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Generic hypersurface singularities

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Abstract. The problem considered here can be viewed as the analogue in higher dimensions of the one variable polynomial interpolation of Lagrange and Newton. Let $x_1, \ldots, x_r$ be closed points in general position in projective space Pn, then the linear subspace V of $H^0(\mathbb{P}^n, \mathcal{O}(d))$ (the space of homogeneous polynomials of degree d on $\mathbb{P}^n$) formed by those polynomials which are singular at each x_i, is given by $r(n+1)$ linear equations in the coefficients, expressing the fact that the polynomial vanishes with its first derivatives at $x_1, \ldots, x_r$. As such, the "expected" value for the dimension of V is $\max(0, h^0(\mathcal{O}(d)) - r(n+1))$. We prove that V has the "expected" dimension for $d \geqslant 5$ (theorem A). This theorem was first proven in [A] using a very complicated induction with many initial cases. Here we give a greatly simplified proof using techniques developed by the authors while treating the corresponding problem in lower degrees.

Keywords. Waving problem; multiple points; interpolation; generic singularities.

1. Introduction

Theorem A. *Let $x_1, \ldots, x_r$ be closed points in general position in n dimensional projective space $\mathbb{P}^n$ over an algebraically closed field k. Then the vector subspace of $H^0(\mathbb{P}^n, \mathcal{O}(d))$ of homogeneous polynomials of degree $d \geqslant 5$ on $\mathbb{P}^n$ having singularites at each x_i has the "expected" dimension* $\max(0, h^0(\mathcal{O}(d)) - r(n+1))$.

We will give a rough outline of the proof. By a standard argument we reduce to a proposition (2.6) concerning unions of double points (i.e. first infinitesimal neighbourhoods of closed points in $\mathbb{P}^n$). This is done in § 2. We attempt to prove this proposition by specializing a number of the double points to have support in a hyperplane H. Each of the specialized double points then induces a double point in H (its trace on H) and has a residual simple point. If the numbers fit (see 'adjusting' 3.2) we obtain a (lower dimensional) trace on H of the kind treated by our proposition and we can then argue by induction.

Unfortunately these numbers do not always fit! Nonetheless, we are able to overcome the problem by using the differential lemma of [AH1] (see lemma 3.3). We begin as above by specializing a number of double points to have support in H. We put in just enough double points to have at least the required number of conditions in H. Any excess of conditions in H, is then considered as coming from the trace of a single double point Z with support in H. The lemma then allows us to use the induction hypothesis (applied in H) to reduce to a residual problem (see degue in 3.3) in degree one less, involving a subscheme $d'Z$ of $Z \cap H$. This $d'Z$ depends on the union of all the residual points of the other double points we specialized to H and although it is well determined (see 3.4 (2)), it is very difficult to locate. In [AH1] we opted for a strategy of specialization, allowing us to identify $d'Z$ at the cost of a more complicated argument. Here, we are able to proceed without identifying $d'Z$ (this means that the degue of 3.3 is true for all subschemes of $Z \cap H$ having the same length as $d'Z$). We do this in 5.1, by ejecting (see 3.2) $d'Z$ out of H and specializing further double points to have support in H. This delays the influence of $d'Z$ to a later stage in the argument, when we dispose of many points in H on whose position, $d'Z$ does not depend. By specialization, we transform the union of $d'Z$ with the right number of simple points, into a double point of $\mathbb{P}^n$. This leads to a statement (6.1) in which there is no unknown component. The proof of 6.1, uses simpler techniques of specialization and ejection.

It is interesting to note that there are numerous examples in degrees $d \leqslant 5$ where theorem A fails. In fact for $d=2$ all values of r with $2 \leqslant r \leqslant n$ give such examples (see [A], §4), while for $d=4$, there is an example for each $n=2$, 3, 4 (see [A], §5, and [AH2]). The only example with $d=3$ is that of a hypercubic in $\mathbb{P}^4$ with seven generic singularities (see [A], §5, [CH] and [AH3]). Of course, unlike the one variable case, the theorem fails miserably in higher dimensions if one abandons the general position hypothesis on the x_i.

From another point of view, one can consider the general problem of determining the dimension of the space of homogeneous polynomials of degree d, having singularites of multiplicity $\geqslant m_i$ at x_i for some fixed sequence $(m_1,\ldots,m_r)$. For $n=2$, this has been persued further in [H3], where the second author has formulated a conjecture giving a clear geometric meaning to those cases where the "expected" dimension is not realized. Eventually, thorem A should be completed by a similar interpretation of the examples in the preceeding paragraph.

2. A reformulation of the theorem

Throughout the rest of the article we work over a fixed algebraically closed base field k. We write $\mathbb{P}^n$ for projective n space over this field.

In this section we formulate clearly theorem A in terms of projective cohomology. Once this has been done we show that theorem A results from proposition 1.6 below. The remaining sections are devoted to proving 1.6.

Let $x_1,\ldots,x_r$ be closed points in general position in $\mathbb{P}^n$ and let Y be the union of the first infinitesimal neighbourhoods of the x_i $(i=1,\ldots,r)$. In any degree d, the homogeneous polynomials of degree d which are singular at each x_i are canonically identified, via the usual exact sequence

$$0 \to H^0(\mathbb{P}^n,\mathscr{I}_Y(d)) \to H^0(\mathbb{P}^n,\mathcal{O}(d)) \to H^0(\mathbb{P}^n,\mathcal{O}_Y(d)) \to H^1(\mathbb{P}^n,\mathscr{I}_Y(d)) \to 0$$

with $H^0(\mathbb{P}^n,\mathscr{I}_Y(d))$, where $\mathscr{I}_Y$ is the ideal sheaf of Y as a closed subscheme of $\mathbb{P}^n$.

To say that $H^0(\mathbb{P}^n, \mathscr{I}_Y(d))$ has the expected dimension is equivalent to saying that the canonical map $H^0(\mathbb{P}^n,\mathcal{O}(d)) \to H^0(\mathbb{P}^n,\mathcal{O}_Y(d))$ has maximal rank as a map of vector spaces or, what amounts to the same thing, that one or the other of the two numbers $h^0(\mathbb{P}^n,\mathscr{I}_Y(d))$ or $h^0(\mathbb{P}^n,\mathscr{I}_Y(d))$ is zero. This motivates the following definition.

