Proc. Indian Acad. Sci. (Math. Sci.), Vol. 107, No. 1, February 1997, pp. 1–12.

On partial sums of mock theta functions of order three

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MS received 19 April 1996

Abstract. The object of this paper is to define and study the properties of partial mock theta functions of order three, on the lines Ramanujan had studied partial θ-functions. These new partial functions have been expressed in terms of basic hypergeometric function ${}_2\Phi_1$. Their continued fractions representations have also been given.

Keywords. Mock theta function; basic hypergeometric series; continued fractions.

1. Introduction

Ramanujan during the last days of his life discovered the mock theta functions which were communicated by him in his last letter to Hardy in February 1920. He classified these functions as of order 3, 5 and 7, without assigning any reason. In some of his unpublished works (now published as the 'Lost Note Book'), a number of identities and expansion formulae for, partial sums of the Jacobi's theta functions are mentioned without proof.

A study of these sums and expansions has been made by Andrews [3]. A little later, Agarwal [1, p.121–128] showed that these relations, in fact, belong to a general class of basic hypergeometric identities and one could find a number of similar relations by making a study of three and four term relations between ${}_3\Phi_2$ series given by Sears [5].

Andrews in his work has shown that some of these partial theta function identities have interesting number theoretic interpretations.

The above considerations and the interest shown by a number of mathematician in the study of partial theta function identities motivated me to define partial mock theta functions and study of their associated identities.

The object of this paper is to define and study the relations between partial mock theta functions of order three. It has been shown that these partial mock theta functions are also interrelated with relations similar to the one for the mock theta function of order three.

It would be interesting to study the number theoretic interpretations, if any, of these partial mock theta functions.

2. Notation

Let

$$(a)_n=(1-a)(1-aq)(1-aq^2),\dots,(1-aq^{n-1}),\quad n>0,$$

$$(a)_0=1.$$

Also

$$(a_1,a_2,\dots,a_r)_n=(a_1)_n(a_2)_n,\dots,(a_r)_n$$

and

$$(a)_\infty=\prod_{n=0}^{\infty}(1-aq^n).$$

A basic hypergeometric series is then defined for $|q| < 1$, as

$$ {}_{r+1}\Phi_r \begin{bmatrix} \alpha_1, \alpha_2, \ldots, \alpha_{r+1}; & \\ & z \\ \rho_1, \rho_2, \ldots, \rho_r & \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(\alpha_1,\alpha_2,\ldots,\alpha_{r+1})_n Z^n}{(q,\rho_1,\rho_2,\ldots,\rho_r)_n}. \tag{2.1} $$

The series (2.1) converges for $|z| < 1$.
Further, let

$$ {}_{r+1}\Phi_r \begin{bmatrix} \alpha_1, \alpha_2, \ldots, \alpha_{r+1}; & \\ & z \\ \rho_1, \rho_2, \ldots, \rho_r & \end{bmatrix}_m $$

denote the partial sum

$$ \sum_{n=0}^{m} \frac{(\alpha_1,\alpha_2,\ldots,\alpha_{r+1})_n z^n}{(q,\rho_1,\rho_2,\ldots,\rho_r)_n}. $$

of the generalized basic hypergeometric series.

3. Definitions of partial mock theta function of order three

Fine [4] represented all the third order mock theta functions as a limiting cases of a

$$ {}_2\Phi_1 \begin{bmatrix} \alpha, q; & \\ & x \\ \beta & \end{bmatrix}. $$

By using Heine's transformation

$$ (x,e)_\infty \Phi \begin{bmatrix} a, b; & \\ & x \\ e & \end{bmatrix} = (b,ax)_\infty \Phi \begin{bmatrix} \frac{e}{b}, x; & \\ & b \\ ax & \end{bmatrix}, \tag{3.1} $$

we have given alternative forms for the third order mock theta functions as

$$ f(q) = 2 - \frac{(q)_\infty}{(-q)_\infty^2} \Phi \begin{bmatrix} -1, -q; & \\ & q \\ 0 & \end{bmatrix}, \tag{3.2} $$

$$ \phi(q) = (1+i) - \frac{i(q)_\infty}{(iq,-iq)_\infty} \Phi \begin{bmatrix} -i, iq; & \\ & q \\ 0 & \end{bmatrix}, \tag{3.3} $$

$$ \Psi(q) = \frac{q(q^2,-q^3;q^2)_\infty}{(q;q^2)_\infty} \phi(q^2) \begin{bmatrix} q, 0; & \\ & q^2 \\ -q^3 & \end{bmatrix}, \tag{3.4} $$

