup H of G we may regard $k[X^H]$ as an object of Funct $(\Delta C^{\mathrm{op}}, k\text{-mod})$. Then every morphism $\hat{g}:G/H \to G/K$ of O_G induces a natural transformation $g:k[X^K] \to k[X^H]$. We define $\underline{\mathrm{Tor}}^{\Delta C^{\mathrm{op}}}_n(k,k[X])(G/H) = \mathrm{Tor}^{\Delta C^{\mathrm{op}}}_n(k, k[X^H])$ for every object G/H of O_G and for a morphism $\hat{g}:G/H \to G/K$ of O_G, $\underline{\mathrm{Tor}}^{\Delta C^{\mathrm{op}}}_n(k,k[X])\ (\hat{g})$: $\mathrm{Tor}^{\Delta C^{\mathrm{op}}}_n(k,k[X^K]) \to \mathrm{Tor}^{\Delta C^{\mathrm{op}}}_n(k,k[X^H])$ is the homomorphism $\mathrm{Tor}^{\Delta C^{\mathrm{op}}}_n(k,g)$, where $g:k[X^K] \to k[X^H]$ is the natural transformation induced by $\hat{g}$.

To describe the functor (ii) we proceed as follows. For any subgroup H of G regard $X^H:\Delta C^{\mathrm{op}} \to \mathscr{S}$ as a functor and let $g:X^K \to X^H$ be the natural transformation induced by $\hat{g}:G/H \to G/K$. Let $\Delta C^{\mathrm{op}}_{X^H}$ denote the Bousfield–Kan category corresponding to the functor X^H. The natural transformation $g:X^K \to X^H$ induces in an obvious way a functor $\Delta C^{\mathrm{op}}_{X^K} \to \Delta C^{\mathrm{op}}_{X^H}$ which we denote again by g and this in turn induces a functor $\underline{g}$: Funct $(\Delta C^{\mathrm{op}}_{X^H}, k\text{-mod}) \to$ Funct $(\Delta C^{\mathrm{op}}_{X^K}, k\text{-mod})$. It is easy to see that $\underline{g}$ is an exact functor. Let colim^H: Funct$(\Delta C^{\mathrm{op}}_{X^H}, k\text{-mod}) \to k$-mod denote the colimit functor 'colim' as mentioned before to distinguish it from the colimit functor $\mathrm{colim}^K:\mathrm{Funct}(\Delta C^{op}_{X^K}$, k-mod$)\to k$-mod. Then $\mathrm{Tor}_i^{\Delta C^{op}_{X^H}}(k, -)$ are the left derived functors of colim^H. Let k^H denote the constant functor $k^H:\Delta C^{op}_{X^H}\to k$-mod at k. Define $\underline{\mathrm{Tor}}_n^{\Delta C^{op}_X}(k,k)(G/H)=\mathrm{Tor}_n^{\Delta C^{op}_{X^H}}(k,k^H)$ for objects G/H of O_G. For a morphism $\hat{g}:G/H\to G/K$ we must define a module homomorphism $\underline{\mathrm{Tor}}_n^{\Delta C^{op}_X}(k,k)(\hat{g}):\mathrm{Tor}_n^{\Delta C^{op}_{X^K}}(k,k^K)\to\mathrm{Tor}_n^{\Delta C^{op}_{X^H}}(k,k^H)$. Note that by the universal property defining 'colim', the functor $\underline{g}$ induces a natural transformation $\tilde{g}$: $\mathrm{colim}^K\circ\underline{g}\to\mathrm{colim}^H$. Let $P.\to k^H\to 0$ be a projective resolution of k^H in Funct $(\Delta C^{op}_{X^H}$, k-mod). Since $\underline{g}$ is exact, $\underline{g}(P.)\to\underline{g}(k^H)=k^K\to 0$ is a resolution of k^K in Funct $(\Delta C^{op}_{X^H}$, k-mod). Let $Q.\to k^K\to 0$ be any projective resolution of k^K. Then there exists a map of complexes $Q.\to\underline{g}(P.)$ (which is a quasi-isomorphism). Therefore we get a map of complexes of k-modules $\mathrm{colim}^K Q.\to\mathrm{colim}^K\underline{g}(P.)$. By composing with $\tilde{g}$ we get a map of complexes of k-modules $\mathrm{colim}^K Q.\to\mathrm{colim}^H P.$, and hence an induced map in homology H_n $(\mathrm{colim}^K Q.)\to H_n$ $(\mathrm{colim}^H P.)$. Note that H_n $(\mathrm{colim}^K Q.)=\mathrm{Tor}_n^{\Delta C^{op}_{X^K}}(k,k^K)$ and H_n $(\mathrm{colim}^H P.)=\mathrm{Tor}_n^{\Delta C^{op}_{X^H}}(k,k^H)$. We define $\underline{\mathrm{Tor}}_n^{\Delta C^{op}_X}(k,k)$ $(\hat{g})$ to be this induced map in homology. This makes $\underline{\mathrm{Tor}}_n^{\Delta C^{op}_X}(k,k)$ a contravariant functor from O_G to k-mod.