DEFINITION 2.1

Let Y be a closed subscheme of $\mathbb{P}^n$. We say that Y has maximal rank in degree d if the canonical map $H^0(\mathbb{P}^n,\mathcal{O}(d)) \to \mathrm{H}^0(\mathbb{P}^n,\mathcal{O}_Y(d))$ has maximal rank as a map of vector spaces. We say that Y has maximal rank if it has maximal rank in all degrees $d \geqslant 0$.

We say that Y is numerically adjusted (resp. adjusted) in degree d if $\chi(\mathbb{P}^n,\mathscr{I}_Y(d))=0$ (resp. $h^i(\mathbb{P}^n,\mathscr{I}_Y(d))=0$ for $i \geqslant 0$).

Clearly if Y is adjusted in degree d then it is both numerically adjusted and of maximal rank in degree d.

The following proposition and its corollary show how these various notions are related for zero dimensional subschemes.

PROPOSITION 1

Let $Y' \subset Y \subset Y''$ be zero dimensional closed subschemes of $\mathbb{P}^n$. Fix $d \geqslant 0$ and consider the following two maps

(i) $H^0(\mathbb{P}^n, \mathcal{O}(d')) \to H^0(\mathbb{P}^n, \mathcal{O}_{Y'}(d'))$; $d' \geqslant 0$,
(ii) $H^0(\mathbb{P}^n, \mathcal{O}(d'')) \to H^0(\mathbb{P}^n, \mathcal{O}_{Y''}(d''))$; $d'' \geqslant 0$.

Then we have (a) *If* (i) *is injective, then $H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(\mathbb{P}^n, \mathcal{O}_Y(d))$ is injective for $d \leqslant d'$.* (b) *If* (ii) *is surjective, then $H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(\mathbb{P}^n, \mathcal{O}_Y(d))$ is surjective for $d \geqslant d''$.* (c) *If* (i) *is injective for $d' = d$ and* (ii) *is surjective for $d'' = d + 1$, then Y has maximal rank.*

Proof. See [H1]. □

COROLLARY 2.2

If Y is a zero dimensional closed subscheme of $\mathbb{P}^n$ which is adjusted in degree d, then Y has maximal rank.

Proof. Put $Y' = Y = Y''$ in the proposition. □

We will apply 2.2 and 2.3 in the following way. We first define as follows.

DEFINITION 2.3

For $n, d \geqslant 1$ let $A_{n,d}$, $B_{n,d}$ be the integers defined by

$$\binom{n+d}{d} = (n+1)A_{n,d} + B_{n,d}$$

where $0 \leqslant B_{n,d} \leqslant n$. Let x_i ($i = 1, \ldots, A_{n,d}$) be generic closed points of $\mathbb{P}^n$ and let $(y_{n,d}, L_{n,d})$ be the generic couple formed by a closed point $y_{n,d}$ and a linear subspace $L_{n,d}$ of dimension $B_{n,d} - 1$ incident with $y_{n,d}$. We define $Y_{n,d}$ to be the union of the first infinitesimal neighbourhoods of $x_1, \ldots, x_{A_{n,d}}$ in $\mathbb{P}^n$ and the first infinitesimal neighbourhood of $y_{n,d}$ in $L_{n,d}$.

We now have the following Lemma.

Lemma 2.4 With the notation of 2.4, $Y_{n,d}$ is *numerically adjusted in degree d and for any $r \geqslant 0$, there is a uniquely determined $\delta \geqslant 0$ such that the generic union Y of r first infinitesimal neighbourhoods of $\mathbb{P}^n$ is inserted in a sequence of inclusions $Y_{n,\delta} \subset Y \subset Y_{n,\delta+1}$. If $Y_{n,\delta}$* and *$Y_{n,\delta+1}$ are adjusted then Y has maximal rank.*

Proof. This is a purely numerical consequence of the definitions and a simple application of 2.2 (c). □

We now show that the following proposition implies theorem A.

PROPOSITION 2

With the notation of 2.4, $Y_{n,d}$ *is adjusted in degree d in* $\mathbb{P}^n$ *for* $d \geq 5$.

Proof of theorem A. Let $x_1,\dots,x_r$ be closed points in general position in projective space $\mathbb{P}^n$ and let Y be the union of the first infinitesimal neighborhoods of the x_i. Let δ be the uniquely determined integer such that $A_{n,\delta} \leqslant r < A_{n,\delta+1}$. If $d \geqslant 5$, then Y has maximal rank by 2.5 and 2.6. If $\delta < 5$, then $Y \subset Y_{n,5}$ so that Y has maximal rank in all degrees $d \geqslant 5$ by 2.2 (b) and 2.6. □

3. Preliminaries

Generally speaking the proof of 2.6 goes as follows (see [H1] for the general setting). We argue by induction on the dimension n and the degree d using the following 'lemma-definition'.

Lemma-definition 3.1. *Let Y be a closed subscheme of* $\mathbb{P}^n$ *numerically adjusted in degree d and let H be a hyperplane of* $\mathbb{P}^n$. *We put* $Y'' = Y \cap H$, *which we call the trace of Y on H and we let Y′ be the closed subscheme of* $\mathbb{P}^n$ *defined by the following canonical exact sequence*

$$0 \to I_{Y'}(-1) \to I_Y \to I_{Y'',H} \to 0, \tag{1}$$

where I_Y *is the ideal of Y in* $\mathbb{P}^n$ and $I_{Y'',H}$, *is the ideal of Y″ in H. We say that Y′ is the residual of Y with respect to H. In view of definition* 1.1, *it follows immediately that Y is adjusted in* P^n *in degree d if the following two conditions are verified:* (i) *Y″ is adjusted in* $H = \mathbb{P}^{n-1}$ *in degree d,* (ii) *Y′ is adjusted in* $\mathbb{P}^n$ *in degree* $d-1$.

This allows us to resolve the proposition "Y is adjusted in $\mathbb{P}^n$ in degree d", into two similar statements, one of which is in lower degree and the other in lower dimension. Note that to exploit 3.3 in this way, it is essential to deal only with (numerically) adjusted subschemes!

3.1 *Adjusting*

Now in practice, Y is the generic member of a nice smooth irreducible flat family T. In this case we can use the upper semicontinuity of the functions $t \mapsto h^i(\mathbb{P}^n, I_{Y_t}(d))$, $t \in T$, to conclude that Y is adjusted in $\mathbb{P}^n$ in degree d if there is some point $t \in T$ where Y_t has this property (in keeping with common usage we call Y_t a specialization of Y). This is the essential element which allows us to employ 3.1. Given a familly T of closed subschemes of $\mathbb{P}^n$ numerically adjusted in degree d, we find a specialization Y_t and a hyperplane H such that (i) $Y_t'' = Y_t \cap H$ is numerically adjusted in H in degree d. By the additivity of the Euler characteristic, this implies that (ii) Y_t' is numerically adjusted in $\mathbb{P}^n$ in degree $d-1$. This necessity to adjust the Euler characteristic of the trace, is the major technical obstacle in the proof. Insufficient techniques for doing this was the source of formidible technical difficulties in the proof of theorem A given in [A]. The big advance is the differential lemma [AH1] which is given in a simplified form in 3.3.

