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Double convolution integral equations involving a general class of multivariable polynomials and the multivariable H-functions

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Abstract. In this paper we have solved a double convolution integral equation whose kernel involves the product of the H-functions of several variables and a general class of multivariable polynomials. Due to general nature of the kernel, we can obtain from it, solutions of a large number of double and single convolution integral equations involving products of several classical orthogonal polynomials and simpler functions. We have also obtained here solutions of two double convolution integral equations as special cases of our main result. Exact reference of three known results, which are obtainable as particular cases of one of these special cases, have also been included.

Keywords. Double Laplace transform; convolution theorem for Laplace transform; the double convolution integral equation; a general class of multivariable polynomials; multivariable H-function.

1. Introduction

A large number of convolution integral equations (double or single) involving various functions as their kernel have been solved by several research workers. An extensive and systematic presentation has been given in the books by Srivastava and Buschman [6, 7]. In the present paper we consider the generalizations of the results obtained earlier by Gupta *et al* [2], Jain [3], and Raina and Koul [4].

We shall require the following definitions and results later on. (i) A general class of multivariable polynomials [9, p. 686, eq. (1.4)] defined as follows

$$S_N^{M_{s+1},\dots,M_n}[-z_{s+1},\dots,-z_n]$$
$$= \sum_{k_{s+1},\dots,k_n=0}^{M_{s+1}k_{s+1}+\dots+M_nk_n\le N} (-N)_{M_{s+1}k_{s+1}+\dots+M_nk_n} A_1(N;k_{s+1},\dots,k_n)$$
$$\times\frac{(-z_{s+1})^{k_{s+1}}}{k_{s+1}!}\dots\frac{(-z_n)^{k_n}}{k_n!}, \tag{1.1}$$

where $M_{s+1},\dots,M_n$ are arbitrary positive integers and the coefficients $A_1(N; k_{s+1},\dots,k_n)$, $(N;\ k_i\ge 0,\ i=s+1,\dots,n)$ are arbitrary constants, real or complex.

Another class of polynomials $S_{N'}^{M_{r+1},\dots,M_m}[-w_{r+1},\dots,-w_m]$ occurring in (2.1) is defined in a similar manner with changed parameters. (ii) A particular case of the multivariable H-function defined by Srivastava and Panda (cf. [10, p. 251, eq. (C.1)]) possessing the following integral representation

$$H_1\begin{bmatrix} z_1\\ \vdots\\ z_s\end{bmatrix} = H^{0,0\,:1,n_1;\dots;1,n_s}_{p^{(1)},q^{(1)}:p_1,q_1+1;\dots;p_s,q_s+1}\begin{bmatrix} z_1\\ \vdots\\ z_s\end{bmatrix}.$$

$$\left.\begin{matrix}(a_j:\alpha_j^{(1)},\ldots,\alpha_j^{(s)})_{1,p^{(1)}}:(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\ldots;(c_j^{(s)},\gamma_j^{(s)})_{1,p_s}\\ \\ (b_j:\beta_j^{(1)},\ldots,\beta_j^{(s)})_{1,q^{(1)}}:(0,1),(d_j^{(1)},\delta_j^{(1)})_{1,q_1};\ldots;(0,1),(d_j^{(s)},\delta_j^{(s)})_{1,q_s}\end{matrix}\right]$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_s}\phi_1(\xi_1)\cdots\phi_s(\xi_s)\psi(\xi_1,\ldots,\xi_s)$$

$$\times\Gamma(-\xi_1)\cdots\Gamma(-\xi_s)z_1^{\xi_1}\cdots z_s^{\xi_s}\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_s,\quad w=\sqrt{-1}\tag{1.2}$$

or, equivalently, in the series form as follows [11, p. 64, eq. (1.3)]

$$H_1\begin{bmatrix}z_1\\ \vdots\\ z_s\end{bmatrix}=\sum_{k_1,\ldots,k_s=0}^{\infty}\phi_1(k_1)\cdots\phi_s(k_s)\psi(k_1,\ldots,k_s)\frac{(-z_1)^{k_1}}{k_1!}\cdots\frac{(-z_s)^{k_s}}{k_s!},\tag{1.3}$$