The functor (iii) is defined as follows. For objects G/H of O_G, $\underline{HC}_n$ $(k[X])$ $(G/H)=HC_n$ $(k[X^H])$ and for a morphism $\hat{g}$, $\underline{HC}_n$ $(k[X])$ $(\hat{g}):HC_n(k[X^K])\to HC_n$ $(k[X^H])$ is the homomorphism of cyclic homologies induced by $g: X^K\to X^H$.

To describe the functor (iv), let ΔC^{op}_X be the Bousfield–Kan category of the underlying cyclic set of the G-cyclic set X. The nerve $B.\Delta C^{op}_X$ of the small category ΔC^{op}_X inherits an obvious G-action as follows. Let u denote a typical element $([n_0],x_0)\xrightarrow{f_1}([n_1],x_1)\xrightarrow{f_2}\cdots\xrightarrow{f_j}([n_j],x_j)$ of $B_j\Delta C^{op}_X$ and $g\in G$, then $gu=([n_0],gx_0)\xrightarrow{f_1}([n_1],gx_1)\to\cdots\xrightarrow{f_j}([n_j],gx_j)$. This makes sense as X is a G-cyclic set. This makes $B.\Delta C^{op}_X$ a G-simplicial set and consequently the classifying space $B\Delta C^{op}_X=|B.\Delta C^{op}_X|$ becomes a G-space. Let $\|X\|^{cy}=B\Delta C^{op}_X$. Moreover we have $(\|X\|^{cy})^H=\|X^H\|^{cy}$ for any subgroup H of G. We may now define the functor (iv) as in (iii) by singular homology. Explicitly, $\underline{H}_n(\|X\|^{cy};k)$ $(G/H)=H_n(\|X^H\|^{cy};k)$ for objects G/H and for a morphism $\hat{g}$, $\underline{H}_n(\|X\|^{cy};k)$ $(\hat{g})$ is the homomorphism induced by $g:X^K\to X^H$. The functor (v) is defined similarly using the G-space $|X|^{cy}$.

Lemma 2.4. *With the above notations, there is a sequence of isomorphisms between the functors*

$$\underline{HC}_n(k[X])\overset{\eta_1}{\cong}\underline{\mathrm{Tor}}_n^{\Delta C^{op}}(k,k[X])\overset{\eta_2}{\cong}\underline{\mathrm{Tor}}_n^{\Delta C^{op}_X}(k,k)\overset{\eta_3}{\cong}\underline{H}_n(\|X\|^{cy};k)$$