3.2 *Ejecting*

One technique for adjusting the Euler characteristic of the trace is what we call ejecting. Fix a hyperplane H in $\mathbb{P}^n$ and a point x in some linear subspace L of H. Let X be the first infinitesimal neighbourhood of x in L and let Z be the first infinitesimal neighbourhood in $\mathbb{P}^n$ of a general point z on a line D through x, with D transverse to H. Let R be the unique, closed subscheme of $\mathbb{P}^n$ obtained as the flat limit of $Z \cup X$, as z is specialized to x along D. The process which consists in replacing $Z \cup X$ by the specialization R, will be referred to as ejecting X by Z. The sense of this terminology resides in the fact that the trace of R on H is the first infinitesimal neighbourhood of x in H, while the residual of R with respect to H is the first infinitesimal neighbourhood of x in the linear subspace spanned by $L \cup D$–loosely speaking, in the limit, X is ejected outside of H and replaced by Z.

3.3 *Extending*

Lemma 3.2. (*see* [AH2] 1.5) *Let Y, Z be a closed subscheme of $\mathbb{P}^n$. We let $Y \cap Z$ and $Y \cup Z$ denote respectively the scheme theoretic intersection and scheme theoretic union of Y and Z. If $h^i(Z, \mathscr{T}_{Y \cap Z, Z}(d)) = 0$ pour $i \geqslant 0$, then Y is adjusted in degree d in P^n if and only if $Y \cap Z$ is adjusted in $\mathbb{P}^n$ in degree d.*

Proof. An immediate consequence of the following, canonical, exact sequence of ideals

$$0 \rightarrow \mathscr{T}_{Y \cup Z} \rightarrow \mathscr{T}_Y \rightarrow \mathscr{T}_{Y \cap Z, Z} \rightarrow 0. \tag{2}$$

The process, in 2.4.1, which consists in replacing Y by $Y \cup Z$, will be referred to as extending Y by Z.

3.4 *The differentielle method*

In §4, we will use the following lemma [AH1, 1.3, 1.5].

Lemma 3.3. *We consider a closed subscheme of $\mathbb{P}^n$, numerically adjusted in degree d which is the disjoint union of two closed subschemes W, Z, with Z of finite support. We suppose that the residual Z' of Z with respect to a fixed hyperplane H is contained in the trace Z'' of Z on H. Then the union Y of W with the generic translate of Z in $\mathbb{P}^n$ is adjusted in $\mathbb{P}^n$ in degree d if the following two conditions are verified:*

- (dime) *There is a subscheme $d''Z$ of Z'' containing Z' such that $W'' \cup d''Z$ is adjusted in H, in degree d.*
- (degue) *There is a closed subscheme $d'Z$ contained in the intersection of the base of the linear system $|H^0(H, \mathscr{T}_{W'' \cup Z', H}(d))|$ with Z'', such that $W' \cup d'Z$ is adjusted in $\mathbb{P}^n$, in degree $d-1$.*

Remark 3.4. 1) The words dime and degue indicate that the corresponding conditions are respectively in lower dimension and lower degree. 2) When Z is the first infinitesimal neighbourhood in $\mathbb{P}^n$ of a closed point in H with

$$\chi(H, \mathscr{T}_{W'' \cup Z', H}(d)) > 0, \quad \chi(H, \mathscr{T}_{W'' \cup Z'', H}(d)) < 0$$

it is easy to see that, if $d''Z$ exists verifying dime, then the intersection $\mathscr{B}$ of the base of the linear system $|H^0(H, \mathscr{T}_{W'' \cap Z', H} \otimes \mathscr{L}_H)|$ with Z'', has degree $\deg(Z) - \deg(d''Z)$. Now this is just the condition that $W' \cup \mathscr{B}$ be numerically adjusted in $\mathbb{P}^n$ in degree $d-1$, so that $\mathscr{B} = d'Z$ is the unique choice for $d'Z$ with $W' \cup d'Z$ numerically adjusted in $\mathbb{P}^n$ in degree $d-1$. What is more, this $d'Z$ depends only on W'' and the point Z'. That $\mathscr{B}$ have degree $\deg(Z) - \deg(d''Z)$, just says that locally at the point Z', the base of the linear system $|H^0(H, \mathscr{T}_{W'' \cup Z',H} \otimes \mathscr{L}_H)|$ is smooth of codimension $\deg(d''Z)$ in H. As such $\mathscr{B}$ is just the intersection of Z'' with some linear subspace L through Z', with L transverse to $d''Z$ and of dimension $n - \deg(d''Z)$. In particular, $\mathscr{B}$ is the first infinitesimal neighbourhood of Z' in L.

4. Proof of 2.6

4.1 *Proof of* 2.6

For $d \geqslant 5$, $Y_{n,d}$ is adjusted in $\mathbb{P}^n$ in degree d.

Proof. We proceed by induction on n, the case $n = 1$ being the classical one variable interpolation.

Let H be the generic hyperplane of $\mathbb{P}^n$ ($n \geqslant 2$). There are two cases to consider in accordance with $B_{n-1,d} = 0$ and $B_{n-1,d} > 0$.

Case $B_{n-1,d} = 0$. In this case we let $S_{n,d}$ be the specialization of $Y_{n,d}$ obtained by specializing the points x_i ($i = 1, \ldots, A_{n-1,d}$) into H. Then the trace $S''_{n,d}$ of $S_{n,d}$ on H is just $Y_{n-1,d}$. By the induction hypothesis $Y_{n-1,d}$ is adjusted in degree d, so by 3.1 it is enough to show that the residual $S'_{n,d}$ of $S_{n,d}$ with respect to H, is adjusted in degree $d-1$.