where

$$\phi_i(k_i)=\frac{\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}k_i)}{\prod_{j=1}^{q_i}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}k_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}k_i)},\quad (i=1,\ldots,s)\tag{1.4}$$

and

$$\psi(k_1,\ldots,k_s)=\left\{\prod_{j=1}^{p^{(1)}}\Gamma\left(a_j-\sum_{i=1}^{s}\alpha_j^{(i)}k_i\right)\prod_{j=1}^{q^{(1)}}\Gamma\left(1-b_j^{(i)}+\sum_{i=1}^{s}\beta_j^{(i)}k_i\right)\right\}^{-1}.\tag{1.5}$$

For the convergence, and existence conditions and other details of the multivariable H-function, the conditions referred to in [10, pp. 251–253, eqs ((C.2)–(C.8))] are assumed to be satisfied throughout the present work.

During our study we shall also need a second multivariable H-function with different variables and parameters. We shall denote it by H_2.

(iii) The results recorded in the book by Ditkin and Prudnikov [1, p. 4, eqs ((1.2), (1.3)); p. 93, eq. (25); p. 99, eq. (68)] as referred to in [3, p. 99, eqs ((1.2)–(1.8))].

2. Main result

Theorem. *The double convolution integral equation*

$$\int_0^x\int_0^y(x-u)^{\rho_1-1}(y-v)^{\rho_2-1}H_1\begin{bmatrix}z_1(x-u)\\ \vdots\\ z_s(x-u)\end{bmatrix}H_2\begin{bmatrix}w_1(y-v)\\ \vdots\\ w_r(y-v)\end{bmatrix}$$

$$\times S_N^{M_{s+1},\ldots,M_n}[-z_{s+1}(x-u),\ldots,-z_n(x-u)]$$

$$\times S_{N'}^{M_{r+1},\ldots,M_m}[-w_{r+1}(y-v),\ldots,-w_m(y-v)]f(u,v)\,\mathrm{d}u\,\mathrm{d}v=g(x,y)\tag{2.1}$$

has the solution given by

$$f(x,y)=\int_0^x\int_0^y(x-u)^{a-\rho_1-\mu_1-1}(y-v)^{b-\rho_2-\mu_2-1}W[x-u,y-v]$$

$$\times g_{u^a,v^b}^{(a+b)}(u,v)\,\mathrm{d}u\,\mathrm{d}v,\tag{2.2}$$

where $\min \operatorname{Re}(\rho_1, \rho_2) > 0$, a and b are positive integers such that $a > \operatorname{Re}(\rho_1 + \mu_1) > 0$, $b > \operatorname{Re}(\rho_2 + \mu_2) > 0$,

$$\bar{g}_{1,y^k}(p,0) = \bar{g}_{2,x^c}(0,q) = g_{x^c,y^k}^{(c+k)}(0,0) = 0 \quad \text{for} \quad 0 \leqslant c < a \quad \text{and} \quad 0 \leqslant k < b \tag{2.3}$$

and

$$W[x,y] = \sum_{\nu_1,\nu_2=0}^{\infty} \frac{H_{\nu_1}^{(1)} H_{\nu_2}^{(2)} x^{\nu_1} y^{\nu_2}}{\Gamma(a+\nu_1-\rho_1-\mu_1)\Gamma(b+\nu_2-\rho_2-\mu_2)}, \tag{2.4}$$

where $H_{\nu_1}^{(1)}$ and $H_{\nu_2}^{(2)}$ are given by the recurrence relations

$$\lambda_{\mu_i}^{(i)} H_0^{(i)} = 1 \quad \text{and for} \quad \mathscr{G}_i > 0, \ \sum_{\nu_i=0}^{\mathscr{G}_i} H_{\nu_i}^{(i)} \lambda_{\mathscr{G}_i+\mu_i-\nu_i}^{(i)} = 0, \quad \mathscr{G}_i = 1,2,3,\ldots, i = 1,2. \tag{2.5}$$

μ_i is the least B_i for which $\lambda_{\mu_i}^{(i)} \neq 0$.