$$\overset{\eta_4}{\cong}\underline{H}_n(|X|^{cy};k).$$

Proof. Recall that for any cyclic module E there is an isomorphism $\mathrm{Tor}_n^{\Delta C^{op}}(k,E)\cong HC_n(E)$ (cf. Theorem 6.2.8, [6]). Define $\eta_1(G/H)$ to be the isomorphism $\mathrm{Tor}_n^{\Delta C^{op}}(k,k[X^H])\cong HC_n(k[X^H])$. That η_1 is a natural equivalence follows from the fact that the above isomorphism is natural with respect to any morphism of cyclic modules $E\to F$. Define $\eta_2(G/H)$: $\mathrm{Tor}_n^{\Delta C^{op}}(k,k[X^H])\cong\mathrm{Tor}_n^{\Delta C^{op}_{X^H}}(k,k^H)$ to be the isomorphism given by the Shapiro's lemma (cf. c.12, p. 414, [6]). One checks that this isomorphism is natural with respect to natural transformation $g:k[X^K]\to k[X^H]$ induced by $\hat{g}:G/H\to G/K$. The natural equivalence η_3 is given by the natural isomorphism $\eta_3(G/H)$: $\mathrm{Tor}_n^{\Delta C^{op}_{X^H}}(k,k^H)\cong H_n$ $(\|X^H\|^{cy};k)$ (cf. [7]). To describe η_4, recall (cf. proposition 7.2.6, p. 233, [6]) that for any cyclic set Y, there is a canonical homotopy equivalence $\|Y\|^{cy} \simeq |Y|^{cy}$. Now if X is a G-cyclic set, then it is straight forward to check that the homotopy equivalence constructed in [6] is actually a G-homotopy equivalence, whose restriction to H-fixed point set is the homotopy equivalence $\|X^H\|^{cy} \simeq |X^H|^{cy}$. Define $\eta_4(G/H)$ to be the isomorphism in homology induced by the homotopy equivalence $\|X^H\|^{cy} \simeq |X^H|^{cy}$. This completes the proof of the lemma. ∎

Proof of Theorem 2.3. We shall prove (2), the proof of (1) is similar. Recall that there is a spectral sequence with E^2-term $E^2_{p,q} = \mathrm{Ext}^p\ (\underline{HC}_q(k[X]), \lambda) \Rightarrow HC_G^{p+q}(k[X]; \lambda)$. We also have a spectral sequence with E^2-term $E^2_{p,q} = \mathrm{Ext}^p(\underline{H}_q(|X|^{cy}; k), \lambda) \Rightarrow H_G^{p+q}(|X|^{cy}; \lambda)$. Now the natural equivalence $\underline{HC}_q(k[X]) \cong \underline{H}_q(|X|^{cy}; k)$ given by Lemma 2.4 extends to an isomorphism between the above spectral sequences and thus we have $HC_G^{p+q}(k[X]; \lambda) \cong H_G^{p+q}(|X|^{cy}; \lambda)$. ∎

3. Examples

Example 3.1. Let Z be an S^1-space as well as a G-space and the two actions commute. For instance, we can take Z to be the free loop space of a G-space X. Set $S_n(Z) = \mathrm{Map}_{S^1}(S^1 \times \Delta_n, Z)$, the S^1-equivariant maps from $S^1 \times \Delta_n$ to Z, where S^1 acts on $S^1 \times \Delta_n$ by multiplication on the first factor. The G-action on Z makes $S_n(Z)$ a G-set so that $S.(Z)$ becomes a G-cyclic set.

To give our next example, we need the following result which is a construction due to Elmendorf [2] adapted in the context of cyclic sets. Let $O_G\mathcal{S}^c$ and $G\mathcal{S}^c$ denote respectively the category of O_G-cyclic sets and G-cyclic sets. Let Φ: $G\mathcal{S}^c \to O_G\mathcal{S}^c$ be the functor given by $\Phi X(G/H) = X^H$ and $\Phi X(\hat{g})$ is the cyclic map g: $X^K \to X^H$.