Case $B_{n-1,d} > 0$. In this case we let $S_{n,d}$ be the specialization of $Y_{n,d}$, obtained by specializing the points x_i ($i = 1, \ldots, A_{n-1,d+1}$) into H. We apply 2.5.1 with Z the first infinitesimal neighbourhood of x_1 and $d''Z$ the intersection of Z with the generic linear subspace of H passing through x_1 of dimension $B_{n-1,d-1}$. The subscheme $(S_{n,d} - Z)'' \cup d''Z$ is just $Y_{n-1,d}$ so that the dime of 2.5.1 results from the induction hypothesis. As indicated in 2.5.2 (2), there is a unique choice for $d'Z$ verifying the numerical part of the condition degue of 4.1, we must show that the union of $(S_{n,d} - Z)'$ and $d'Z$ is adjusted in degree $d-1$.

Now we define $G_{n,d}$ to be $S'_{n,d}$ if $B_{n-1,d} = 0$ (resp. $(S_{n,d} - Z)'$ if $B_{n-1,d} > 0$). This is just the generic union of $A_{n,d} - A_{n-1,d}$ (resp. $A_{n,d} - A_{n-1,d-1}$) double points in $\mathbb{P}^n$ with $A_{n-1,d}$ points in H. To complete the two cases it will be enough to prove that if Z is a generic double point of H and Z_0 is any subscheme of Z such that $T_{n,d} = G'_{n,d} Z_0$ is numerically adjusted in degree $d-1$, then $T_{n,d}$ is adjusted in degree $d-1$. This will be proven in 4. This result is more general than the degue, but, as we do not have any useful information about $d'Z$, we are obliged, in practice, to prove the more general result. □

5. The Degue

In the proof of the following proposition we simply show how the proposition can be deduced from results in lower degree. These results in lower degree are then proven in 6.

PROPOSITION 3

For $d \geqslant 5$, *let* $G_{n,d}$ *be the generic union of* $A_{n,d} - A_{n-1,d} - \varepsilon$ *double points of* $\mathbb{P}^n$ *with* $A_{n-1,d}$ *points in* H, *where* $\varepsilon = 0$ *if* $B_{n-1,d} = 0$ *and* $\varepsilon = 1$ *if* $Bn - 1,d > 0$. *Let* Z *be a generic* *double point of H and let Z_0 be any subscheme of Z such that $T_{n,d} = G_{n,d} \cup Z_0$ is numerically adjusted in degree $d-1$, then $T_{n,d}$ is adjusted in degree $d-1$.*

Proof. If $n = 1$ this follows by classical one variable interpolation. We will now treat the cases $n = 2$ and $n \geqslant 2$ separately.

Case (1). $n = 2$. In this case we have $B_{2,d} = 0$ or 1, since $(d+2)(d+1)/2 \neq 0$ or $1 \pmod 2$. If $d = 5$, then $B_{1,5} = 0 = B_{2,5}$ and $T_{2,5}$ is adjusted by 5.1. If $d = 6$ then $T_{2,6}$ is the union of five first infinitesimal neighbourhoods and one closed point in $\mathbb{P}^2$ with three closed points and one infinitesimal neighbourhood in $H = \mathbb{P}^1$. By ejecting one closed point in H with one infinitesimal neighbourhood of $\mathbb{P}^2$, the resulting specialization S has an adjusted trace on $H = \mathbb{P}^1$. The residual S' of S with respect to H, can be specialized to the generic union of four infinitesimal neighbourhoods of $\mathbb{P}^2$ with one closed point in H and one degree two closed subscheme of $\mathbb{P}^2$, transverse to H with support a closed point of H. It follows from 5.1 that S' is adjusted in $\mathbb{P}^2$ in degree $d = 4$.

Lemma 3.1. *Let s,t be the unique integers satisfying*

$$2s - t = d - (d+1)/2 + B_{1,d}/2 - B_{2,d} - B_{1,d}(3 - B_{1,d}); \quad 0 \leqslant t \leqslant 1.$$

Then for $d \geqslant 7$, we have $(s-t) \geqslant 1$ et $s \leqslant d/2 + 1/2$.

Proof of lemma. If $d = 7$ then $B_{1,d} = 0 = B_{2,d}$ and $s = 2, t = 1$. If $d = 8$, we have $B_{1,d} = 1$, $B_{2,d} = 0$ giving $s = 1$, $t = 0$. So we have the result in these cases. For $d \geqslant 9$, we consider the expression

$$2(s-t) = d/2 - 1/2 + B_{1,d}/2 - B_{1,d}(3 - B_{1,d}) - B_{2,d} - t. \tag{3}$$

Noting that $B_{1,d}$, $B_{2,d}$ and t are all less than or equal to one, we find

$$2(s-t) \geqslant 1/2 > 0$$

which gives the first part of the lemma. The second part is trivial. □

Now suppose that $d \geqslant 7$ and let s,t be as in lemma 5.1.1. Let S be the specialization of $T_{2,d}$ obtained by specializing $(s-t) \geqslant 0$ first infinitesimal neighbourhoods to have support in H, ejecting $t \leqslant 1$ of the $A_{1,d} \geqslant 1$ closed points of H with a further t infinitesimal neighbourhoods and finally specializing $y_{2,d}$ to H in the case $B_{2,d} = 1$. By (5.1.1), the trace of S on H is numerically adjusted in degree d in $H = \mathbb{P}^1$, hence it is adjusted. Now the residual S' of S with respect to H is the numerically adjusted, generic union of a number of infinitesimal neghbourhoods of $\mathbb{P}^2$ with $(s-t) \geqslant 1$ closed points in H and t finite degree two closed subschemes transverse to H with support a closed point of H. Since, by 5.1.1, $s \leqslant d/2 + 1/2$, we can conclude by 6.1, that S' is adjusted in $\mathbb{P}^2$ in degree $d-2$. This gives the result by 3.1.

Case (2). $n \geqslant 3$. The scheme $T_{n,d}$ contains the subscheme $d'Z$ which depends on the $A_{n-1,d}$ closed points in the trace $T'_{n,d}$ (see 3.4(b)). With $n \geqslant 3$ we have very little information about the linear subspace L spanned by $d'Z$ other than the fact that it is transverse to $d''Z$. Any specialization of $T_{n,d}$ which modifies the closed points in H will modify the subscheme $d'Z$. We avoid this unknown quantity by ejecting $d'Z$ by one of the first infinitesimal neighbourhoods in $T_{n,d}$. We then specialize further first infinitesimal neighbourhoods to H, ejecting a number of closed points, in such a way that the trace of this specialization is numerically adjusted in H in degree $d-1$. In this way $d'Z$ does not appear in the trace and, when it appears in the residual, it no longer depends on the closed points in H.

As usual we begin with an accounting lemma.