$$\chi(q)=(1+\omega)-\frac{\omega(q)_\infty}{(-\omega q,-\omega^2 q)_\infty}\Phi\begin{bmatrix}-\omega,\ -\omega^2\ q; & \\ & q\\ 0 & \end{bmatrix}, \tag{3.5}$$

$$\omega(q)=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^2}\Phi^{(q^2)}\begin{bmatrix}q,\ q; & \\ & q^2\\ 0 & \end{bmatrix}, \tag{3.6}$$

$$\gamma(q)=\frac{(q^2,-q^2;q^2)_\infty}{(-q;q^2)_\infty}\Phi^{(q^2)}\begin{bmatrix}-q,\ 0; & \\ & q^2\\ -q^2 & \end{bmatrix}, \tag{3.7}$$

and

$$\rho(q)=\frac{(q^2;q^2)_\infty}{(q\omega;q\omega^2;q^2)_\infty}\Phi^{(q^2)}\begin{bmatrix}\omega q,\ \omega^2 q; & \\ & q^2\\ 0 & \end{bmatrix}. \tag{3.8}$$

Denoting the series

$$\sum_{n=0}^{m}\frac{(a,b)_n x^n}{(c,q)_n}\quad\text{as}\quad\Phi\begin{bmatrix}a,\ b; & \\ & x\\ c & \end{bmatrix}_m$$

and by using the following transformation

$$\Phi\begin{bmatrix}D,\ E; & \\ & q\\ C & \end{bmatrix}_N=\frac{(DEq^{N+1})_\infty(q^{N+1})_\infty}{(Eq^{N+1})_\infty(Dq^{N+1})_\infty}\Phi\begin{bmatrix}Cq^N,\ D,\ E; & \\ & q\\ C,\ DEq^{N+1} & \end{bmatrix}, \tag{3.9}$$

and (3.1) we have defined the partial mock theta functions of order three as

(i)
$$f_N(q)=2-\frac{(q)_\infty}{(-q)^2{}_\infty}\Phi\begin{bmatrix}-1,\ -q; & \\ & q\\ 0 & \end{bmatrix}_N, \tag{3.10}$$

$$=2-\frac{(q)_\infty}{(-q)_\infty^2}\frac{(-q)_{N+1}(-q)_N}{(q)_N}\Phi\begin{bmatrix}-q^{1+N},\ q; & \\ & -q\\ -q & \end{bmatrix}, \tag{3.11}$$

(ii)
$$\phi_N(q)=(1+i)-\frac{i\,(q)_\infty}{(iq,-iq)_\infty}\Phi\begin{bmatrix}-i,\ iq; & \\ & q\\ 0 & \end{bmatrix}_N, \tag{3.12}$$

$$=(1+i)-\frac{i(q)_\infty}{(iq,-iq)_\infty}\frac{i(q)_{N+1}(-iq)_N}{(q)_N}\Phi\begin{bmatrix}-iq^{1+N},\ q; & \\ & iq\\ -iq & \end{bmatrix}, \tag{3.13}$$

(iii)
$$\Psi_N(q)=\frac{q(q^2,-q^3;q^2)_\infty}{(q;q^2)_\infty}\Phi^{(q^2)}\begin{bmatrix} q, & 0; & \\ & & q^2 \\ -q^3 & & \end{bmatrix}_N, \tag{3.14}$$

$$=\frac{q(q^2,-q^3;q^2)_\infty}{(q;q^2)_\infty}\frac{(q;q^2)_{N+1}}{(q^2;q^2)_N(-q^3;q^2)_{N+1}}\Phi\begin{bmatrix} -q^2, & q^2; & \\ & & q \\ -q^{2N+5} & & \end{bmatrix}, \tag{3.15}$$

(iv)
$$\chi_N(q)=(1+\omega)-\frac{\omega(q)_\infty}{(-\omega q,-\omega^2 q)_\infty}\Phi\begin{bmatrix} -\omega, & -\omega^2 q; & \\ & & q \\ 0 & & \end{bmatrix}_N, \tag{3.16}$$

$$=(1+\omega)-\frac{\omega(q)_\infty}{(-\omega q,-\omega^2 q)_\infty}\frac{(-\omega q)_N(-\omega^2 q)_{N+1}}{(q)_N}$$

$$\times\Phi\begin{bmatrix} -\omega q^{N+1}, & q; & \\ & & -\omega^2 q \\ -\omega q & & \end{bmatrix}, \tag{3.17}$$