$$\lambda_{B_1}^{(1)} = (-1)^{B_1} \sum_{k_1+\cdots+k_n=B_1} \Delta_1(k_1,\ldots,k_n) \frac{(z_1)^{k_1}}{k_1!} \cdots \frac{(z_n)^{k_n}}{k_n!}, \tag{2.6}$$

where

$$\Delta_1(k_1,\ldots,k_n) = \phi_1(k_1)\cdots\phi_s(k_s)\psi(k_1,\ldots,k_s)\phi(k_{s+1},\ldots,k_n) \times \Gamma\left(\rho_1 + \sum_{i=1}^{n} k_i\right). \tag{2.7}$$

$\phi_i(k_i)$ for $i = 1,\ldots,s$ and $\psi(k_1,\ldots,k_s)$ are defined by (1.4) and (1.5) respectively, and

$$\phi(k_{s+1},\ldots,k_n) = \begin{cases} (-N)_{\sum_{i=s+1}^{n}(M_i k_i)} A_1(N; k_{s+1},\ldots,k_n), & 0 \leqslant \sum_{i=s+1}^{n} M_i k_i \leqslant N \\ 0, & \sum_{i=s+1}^{n} M_i k_i > N. \end{cases} \tag{2.8}$$

Similarly,

$$\lambda_{B_2}^{(2)} = (-1)^{B_2} \sum_{K_1+\cdots+K_m=B_2} \Delta_2(K_1,\ldots,K_m) \frac{w_1^{K_1}}{K_1!} \cdots \frac{w_m^{K_m}}{K_m!}, \tag{2.9}$$

where

$$\Delta_2(K_1,\ldots,K_m) = \Phi_1(K_1)\cdots\Phi_r(K_r)\Psi(K_1,\ldots,K_r)\Phi(K_{r+1},\ldots,K_m) \times \Gamma\left(\rho_2 + \sum_{i=1}^{m} K_i\right), \tag{2.10}$$

$$\Phi_i(K_i) = \frac{\prod_{j=1}^{N_i} \Gamma(1-u_j^{(i)} + U_j^{(i)} K_i)}{\prod_{j=1}^{Q_i} \Gamma(1-v_j^{(i)}) + V_j^{(i)} K_i) \prod_{j=N_i+1}^{P_i} \Gamma(u_j + U_j^{(i)} K_i)} \quad (i = 1,\ldots,r), \tag{2.11}$$

$$\Psi(K_1,\ldots,K_r)=\left\{\prod_{j=1}^{P}\Gamma\left(g_j-\sum_{i=1}^{r}G_j^{(i)}K_i\right)\prod_{j=1}^{Q}\Gamma\left(1-h_j+\sum_{i=1}^{r}H_j^{(i)}K_i\right)\right\}^{-1} \tag{2.12}$$

and

$$\Phi(K_{r+1},\ldots,K_m)=\begin{cases}(-N')_{\sum_{i=r+1}^{m}(M_iK_i)}A_2(N';K_{r+1},\ldots,K_m), & 0\leqslant \sum\limits_{i=r+1}^{m} M_iK_i\leqslant N' \\ 0, & \sum\limits_{i=r+1}^{m} M_iK_i>N'.\end{cases} \tag{2.13}$$

Also

$$\bar{g}_{1,y^k}(p,0)=\int_0^\infty \mathrm{e}^{-p\xi}\frac{\partial^k}{\partial y^k}g(\xi,y)\bigg|_{y=0}\mathrm{d}\xi\,, \tag{2.14}$$

$$\bar{g}_{2,x^c}(0,q)=\int_0^\infty \mathrm{e}^{-q\eta}\frac{\partial^c}{\partial x^c}g(x,\eta)\bigg|_{x=0}\mathrm{d}\eta \tag{2.15}$$

and

$$g_{x^c,y^k}^{(c+k)}(x,y)=\frac{\partial^{c+k}}{\partial x^c\,\partial y^k}g(x,y). \tag{2.16}$$