Lemma 3.2. *For $n \geqslant 3, d \geqslant 5$, let s,t be the integers determined by the expression*

$$ns - t = \binom{n+d-2}{d-1} - A_{n-1,d} - B_{n,d} - n\delta_{n-1,d}; \quad 0 \leqslant t < n, \tag{4}$$

where $\delta_{n-1,d} = 0$ if $B_{n-1,d} = 0$ and $\delta_{n-1,d} = 1$ otherwise. Then we have the following inequalities:

(a) $s - t \geqslant (n-1) + \delta_{n-1,d} B_{n-1,d} - 1,$
(b) $s \geqslant n(n-1)/2 + \delta_{n-1,d} B_{n-1,d} - 1$

or one of the following

$$n = 4, d = 5 \quad \textit{where } s = 5, \ t = 0, \ B_{4,5} = 1, \ B_{3,5} = 0,$$

$$n = 5, d = 5 \quad \textit{where } s = 8, \ t = 0, \ B_{5,5} = 0,$$

$$n = 6, d = 5 \quad \textit{where } s = 14, \ t = 0, \ B_{6,5} = 0,$$

(c) $\dfrac{1}{n}\dbinom{n+d-2}{d-1} + (n-1)/2 \geqslant s.$

Equally,

$$A_{n-1,d} \geqslant n(n-1)/2 + d_{n,d}(n - B_{n,d}).$$

Proof of the lemma. We have $A_{n-1,d} = \dfrac{1}{n}\dbinom{n+d-1}{d} - B_{n-1,d}/n$, so that

$$ns - t = \frac{(n-1)(d-1)}{nd}\binom{n+d-2}{d-1} - B_{n,d} - n\delta_{n-1,d} + \frac{B_{n-1,d}}{n}. \tag{5}$$

Since $t \leqslant (n-1)$, we see that (a) is implied by (b) for $n \geqslant 4$. We will prove (a) for $n = 3$. From (5) we get

$$3(s-t) = \frac{(d+1)(d-1)}{3} - B_{3,d} - 3\delta_{2,d} + \frac{B_{2,d}}{3} - 2t$$

so that (a) is equivalent, in its strict inequality form, to

$$\frac{(d+1)(d-1)}{3} > 3(1 + \delta_{2,d} B_{2,d-1}) + B_{3,d} + 3\delta_{2,d} - \frac{B_{2,d}}{3} + 2t$$

$$= 3 + B_{3,d} + (3\delta_{2,d} - 1/3) B_{2,d} + 2t. \tag{6}$$

Using the inequalities $t \leqslant 2$, $B_{3,d} \leqslant 3$, $B_{2,d} \leqslant 2$ we see that it is enough to show that

$$(d+1)(d-1)/3 > 46/3.$$

This is the case for $d \geqslant 7$. For $d=5$ we have $\delta_{2,5}=B_{2,5}=B_{3,5}=0$ and $s=3$, $t=1$. For $d=6$ we have $\delta_{2,6}=B_{2,6}=1$, $B_{3,6}=0$ and $s=3$, $t=0$. This gives (a) for $n=3$ as well.

We will now prove (b). By (5) we have

$$ns=\frac{(n-1)(d-1)}{nd}\binom{n+d-2}{d-1}-B_{n,d}-n\delta_{n-1,d}+\frac{B_{n-1,d}}{n}+t \tag{7}$$

so that (b) is equivalent to

$$\begin{aligned}\frac{(n-1)(d-1)}{nd}\binom{n+d-2}{d-1} &> \frac{n^2(n-1)}{2}+n\delta_{n-1,d}(B_{n-1,d}-1)-n\\ &\quad +B_{n,d}-\frac{B_{n-1,d}}{n}+n\,\delta_{n-1,d}-t\\ &=\frac{n^2(n-1)}{2}+B_{n,d}+\left(n\delta_{n-1,d}-\frac{1}{n}\right)B_{n-1,d}-n-t.\end{aligned}$$

Using the upper bounds for $B_{n,d}$, $B_{n-1,d}$ one easily sees that it is sufficient to prove the following inequality

$$\frac{(n-1)(d-1)}{nd}\binom{n+d-2}{d-1}>\frac{n^2(n-1)}{2}+\frac{(n+1)(n-1)^2}{n}.$$

Simplifying, this becomes

$$(d-1)\binom{n+d-2}{d}>\frac{1}{2}(n-1)(n^3+2n^2-2) \tag{8}$$

and one easily verifies (8) for $d=7, n\geqslant 3$; $d=6, n\geqslant 6$ and $d=5, n\geqslant 11$. Now using the fact that the left hand side of (8) is an increasing function of d, we conclude that to prove (b) it is enough to treat the particular cases $d=6$, $n=3,4$; $d=5$, $n=3,\ldots,10$.

For $d=6$ we have for $n=3$ (resp. $n=4$) $s=3$, $t=0$ (resp. $s=9$, $t=1$) giving (b) in this case. For $d=5$ we have the following triples (n,s,t): $(3,3,1),(4,5,0),(5,8,0),(6,14,0),(7,21,4),(8,29,1),(9,39,1),(10,51,5)$, giving (b) in these cases as well.

One easily proves (c) using the expression (7).

Finally to prove the last part of the lemma we have

$$A_{n-1,d}=\frac{1}{n}\binom{n+d-1}{d}-B_{n-1,d}/n$$

so we must show that

$$\frac{1}{n}\binom{n+d-1}{d}>\frac{n(n-1)}{2}+\delta_{n,d}(n-B_{n,d})+\frac{B_{n-1,d}}{n}-1.$$

Using the upper bound for $B_{n-1,d}$ and the lower bound $1\leqslant B_{n,d}$ for $\delta_{n,d}\neq 0$, we see that, since the left hand side is an increasing function of n, it is enough to show that

$$\binom{n+4}{5}>\frac{n^2(n-1)}{2}+n(n-1)$$

or that

$$(n+4)(n+3)(n+1)>60(n-1).$$

One easily verifies that this holds for $n\geqslant 3$. □

Now let S be the specialization of $T_{n,d}$ obtained by specializing $(s-t)$ first infinitesimal neighbourhoods to have support in H, specializing $(y_{n,d}, L_{n,d})$ into H, then ejecting $t \leqslant n-1$ closed points of H by a further t infinitesimal neighbourhoods (one easily verifies that one has enough). In the case where $d'Z$ is non-empty we eject $d'Z$ by yet another infinitesimal neighbourhood. By (4) the trace S'' of S on H is numerically adjusted in degree $d-1$. We will apply lemma 3.1 to S.