(v)
$$\omega_N(q)=\frac{(q^2;q^2)_\infty}{(q;q^2)^2_\infty}\Phi^{(q^2)}\begin{bmatrix} q, & q; & \\ & & q^2 \\ 0 & & \end{bmatrix}_N, \tag{3.18}$$

$$=\frac{(q^2;q^2)_\infty}{(q;q^2)^2_\infty}\frac{(q;q^2)_{N+1}(q^3;q^2)_N}{(q^2;q^2)_N}\Phi^{(q^2)}\begin{bmatrix} q^{3+2N}, & q^2; & \\ & & q \\ q^3 & & \end{bmatrix}, \tag{3.19}$$

(vi)
$$\gamma_N(q)=\frac{q^2,-q^2;q^2)_\infty}{(-q;q^2)_\infty}\Phi^{(q^2)}\begin{bmatrix} -q, & 0; & \\ & & q^2 \\ -q^2 & & \end{bmatrix}_N, \tag{3.20}$$

$$=\frac{(q^2,-q^2;q^2)_\infty}{(-q;q^2)_\infty}\frac{(-q;q^2)_{N+1}}{(q^2;q^2)_N(-q^2;q^2)_{N+1}}\Phi^{(q^2)}\begin{bmatrix} q^2, & q; & \\ & & -q \\ -q^{4+2N} & & \end{bmatrix}, \tag{3.21}$$

and

(vii)
$$\rho_N(q)=\frac{(q^2;q^2)_\infty}{(q\omega,q\omega^2;q^2)_\infty}\Phi^{(q^2)}\begin{bmatrix} \omega q, & \omega^2 q; & \\ & & q^2 \\ 0 & & \end{bmatrix}_N, \tag{3.22}$$

$$=\frac{(q^2;q^2)_\infty}{(q\omega;q\omega^2;q^2)_\infty}\frac{(q\omega^2;q^2)_{N+1}(q^3\omega;q^2)_N}{(q^2;q^2)_N}\Phi\begin{bmatrix} \omega q^{3+2N,q^2}; & \\ & \omega^2 q \\ \omega q^3 & \end{bmatrix}\cdot \tag{3.23}$$

4. A summation formula and a lemma

A transformation formula of Sears [6, p.167] is

$$
{}_4\Phi_3\left[\begin{matrix} q^{-n},\ a,\ b,\ c; \\ d,\ e,\ f \end{matrix}\ q\right] = \frac{\left(\frac{aq^{1-n}}{e}, \frac{aq^{1-n}}{f}\right)_n}{(e,f)_n}\left(\frac{bc}{d}\right)^n \ {}_4\Phi_3\left[\begin{matrix} q^{-n}, a, d/b, d/c; \\ d,\ \frac{aq^{1-n}}{e},\ \frac{aq^{1-n}}{f} \end{matrix}\right] \tag{4.1}
$$

Putting $d=q^{-n}$ and $c=q$ in (4.1), we get for $ef=abq^2$ and using q-Saalschütz's theorem

$$
\begin{aligned}
{}_3\Phi_2\left[\begin{matrix} a,\ b,\ q; \\ e,\ f \end{matrix}\ q\right]_n &= \frac{\left(\frac{aq^{1-n}}{e}, \frac{aq^{1-n}}{f}\right)_n}{(e,f)_n}(bq^{1+n})^n \\
&\quad {}_3\Phi_2\left[\begin{matrix} a, & q^{-n}/b, & q^{-n-1} \\ aq^{1-n}/e, & aq^{1-n}/f & \end{matrix}\ ;q\right]_n \\
&= \frac{\left(\frac{aq^{1-n}}{e}, \frac{aq^{1-n}}{f}\right)_n}{(e,f)_n}(bq^{1+n})^n \\
&\quad \left\{ {}_3\Phi_2\left[\begin{matrix} a, & q^{-n}/b, & q^{-n-1} \\ aq^{1-n}/e, & aq^{1-n}/f & \end{matrix}\ q\right]_{(n+1)} \right. \\
&\quad \left. - \frac{(a)_{n+1}\left(\frac{q^{-n}}{b}\right)_{n+1}(q^{-n-1})_{n+1}q^{n+1}}{\left(\frac{aq^{1-n}}{e}\right)_{n+1}\left(\frac{aq^{1-n}}{f}\right)_{n+1}(q)_{n+1}} \right\} \\
&= \frac{\left(\frac{aq^{1-n}}{e}, \frac{aq^{1-n}}{f}\right)_n}{(e,f)_n}(bq^{1+n})^n \left\{ \frac{\left(\frac{q^{1-n}}{e}\right)_{n+1}\left(\frac{abq}{e}\right)_{n+1}}{\left(\frac{aq^{1-n}}{e}\right)_{n+1}\left(\frac{bq}{e}\right)_{n+1}} \right. \\
&\quad \left. - \frac{(a)_{n+1}\left(\frac{q^{-n}}{b}\right)_{n+1}(q^{-n-1})_{n+1}q^{n+1}}{\left(\frac{aq^{1-n}}{e}\right)_{n+1}\left(\frac{aq^{1-n}}{f}\right)_{n+1}(q)_{n+1}} \right\} \\
&= \frac{(q-e)(e-abq)}{(aq-e)(e-bq)}\left[1 - \frac{(a,b)_{n+1}}{\left(\frac{e}{q}, \frac{abq}{e}\right)_{n+1}}\right].
\end{aligned} \tag{4.2}
$$