Theorem 3.2. (*Elmendorf*). *There is a functor* $\mathscr{E}: O_G\mathcal{S}^c \to G\mathcal{S}^c$, *and a natural transformation* η: $\Phi\mathscr{E} \to id$ *such that for each* O_G*-cyclic set* T *and each subgroup* H, $\eta(T)(G/H)$: $(\mathscr{E}T)^H \to T(G/H)$ *is a homotopy equivalence.*

Proof. Given an O_G-cyclic set T, we form the bar complex $B_*(T, O_G)$, the n-simplexes of which are $(n+2)$-tuples $(u; f_1, f_2, \ldots, f_n; x)$ where $G/H_n \xrightarrow{f_n} G/H_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} G/H_1 \xrightarrow{f_1} G/H_0$ are composable morphisms in O_G, $u \in T(G/H_0)_n$ and $x \in G/H_n$. The face and degeneracy maps are given by

$$d_i(u; f_1, \ldots, f_n; x) = \begin{cases} (d_0\, T(f_1)(u); f_2, \ldots, f_n; x) & i = 0 \\ (d_i u; f_1, \ldots, f_i \circ f_{i+1}, \ldots, f_n; x) & 0 < i < n \\ (d_n u; f_1, \ldots, f_{n-1}; f_n(x)) & i = n \end{cases}$$

$$s_i(u; f_1, \ldots, f_n; x) = (s_i u; f_1, \ldots, f_i, id, f_{i+1}, \ldots, f_n; x),$$

where the face and the degeneracy maps on the right are those of the appropriate complexes. There is an obvious action of G on the last coordinate which makes $B_*(T, O_G)$ a G-simplicial set. Define $\mathscr{E}T = B_*(T, O_G)$. For an n-simplex $(u; f_1, \ldots, f_n; x)$ the action of $\mathbb{Z}_{n+1}$ is given by

$$t_n(u; f_1, \ldots, f_n; x) = (t_n u; f_1, \ldots, f_n; x),$$

where the action of t_n on the right hand side is the action of $\mathbb{Z}_{n+1}$ on $T(G/H_0)_n$. This makes $\mathscr{E}T$ a G-cyclic set. For a subgroup H, the n-simplexes of the cyclic set $(\mathscr{E}T)^H$ are of the form $(u; f_1, \ldots, f_n; f)$ where $f: G/H \to G/H_n$ is a morphism in O_G. There are cyclic maps η^T_H: $(\mathscr{E}T)^H \to T(G/H)$ and ξ^T_H: $T(G/H) \to (\mathscr{E}T)^H$ given by $\eta^T_H(u; f_1, \ldots, f_n; f) = T(f_1 \circ f_2 \circ \cdots \circ f_n \circ f)(u)$ and $\xi^T_H(u) = (u;\ id, \ldots, id; id)$ where id is the identity map $G/H \to G/H$. Then $\eta^T_H \circ \xi^T_H = id$ and $\xi^T_H \circ \eta^T_H \simeq id$. The homotopy is given by the functions $h_i: (\mathscr{E}T)^H_n \to (\mathscr{E}T)^H_{n+1}$, $0 \leqslant i \leqslant n$, where $h_i\ (u; f_1, \ldots, f_n; f) = (s_i u, f_1, \ldots, f_i, f_{i+1} \circ \cdots \circ f_n \circ f, id, \ldots, id; id)$. This completes the proof. ■

Example 3.3. Let λ be an abelian O_G-group, that is, $\lambda(G/H)$ is abelian for all subgroups H. Let $B.\lambda(G/H)$ be the nerve of $\lambda(G/H)$, where $B_n\lambda(G/H) = \lambda(G/H)^n$. The assignment $G/H \to B.\lambda(G/H)$ yields an O_G-simplicial set $B.\lambda$. Let $z_G \in \lambda(G/G)$. For any subgroup H of G, let $z_H \in \lambda(G/H)$ be the element $z_H = \lambda(G/H \to G/G)\,(z_G)$, where $G/H \to G/G$ is induced by the inclusion $H \subseteq G$. Then for any morphism $\hat{g}$: $G/H \to G/K$, $\lambda(\hat{g})\,(z_K) = z_H$. Now define an action of the cyclic operator t_n on $B_n\lambda(G/H)$ by $t_n(a_1, \ldots, a_n) = (z_H(a_1 a_2, \ldots, a_n)^{-1}, a_1, \ldots, a_{n-1})$ (cf. [6], 7.3.3). This makes $B.\lambda$ an O_G-cyclic set. Therefore by Theorem 3.2, $\mathscr{E}B.\lambda$ is a G-cyclic set. The construction of $\mathscr{E}B.\lambda$ from λ is of course functorial.