We first show that S'' is adjusted in degree $d-1$ in $H = \mathbb{P}^{n-1}$. For this it is enough to show that S'' admits a specialization to a closed subscheme of H of the type figuring in 6.1.

If $B_{n,d} = 0$, then S'' is a generic union of closed points and first infinitesimal neighbourhoods of closed points of H. If $B_{n,d} \neq 0$, then we can specialize $\delta_{n,d}(n - B_{n,d})$ of the closed points in H to the point $y_{n,d}$, completing the first infinitesimal neighbourhood of $y_{n,d}$ in $L_{n,d}$ to a first infinitesimal neighbourhood of $y_{n,d}$ in H. This means that S''admits a specialization which is a generic union of first infinitesimal neighbourhoods of closed points in H and a number c of closed points of H where

$$c = A_{n-1,d} - \delta_{n,d}(n - B_{n,d}) - t \geqslant \frac{(n-1)(n-2)}{2}.$$

By specializing further closed points to infinitesimal neighbourhoods we obtain a specialization where the number c of closed points satisfies.

$$\frac{(n-1)(n-2)}{2} + (n-1) \geqslant c \geqslant \frac{(n-1)(n-2)}{2}$$

and we conclude that S'' admits a specialization to a closed subscheme of the type figuring in 6.1, so that S'' is adjusted in degree $d-1$.

We will now show that S' is adjusted in degree $d-2$ in $\mathbb{P}^n$. The residual S' of S with respect to H is the generic union, numerically adjusted in $\mathbb{P}^n$ in degree $d-2$, of first infinitesimal neighbourhoods of closed points in $\mathbb{P}^n$ with $t \leqslant n-1$ finite, degree two, closed subschemes with support in H, but transverse to H, $s-t$ closed points of H and res $(d'Z) \subset H$ (this is the residual of the ejected $d'Z$ which we consider as empty if $B_{n-1,d}$ is zero).

Now the subscheme res$(d'Z)$ is independent of the

$$s - t \geqslant (n-1) + \delta_{n-1,d}(B_{n-1,d} - 1)$$

closed points in H, so if res $(d'\,Z)$ is non-empty we can specialize $(B_{n-1,d} - 1)$ of these closed points to $d'Z$ to make a first infinitesimal neighbourhood. The conditions of lemma 5.1.2, assure that this specialization of S' satisfies the hypotheses of 6.1, where the variable v is zero if $B_{n-1,d}$ is zero and one otherwise. As such S' is adjusted in $\mathbb{P}^n$ in degree $d-2$ and we have completed the proof of 5.1. □

6. Results and initial cases

PROPOSITION 4

Let $H = \mathbb{P}^{n-1}$ be a hyperplane of $\mathbb{P}^n$. In $\mathbb{P}^n$ we consider, for $d \geqslant 3$, the closed subscheme Y formed by the generic union, numerically adjusted in $\mathbb{P}^n$ in degree d, of

(a) *the first infinitesimal neighbourhoods of u closed points of* $\mathbb{P}^n$,
(b) *a closed points of H*,
(c) *b finite closed subschemes of degree two of* $\mathbb{P}^n$ *transverse to H with support a closed point of H.*
(d) *the first infinitesimal neighbourhood in* $\mathbb{P}^n$ *of* $v \leqslant 1$ *closed points of H where*

(i) $a \geqslant n-1$, *or* $d=3$ *and we have one of the following cases* $n=4, a=5, b=0=v;\ n=5, a=8, b=0, v=1; n=6, a=14, b=0=v$

(ii) $\frac{1}{n}\binom{n+d}{d+1}+\frac{n-1}{2} \geqslant a+b \geqslant \frac{n(n-1)}{2}$

(iii) $0 \leqslant b < n$

(iv) *if* $n=2$ *and* $d=3$, *then* $v=0$.

Then Y is adjusted in $\mathbb{P}^n$ *in degree d.*

Proof. The proposition is true for $n=1$ by classical one variable interpolation. For $d=3$ the proposition results from 5 below. Henceforth we suppose that $n \geqslant 2$ and $d \geqslant 4$ and we argue by induction on n and d. Firstly we need the following accounting lemma.

Lemma 4.1. *With the same hypotheses as in* 6.1, *let s,t be the integers uniquely determined by the following conditions,*

$$sn-t=(n+d-1)!/(n-1)!d!-(a+b); 0 \leqslant t < n. \tag{9}$$

Then for $n \geqslant 2$, *we have* (a) $s-t+b \geqslant n-1$, (b) $s+b \geqslant (n(n-1))/2$, *or* $n=4$, $a=15$, $b=0$ *in which case* $t=0$ and $s=5$.

Proof of lemma. For (a) we have

$$s-t=\frac{1}{n}\binom{n+d-1}{d}-\frac{(a+b)}{n}-\frac{(n-1)t}{n}$$

and using (ii) of the proposition, we have

$$\begin{aligned} s-t &\geqslant \frac{1}{n}\binom{n+d-1}{d}-\frac{1}{n^2}\binom{n+d}{d+1}-\frac{n-1}{2n}-\frac{(n-1)t}{n} \\ &= \frac{(n-1)d}{n^2(d+1)}\binom{n+d-1}{d}-\frac{n-1}{2n}-\frac{(n-1)t}{n} \\ &= R(n,d,t). \end{aligned} \tag{10}$$

We want $(s-t) > (n-2)$. Since $t \leqslant (n-1)$ and $R(n,d,t)$ is a strictly increasing function of d, it is enough to show that $R(n,4,n-1) > (n-2)$, i.e.

$$\frac{4(n-1)}{5n^2}\binom{n+3}{4} > \frac{4n-7}{2}.$$

This is the case for $n=2$ or $n \geqslant 4$.

If $n=3$, then (10) becomes $s-t \geqslant (7-2t)/3 \geqslant 1$ and the last inequality is strict for $t<2$. When $t=2$, $s-t \geqslant 1$ with strict inequality if $s \geqslant 3$. Finally if $s=3$ and $t=2$, we find by (9) that $(a+b)=8$. Using the fact that Y is adjusted in $\mathbb{P}^3$ in degree three, we find $b+8+4(u+v)=35$, so that $b\geqslant 3$ contradicting (iii) of the proposition. This gives (a).