Equation (4.2) gives the required summation formula.

Next we prove the following Lemma:

Lemma. We know that

$$\sum_{m=0}^{p}\delta_m\sum_{r=0}^{m}\alpha_r=\sum_{r=0}^{p}\alpha_r\sum_{m=r}^{p}\delta_m=\sum_{r=0}^{p}\alpha_r\sum_{m=0}^{p}\delta_m-\sum_{r=0}^{p}\alpha_r\sum_{m=0}^{r-1}\delta_m. \tag{4.3}$$

COROLLARY

Taking $p\to\infty$ in the Lemma, subject to convergence, we get

$$\sum_{m=0}^{\infty}\delta_m\sum_{r=0}^{m}\alpha_r=\left(\sum_{r=0}^{\infty}\alpha_r\right)\left(\sum_{m=0}^{\infty}\delta_m\right)-\sum_{r=1}^{\infty}\alpha_r\sum_{m=0}^{r-1}\delta_m. \tag{4.4}$$

Choosing α_r such that $\sum_{r=0}^{m}\alpha_r$ is some partial mock theta function we choose δ_m such that $\sum_{m=0}^{r-1}\delta_m$ is summable to β_m (say) and $\sum_{r=1}^{\infty}\alpha_r\beta_r$ is some other mock theta function. Such a choice of δ_r and α_r gives us a relation between a given partial mock theta function, which is a complete form and possibly another mock theta function.

5. A general theorem

In this section we give two general theorems from which we shall derive a partial mock theta function identities.

Theorem. *Choosing*

$$\delta_m=\frac{(aq-e)(e-bq)}{(q-e)(e-abq)}\frac{(a,b)_m q^m}{\left(e,\dfrac{abq^2}{e}\right)_m}$$

in (4.4) *and evaluating β_r by* (4.2) *gives the following theorem*:

$$\begin{aligned}&\frac{(aq-e)(e-bq)}{(q-e)(e-abq)}\sum_{m=0}^{\infty}\frac{(a,b)_m q^m}{\left(e,\dfrac{abq^2}{e}\right)_m}\sum_{r=0}^{m}\alpha_r\\&=\left[1-\frac{(a,b)_\infty}{\left(\dfrac{e}{q},\dfrac{abq}{e}\right)_\infty}\right]\sum_{m=0}^{\infty}\alpha_m-\sum_{m=1}^{\infty}\alpha_m\left[1-\frac{(a,b)_m}{\left(\dfrac{e}{q},\dfrac{abq}{e}\right)_m}\right]\\&=-\frac{(a,b)_\infty}{\left(\dfrac{e}{q},\dfrac{abq}{e}\right)_\infty}\sum_{m=0}^{\infty}\alpha_m+\sum_{m=0}^{\infty}\frac{(a,b)_m}{\left(\dfrac{e}{q},\dfrac{abq}{e}\right)_m}\alpha_m.\end{aligned} \tag{5.1}$$

6. Relations between partial mock theta function of order three

We use Theorem 1 (5·1) for particular values of α_r to get a number of partial mock theta function identities