Proof. To solve the double convolution integral equation (2.1), we first take the double Laplace transform of both its sides, use the convolution theorem for the two dimensional Laplace transform and arrive at the following:

$$\left\{\int_0^\infty\int_0^\infty \mathrm{e}^{-px-qy}x^{\rho_1-1}y^{\rho_2-1}H_1\begin{bmatrix}z_1x\\ \vdots\\ z_sx\end{bmatrix}H_2\begin{bmatrix}w_1y\\ \vdots\\ w_ry\end{bmatrix}\right.$$
$$\times\, S_N^{M_{s+1},\ldots,M_n}[-z_{s+1}x,\ldots,-z_nx]S_{N'}^{M_{r+1},\ldots,M_m}$$
$$\left.\times\,[-w_{r+1}y,\ldots,-w_my]\,\mathrm{d}x\,\mathrm{d}y\right\}\cdot\bar{f}(p,q)=\bar{g}(p,q), \tag{2.17}$$

where $\bar{f}(p,q)$ and $\bar{g}(p,q)$ denote the double Laplace transform of functions $f(x,y)$ and $g(x,y)$, respectively.

Now we express the H-function of several variables and the general class of polynomials occurring in the left-hand side of (2.17) in series form using (1.3) and (1.1), interchange the order of series and integrations and evaluate the x and y integrals. By making use of a known formula [11, p. 67, eq. (2.3)], expressing the equation in terms of $\bar{f}(p,q)$, reciprocating the two series occurring on the right-hand side, proceeding in a manner as given in a recent paper [2] and taking the inverse of double Laplace transform of both sides of the resulting equation and applying convolution theorem in its right-hand side, we arrive at the desired result. The details of the proof can be seen in [2].

3. Special cases

(i) If we take

$$A_1(N;k_{s+1},\ldots,k_n)=\frac{\prod_{j=1}^{E_1}(e_j)_{k_{s+1}\theta_j^{(s+1)}+\cdots+k_n\theta_j^{(n)}}}{\prod_{j=1}^{G_1}(g_j)_{k_{s+1}\psi_j^{(s+1)}+\cdots+k_n\psi_j^{(n)}}} \times\frac{\prod_{j=1}^{U_1^{(s+1)}}(u_j^{(s+1)})_{k_{s+1}\phi_j^{s+1}}\cdots\prod_{j=1}^{U_1^{(n)}}(u_j^{(n)})_{k_n\phi_j^{(n)}}}{\prod_{j=1}^{V_1^{(s+1)}}(v_j^{(s+1)})_{k_{s+1}\zeta_j^{s+1}}\cdots\prod_{j=1}^{V_1^{(n)}}(v_j^{(n)})_{k_n\zeta_j^{(n)}}} \tag{3.1}$$

in (2.1), $S_N^{M_{s+1},\ldots,M_n}[-z_{s+1},\ldots,-z_n]$ reduces to the generalized Lauricella function of Srivastava and Daoust [8, p. 454] as follows:

$$S_N^{M_{s+1},\ldots,M_n}[-z_{s+1},\ldots,-z_n]=F_{G_1\ \ :V_1^{(s+1)},\ldots,V_1^{(n)}}^{1+E_1:U_1^{(s+1)},\ldots,U_1^{(n)}}$$

$$\left[\begin{matrix}(-N:M_{s+1},\ldots,M_n),(e:\theta_1^{(s+1)},\ldots,\theta_1^{(n)}):((u^{(s+1)}):\phi^{(s+1)};\ldots;((u^{(n)}):\phi^{(n)});\\ (g:\psi_1^{(s+1)},\ldots,\psi_1^{(n)}):((v^{(s+1)}):\zeta^{(s+1)};\ldots;((v^{(n)}):\zeta^{(n)});\end{matrix}\ -z_{s+1},\ldots,-z_n\right]. \tag{3.2}$$