Let $\mathscr{E}B\lambda = |\mathscr{E}B.\lambda|$. Then $\mathscr{E}B\lambda$ is a G-space and an S^1-space and the two actions commute. The S^1-action on $\mathscr{E}B\lambda$ may be described as follows. Let $\bar{\mathbb{Z}}$ be the constant O_G-group and $z = \{z_H\}$ be a natural family of elements from λ chosen as above. Then there is an action of $\bar{\mathbb{Z}}$ on λ given by the natural transformation $\gamma_z: \bar{\mathbb{Z}} \times \lambda \to \lambda$, where $\gamma_z(G/H)\ (n, a) = z^n_H a$, $a \in \lambda(G/H)$. Then it follows from Proposition 7.3.4, [6] and Theorem 3.2, that the S^1-action is given by the composition $S^1 \times \mathscr{E}B\lambda \simeq_G \mathscr{E}B(\bar{\mathbb{Z}} \times \lambda) \xrightarrow{\mathscr{E}B(\gamma_z)} \mathscr{E}B\lambda$. Let $X(\lambda, z) = ES^1 \times_{S^1} \mathscr{E}B\lambda$. Thus $X(\lambda, z) = |\mathscr{E}B.\lambda|^{cy}$. Suppose now that z_H is of infinite order for every subgroup H. This happens, for instance, if z_G is of infinite order and the subgroup $\{z_G\}$ of $\lambda(G/G)$ does not intersect the kernel of the homomorphism $\lambda(G/H \to G/G)$ for any subgroup H, except at identity. Then it follows that $X(\lambda, z)$ is an equivariant Eilenberg–MacLane space $K(\lambda/\{z\}, 1)$, where $\lambda/\{z\}$ is the O_G-group $\lambda/\{z\}\ (G/H) = \lambda(G/H)/\{z_H\}$. If some z_H is of finite order then $X(\lambda, z)^H$ has only two non-trivial homotopy groups, $\pi_1(X(\lambda, z)^H) = \lambda(G/H)/\{z_H\}$, $\pi_2(X(\lambda, z)^H) = \mathbb{Z}$, (cf. 7.3.5, [6], and § 2, [2]). From Theorem 2.3, we get as follows.

COROLLORY 3.4

Let λ be an abelian O_G-group, $z = \{z_H\}$ be a natural family of elements as described above. Let $\lambda' \in \mathrm{Vec}_G$. Then

$$HC^*_G(k[\mathscr{E}B.\lambda]; \lambda') \cong H^*_G(X(\lambda, z); \lambda').$$

Moreover, if each z_H is of infinite order, then

$$HC^*_G(k[\mathscr{E}B.\lambda]; \lambda') \cong H^*_G(K(\lambda/\{z\}, 1); \lambda') \cong [K(\lambda/\{z\}, 1), K(\lambda', *)]_G.$$ ■

The last identification follows from the fact that equivariant Eilenberg–MacLane spaces classify Bredon cohomology. In particular, we have

$$HC^1_G(k[\mathscr{E}B.\lambda]; \lambda') \cong \mathrm{Hom}(\lambda/\{z\}, \lambda').$$

As an example consider $\lambda = \bar{\mathbb{Z}}_p$, where $p > 1$ a prime, and let z_G be any element other than the identity. Then it is easy to see that $X(\lambda, z) = K(\bar{\mathbb{Z}}, 2)$. Therefore it follows from

Corollary 3.4 that

$$HC^1_G(k[\mathscr{E}B.\lambda];\lambda') = 0,$$
$$HC^2_G(k[\mathscr{E}B.\lambda];\lambda') \cong \mathrm{Hom}(\bar{\mathbb{Z}}, \lambda').$$

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