For (b) we have

$$s=\frac{1}{n}\binom{n+d-1}{d}-\frac{a+b}{n}+\frac{t}{n} \tag{11}$$

and using (ii) of the proposition we find

$$\begin{aligned} s &\geqslant \frac{(n-1)d}{n^2(d+1)}\binom{n+d-1}{d}-\frac{n-1}{2n}+\frac{t}{n} \\ &\geqslant \frac{4(n-1)}{5n^2}\binom{n+3}{4}-\frac{n-1}{2n}+\frac{t}{n}. \end{aligned} \tag{12}$$

Since $t\geqslant 0$ and we need $s+b>\dfrac{n(n-1)}{2}-1$, it is enough to show that

$$\frac{4(n-1)}{5n^2}\binom{n+3}{4}>\frac{n(n-1)}{2}-1-\frac{n-1}{2n}.$$

This is the case for $n=2,3$ or $n\geqslant 5$.

For $n=4$, (12) becomes $s\geqslant(39+2t)/8$, so that $s>5$ for $t>0$. If $t=0$, the same expression gives $s\geqslant 5$. Finally if $s=5$ and $t=0$, then (11) gives $a+b=15$. If $b>0$, then $s+b\geqslant 6$ as required and we are left with the case $a=15, b=0=t, s=5$. This gives (b) using the fact that the right hand side of (12) is a strictly increasing function of d. □

Now to finish the proof of the proposition we use lemma 2.1. Let s,t be as in (9). We consider the specialization S of Y obtained by specializing $(s-t)-v(\geqslant 0$ by the lemma) of the u first infinitesimal neighbourhoods to have support in H and ejecting t of the $a\geqslant(n-1)$ points in H by a further t of the infinitesimal neighbourhoods. By (9), the trace S'' of S on H is numerically adjusted in H in degree d with

$$a'=a+b-t\geqslant\frac{(n-2)(n-1)}{2} \tag{13}$$

closed points contained in S''.

For $n\geqslant 2$, $S''\subset H=\mathbb{P}^{n-1}$ can be specialized to a subscheme of the same type as figures in the proposition. In fact, S'' is a generic union of closed points and first infinitesimal neighbourhoods of $\mathbb{P}^{n-1}$ and by specializing closed points to infinitesimal neighbourhoods, we can suppose that the number of closed points satisfies

$$\frac{(n-2)(n-1)}{2}+(n-1)\geqslant a'\geqslant\frac{(n-2)(n-1)}{2}. \tag{14}$$

One then verifies that

$$\binom{n+d-2}{d}+\frac{n-1}{2}\geqslant\frac{n(n-1)}{2}+\frac{n-1}{2}\geqslant\frac{(n-2)(n-1)}{2}+(n-1) \tag{15}$$

for $d\geqslant 2$. Now we conclude by the induction hypothesis that S'' is adjusted in degree d in $\mathbb{P}^{n-1}$.

The residual S' of S with respect to H is numerically adjusted in $\mathbb{P}^n$ in degree $d-1$. It is the generic union of $(s-t+b)$ closed points in H, plus $t \leqslant (n-1)$ finite, degree two, closed subschemes of $\mathbb{P}^n$ with support in H, but transverse to H (these are the residuals of the ejections) and $(u-s)$ first infinitesimal neighbourhoods of closed points in $\mathbb{P}^n$. In view of (9), (ii) and (iii) of the proposition and lemma 4.1, we have

$$(s-t+b) \geqslant (n-1)$$

$$\frac{1}{n}\binom{n+d-1}{d} + \frac{n-1}{2} \geqslant (s+b) \geqslant \frac{n-1}{2}$$

so that S' is of the type figuring in the theorem for $d \geqslant 4$. By the induction hypothesis we conclude that S' is adjusted in degree $d-1$ for $d \geqslant 4$.

The proposition now follows by 3.1. □

DEFINITION 6.1

Let $x_1, \ldots, x_{s+2}$ be closed points in $\mathbb{P}^n$ with $s \leqslant n-1$ and consider the union U of the $((s+2)(s+1))/2$ lines which join these points two by two. If H is a hyperplane in $\mathbb{P}^n$ not containing any of the points, then we say that the intersection $U \cap H$ is an s-complex. The points of an s-complex span a linear subspace of dimension s and we will admit the empty set as a (-1)-complex.

PROPOSITION 5

Let Y be the generic union, numerically adjusted in $\mathbb{P}^n$ in degree $d=3$, of
(a) *the first infinitesimal neighbourhoods of u closed points of $\mathbb{P}^n$,*
(b) *an $(l-2)$-complex,*
(c) *a closed points of H,*
(d) *b finite closed subschemes of degree two of $\mathbb{P}^n$* transverse to H with support a closed point of H,
(e) *the first infinitesimal neighbourhood in $\mathbb{P}^n$ of $v \leqslant 1$ closed points of H, with $v=0$. If $n=2$, where*

(i) $a \geqslant n-1$,

(ii) $\binom{n+3}{4} + \frac{n-1}{2} \geqslant a+b \geqslant \frac{n(n-1)}{2}$ *or one of the following cases*

$n=4,\ a=5,\ b=0=v,\ l=-1,$

$n=5,\ a=8,\ b=0, v=1,\ l=-1,$

$n=6, a=14, b=0=v, l=-1,$

(iii) $0 \leqslant b < n$,

(iv) $-1 \leqslant l \leqslant (n+2)/2$.

Then Y is adjusted in $\mathbb{P}^n$ in degree $d=3$.

Proof. For $n = 1$ this is just classical one variable interpolation, so from now on we suppose that $n \geqslant 2$ and we argue by induction on n using lemma 3.1. We begin with the following accounting lemma.

Lemma 5.1. *With the hypotheses of the proposition, let $w = u + v$, then there exists two integers s, t verifying the following conditions*

(α) $\quad (w - s)n - t = \binom{n+2}{3} - (a + b) - \binom{s}{2},$

(β) $\quad 0 \leqslant t \leqslant n,$

(γ) $\quad 1 \leqslant s \leqslant (n + 1)/2,$

(δ) $\quad (w - s) \geqslant (t + v).$

Proof of lemma. We begin with the exceptional cases of 6.3(ii). For $n = 4$, $a = 5$, $b = 0 = v$, $l = -1$ we have $u = w = 6$ and it is enough to take $s = 2$ and $t = 2$. For $n = 5$, $a = 8$, $b = 0$, $v = 1$, $l = -1$ we have $u = 7$ and it is enough to take $s = 3$, $t = 1$. For $n = 6$, $a = 14$, $b = 0 = v$, $l = -1$ we have $u = w = 10$ and it is enough to take $s = 3$, $t = 3$. Now, in the general case, we consider the following function of s,

$$F(s) = \frac{(n + 2)(n + 1)n}{6} - (w - s)n - (a + b) - \frac{s(s - 1)}{2} \tag{16}$$

which is a decreasing function of s on the interval $[1, \ldots, n]$, with $0 \leqslant F(s + 1) - F(s) = n - m \leqslant n - 1$.