Similarly choosing the coefficients $A_2(N';K_{r+1},\ldots,K_m)$ appropriately $S_{N'}^{M_{r+1},\ldots,M_m}[-w_{r+1},\ldots,-w_m]$ reduces to another generalized Lauricella function given by

$$S_{N'}^{M_{r+1},\ldots,M_m}[-w_{r+1},\ldots,-w_m]=F_{G_2\ \ :V_2^{(r+1)},\ldots,V_2^{(m)}}^{1+E_2:U_2^{(r+1)},\ldots,U_2^{(m)}}$$

$$\times\left[\begin{matrix}(-N':M_{r+1},\ldots,M_m),(e':\theta_2^{(r+1)},\ldots,\theta_2^{(m)}):((f^{(r+1)}):\eta^{(r+1)});\ldots;((f^{(m)}):\eta^{(m)});\\ (g':\psi_2^{(r+1)},\ldots,\psi_2^{(m)}):((t^{(r+1)}):\nu^{(r+1)});\ldots;((t^{(m)}):\nu^{(m)});\end{matrix}\ -w_{r+1},\ldots,-w_m\right] \tag{3.3}$$

and our theorem yields the following result. (For the sake of brevity, here we shall write the functions occurring on the right-hand side of (3.2) and (3.3) as $F(-z_{s+1},\ldots,-z_n)$ and $F^*(-w_{r+1},\ldots,-w_m)$, respectively.

1. COROLLARY

The double convolution integral equation

$$\int_0^x\int_0^y (x-u)^{\rho_1-1}(y-v)^{\rho_2-1}H_1\begin{bmatrix}z_1(x-u)\\ \vdots\\ z_s(x-u)\end{bmatrix}H_2\begin{bmatrix}w_1(y-v)\\ \vdots\\ w_r(y-v)\end{bmatrix}$$

$$\times F[-z_{s+1}(x-u),\ldots,-z_n(x-u)]F^*[-w_{r+1}(y-v),\ldots,-w_m(y-v)]$$

$$\times f(u,v)\,\mathrm{d}u\,\mathrm{d}v=g(x,y) \tag{3.4}$$

possesses the solution

$$f(x,y)=\int_0^x\int_0^y (x-u)^{a-\rho_1-\mu_1-1}(y-v)^{b-\rho_2-\mu_2-1}\,W_1[x-u,y-v]$$

$$\times g_{u^a,v^b}^{(a+b)}(u,v)\,\mathrm{d}u\,\mathrm{d}v, \tag{3.5}$$

where

$$W_1[x,y]=\sum_{\nu_1,\nu_2=0}^{\infty}\frac{G_{\nu_1}^{(1)}G_{\nu_2}^{(2)}x^{\nu_1}y^{\nu_2}}{\Gamma(a+\nu_1-\rho_1-\mu_1)\Gamma(b+\nu_2-\rho_2-\mu_2)}, \tag{3.6}$$

the coefficients $G_{v_1}^{(1)}$ and $G_{v_2}^{(2)}$ can easily be obtained from $H_{v_1}^{(1)}$ and $H_{v_2}^{(2)}$, *respectively. The result holds under the conditions easily derived from the conditions mentioned with the main result.*

(ii) Again if we take $M_i = 0$, $z_i \to 0$ $(i = s+2,\ldots,n)$ in (2.1) and replace $A_1(N; k_{s+1}, 0,\ldots,0)$ by $A_{N,k_{s+1}}$. The general class of multi-variable polynomials $S_N^{M_{s+1},\ldots,M_n}[-z_{s+1},\ldots,-z_n]$ reduces to a general class of polynomials $S_N^{M_{s+1}}[-z_{s+1}]$ [5, p. 1, eq. (1)] which includes almost all the well known orthogonal polynomials as its special cases.

Similarly the multivariable polynomial $S_{N'}^{M_{r+1},\ldots,M_m}[-w_{r+1},\ldots,-w_m]$ can be reduced to $S_{N'}^{M_{r+1}}[-w_{r+1}]$ and we arrive at the following result:

2. COROLLARY

The double convolution integral equation

$$\int_0^x \int_0^y (x-u)^{\rho_1-1}(y-v)^{\rho_2-1} S_N^{M_{s+1}}[-z_{s+1}(x-u)]\, S_{N'}^{M_{r+1}}[-w_{r+1}(y-v)]$$

$$\times H_1\begin{bmatrix} z_1(x-u) \\ \vdots \\ z_s(x-u) \end{bmatrix} H_2\begin{bmatrix} w_1(y-v) \\ \vdots \\ w_r(y-v) \end{bmatrix} f(u,v)\,du\,dv = g(x,y) \tag{3.7}$$

has the solution given by

$$f(x,y) = \int_0^x \int_0^y (x-u)^{a-\rho_1-\mu_1-1}(y-v)^{b-\rho_2-\mu_2-1} W_2[x-u, y-v]$$

$$\times g_{u^a,v^b}^{(a+b)}(u,v)\,du\,dv, \tag{3.8}$$

where

$$W_2[x,y] = \sum_{v_1,v_2=0}^{\infty} \frac{J_{v_1}^{(1)} J_{v_2}^{(2)} x^{v_1} y^{v_2}}{\Gamma(a+v_1-\rho_1-\mu_1)\Gamma(b+v_2-\rho_2-\mu_2)}, \tag{3.9}$$

the coefficients $J_{v_1}^{(1)}$ and $J_{v_2}^{(2)}$ *can easily be derived from* $H_{v_1}^{(1)}$ and $H_{v_2}^{(2)}$, *respectively. The result holds under the conditions easily obtained from the conditions mentioned with the main result.*

Corollary 2 itself is quite general in nature and includes the following results as it special cases.

(iii) If we take $s = r = 1$, $p^{(1)} = q^{(1)} = 0 = P = Q$ in (ii), then each of the multivariable H_1 and H_2-functions reduce to Fox's H-function. Also, if we take $N = N' = 0$, then the polynomials $S_0^{M_2}[-z_2]$ and $S_0^{M_2}[-w_2]$ will reduce to $A_1(0,0)$ and $A_2(0,0)$, which can be taken to be unity without loss of generality and we arrive at a result which in essence is same as that given by Raina and Koul [4, p. 366, Cor. (1)].

(iv) If we take $s = r = 1$, $z_2 = w_2 = -1$, $p^{(1)} = q^{(1)} = 0 = P = Q$ in (ii) and further reduce the Fox's H-functions thus obtained to $\exp(-z_1)$ and $\exp(-w_1)$ [10, p. 18, eq. (2.6.2)] and let $z_1 \to 0$ and $w_1 \to 0$, these functions reduce to unity and we obtain a result given by Jain [3, p. 100, eq. (2.1)].

(v) The one variable-analogue of Corollary 2 corresponds to the result obtained recently by Gupta *et al* [2, p. 188, eq. (2.1)].

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References

[1] Ditkin V A and Prudnikov A P, *Operational calculus in two variables and its applications* (1962) (New York: Pergamon Press)
[2] Gupta K C, Jain Rashmi and Agrawal Pawan, Convolution integral equations involving a general class of polynomials and the multivariable H-function, *Proc. Indian Acad. Sci.* **105** (1995) 187–192
[3] Jain Rashmi, Double convolution integral equations involving the general polynomials, *Ganita Sandesh* **4** (1990) 99–103
[4] Raina R K and Koul C L, On convolution integral equations, *Indian J. Pure Appl. Math.* **13** (1982) 362–369
[5] Srivastava H M, A contour integral involving Fox's H-function, *Indian J. Math.* **14** (1972) 1–6
[6] Srivastava H M and Buschman R G, *Convolution integral equations with special function kernels* (1977) (New York/London/Sydney/Toronto: Halsted Press)
[7] Srivastava H M and Buschman R G, *Theory and applications of convolution integral equations* (1992) (Dordrecht/Boston/London: Kluwer Academic Publishers)
[8] Srivastava H M and Daoust M C, Certain generalized Neumann expansions associated with the Kampé de Fériet function, *Nederl. Akad. Wetensch. Indag. Math.* **31** (1969) 449–457
[9] Srivastava H M and Garg M, Some integrals involving a general class of polynomials and the multivariable H-function, *Rev. Roumaine Phys.* **32** (1987) 685–692
[10] Srivastava H M, Gupta K C and Goyal S P, *The H-functions of one and two variables with applications* (1982) (New Delhi: South Asian Publishers)
[11] Srivastava H M, Koul C L and Raina R K, A class of convolution integral equations, *J. Math. Anal. Appl.* **108** (1985) 63–71