As such, there is an s verifying (α), (β) and (γ) if we know that the following two conditions are satisfied

(1) $F(1) \leqslant 0$,

(2) $F[\![(n + 1)/2]\!] \geqslant -n + 1$,

where $[\![\ ]\!]$ denotes the integer part. The fact that Y is adjusted in degree $d = 3$ in $\mathbb{P}^n$, allows us to express w as

$$w = (n + 3)(n + 2)/6 - (a + b)/(n + 1) - l(l - 1)/2(n + 1)$$

and this gives

$$\begin{aligned} F(s) = &-n(n + 2)/3 - (a + b)/(n + 1) + nb/(n + 1) + ns \\ &+ nl(l - 1)/2(n + 1) - s(s - 1)/2. \end{aligned}$$

Now we have

$$\begin{aligned} (n + 1)(F(1) - 1) = &-n(n + 1)(n + 2)/3 - (a + b) + nb + n(n + 1) \\ &+ nl(l - 1)/2 - (n + 1) \end{aligned}$$

and using (ii) and (iv) of the proposition we find

$$\begin{aligned} (n + 1)(F(1) - 1) &\leq -n(n + 1)(n + 2)/3 - n(n - 1)/2 + n(n - 1) + n(n + 1) \\ &\quad + n2(n + 2)/8 - (n + 1) \\ &= (-5n3 + 18n2 - 28n - 24)/24 \\ &< 0 \end{aligned}$$

pour $n \geqslant 2$. On the other hand,

$$\begin{aligned}(n+1)(F(n/2)+n) &= -n(n+1)(n+2)/3-(a+b)+bn+n^2(n+1)/2\\ &\quad +nl(l-1)/2-n(n+1)(n-2)/8+n(n+1)\\ &\geqslant -n(n+1)(n+2)/3-((n+3)(n+2)(n+1)/24\\ &\quad +(n-1))n^2(n+1)/2-n(n+1)(n-2)/8+n(n+1)\\ &=(9n^2-9n+6)/24\\ &>0\end{aligned}$$

pour $n \geqslant 1$.

Finally we must show that if s satisfies (α), (β) and (γ), then $(w-s) \geqslant (t+v)$. By (α), we have, using (ii) and (iv) of the proposition

$$\begin{aligned}w-s &= \tfrac{1}{6}(n+2)(n+1)-\frac{1}{n}(a+b)-\frac{1}{2n}s(s-1)+\frac{t}{n}\\ &\geqslant \tfrac{1}{6}(n+2)(n+1)-\frac{1}{n}(1/24(n+3)(n+2)(n+1)+\tfrac{1}{2}(n-1))\\ &\quad -\frac{1}{8n}(n^2-1)+\frac{t}{n}\\ &=\tfrac{3}{4}(n+2)(n+1)(n-1)-\frac{n}{8}+\frac{1}{2n}+\frac{t}{n}\\ &=V(n)+\frac{t}{n}.\end{aligned}$$

We will show that $V(n)+t/n > t+v-1$. Since $(n-1) \geqslant t$, it will be enough to show that

$$W(n)=V(n)-((n-1)2/n)-1>v.$$

For $n=2$, we have $v=0$ and $W(2)=1$. Now $W(n)=1/8(n^3+2n^2-10n+14)-(1/2n)$, is a strictly increasing function of n for $n \geqslant 2$ so that, since $v \leqslant 1$, $W(n) > v$ for $n \geqslant 2$. This completes the proof of the lemma. □

Now to complete the proof of the proposition we use lemma 2.1. Fix s, t verifying the conditions of lemma 6.3.4. Let S be the specialization of Y obtained by specilizing $u-s-t$ (which is $\geqslant 0$ by lemma 6.3.4 (d)) of the u first infinitesimal neighbourhoods to have support in H, and ejecting t of the a closed points in H with a further t infinitesimal neighbourhoods. We will show that S is adjusted in $\mathbb{P}^n$ in degree $d=3$.

Let S_0 be obtained from S by extending the s infinitesimal neighbourhoods remaining outside of H by the union Z of the $s(s-1)/2$ lines which join them two by two. Clearly $S \subset D$ is adjusted in each of the lines D contained in Z so, by lemma 3.2, it is enough to show that S_0 is adjusted in $\mathbb{P}^n$ in degree d. We will apply lemma 3.1 to S_0.

The trace S_0'' of S_0 on H, is the generic union of $u+v-s$ first infinitesimal neighbourhoods of closed points in H with

$$a'=a-t+b \geqslant (n-1)(n-2)/2$$

closed points and an $(s-2)$-complex. The scheme S_0'' is numerically adjusted in degree $d=3$ in $H=\mathbb{P}^{n-1}$ by (α) of lemma 6.3.4. By specializing closed points to first infinitesimal neighbourhoods we can suppose that

$$((n-1)(n-2)/2)+(n-1)\geqslant a'\geqslant(n-1)(n-2)/2. \tag{17}$$

Hence by (15) we conclude that S_0'' admits a specilization to a subscheme of H satisfying the conditions of the proposition in dimension $n-1$. Using the induction hypothesis we conclude that S_0'' is adjusted in H in degree $d=3$.

By lemma 3.1, the residual S_0' of S_0 with respect to H is numerically adjusted in $\mathbb{P}^n$ in degree $d=2$. It is the generic union of the first infinitesimal neighbourhoods of $1\leqslant s\leqslant(n+1)/2$ closed points of $\mathbb{P}^n$ and the $s(s-1)/2$ lines which join them two by two, with an $(l-2)$-complex, $u+v-s+b$ closed points in H and t finite, degree two, closed subschemes of $\mathbb{P}^n$ with support in H, but transverse to H. Let T be the specialization of S_0' obtained by specializing all but one of the infinitesimal neighbourhoods to have support in H and specializing all the degree two finite closed subschemes into H. By [AH2] (4.9) the trace T'' on H is adjusted in H in degree $d=2$ and it is clear that the residual T' of T with respect to H (which is the union of a first infinitesimal neighbourhood of a closed point of $\mathbb{P}^n$ with $(s-1)$ lines through the point) is adjusted in degree one in $\mathbb{P}^n$. Applying lemma 3.1 to T we conclude that S_0' is adjusted in degree $d=2$ in $\mathbb{P}^n$.

Now applying lemma 3.1 to S_0 we conclude that S_0 is adjusted in degree $d=3$ in P^n. □

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