(i) For $\alpha_r=(-1,-q)_r q^r/(q)_r$ in Theorem 1, we get

$$\frac{(aq-e)(e-bq)}{(q-e)(e-abq)}\sum_{m=0}^{\infty}\frac{(a,b)_m q^m}{\left(e,\frac{abq^2}{e}\right)_m}\bar{f}_m(q)=-\frac{(a,b)_\infty}{\left(\frac{e}{q},\frac{abq}{e}\right)_\infty}\bar{f}(q)$$
$$+\frac{(q)_\infty}{(-q)_\infty^2}\sum_{m=0}^{\infty}\frac{(a,b,-1,-q)_m q^m}{\left(\frac{e}{q},\frac{abq}{e},q\right)_m}, \tag{6.1}$$

where $2-f_m(q)=\bar{f}_m(q), 2-f(q)=\bar{f}(q)$.

(ii) For $\alpha_r=(-i,iq)_r q^r/(q)_r$ in (5.1), we get

$$\frac{(aq-e)(e-bq)}{(q-e)(e-abq)}\sum_{m=0}^{\infty}\frac{(a,b)_m q^m}{\left(e,\frac{abq^2}{e}\right)_m}\overline{\phi_m(q)}=-\frac{(a,b)_\infty}{\left(\frac{e}{q},\frac{abq}{e}\right)_\infty}\overline{\phi(q)}$$
$$+\frac{i(q)_\infty}{(iq,-iq)_\infty}\sum_{m=0}^{\infty}\frac{(a,b,-i,iq)_m q^m}{\left(\frac{e}{q},\frac{abq}{e},q\right)_m}, \tag{6.2}$$

where $\overline{\phi_m(q)}=(1+i)-\phi_m(q)$ and $\overline{\phi(q)}=(1+i)-\phi(q)$.

(iii) Letting $q\to q^2$ and then taking $\alpha_r=(q;q^2)_r q^{2r}/(q^2,-q^3;q^2)_r$ in (5.1), we get

$$\frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)}\sum_{m=0}^{\infty}\frac{(a,b;q^2)_m q^{2m}}{\left(e,\frac{abq^4}{\varepsilon};q^2\right)_m}\Psi_m(q)=-\frac{(a,b;q^2)_\infty}{\left(\frac{e}{q^2},\frac{abq^2}{e};q^2\right)_\infty}\Psi(q)$$
$$+\frac{q(q^2,-q^3;q^2)_\infty}{(q;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(a,b;q;q^2)_m q^{2m}}{\left(\frac{e}{q^2},\frac{abq^2}{e},q^2,-q^3;q^2\right)_m}. \tag{6.3}$$

(iv) Letting $q\to q^2$ and then taking $\alpha_r=(-q;q^2)_r q^{2r}/(-q^2,q^2;q^2)_r$ in (5.1), we get

$$\frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)}\sum_{m=0}^{\infty}\frac{(a,b;q^2)_m q^{2m}}{\left(e,\frac{abq^4}{e};q^2\right)_m}\gamma_m(q)=-\frac{(a,b;q^2)_\infty}{\left(\frac{e}{q^2},\frac{abq^2}{e};q^2\right)_\infty}\gamma(q)$$
$$+\frac{(q^2,-q^2;q^2)_\infty}{(-q;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(a,b;-q;q^2)_m q^{2m}}{\left(\frac{e}{q^2},\frac{abq^2}{e},-q^2,q^2;q^2\right)_m}. \tag{6.4}$$

(v) Letting $q\to q^2$ and then taking $\alpha_r=(q;q^2)_r^2 q^{2r}/(q^2;q^2)_r$ in (5.1), we have

$$\frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)}\sum_{m=0}^{\infty}\frac{(a,b;q^2)_m q^{2m}}{\left(e,\frac{abq^4}{e};q^2\right)_m}\omega_m(q)=-\frac{(a,b;q^2)_\infty}{\left(\frac{e}{q^2},\frac{abq^2}{e};q^2\right)_\infty}\omega(q)$$
$$+\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty^2}\sum_{m=0}^{\infty}\frac{(a,b,q,q;q^2)_m q^{2m}}{\left(\frac{e}{q^2},\frac{abq^2}{e},q^2;q^2\right)_m}. \tag{6.5}$$

(vi) Letting $q \to q^2$ and then taking $\alpha_r = (q\omega, q\omega^2; q^2)_r q^{2r}/(q^2;q^2)_r$ in (5.1), we get

$$\frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)} \sum_{m=0}^{\infty} \frac{(a,b;q^2)_m q^{2m}}{\left(e, \frac{abq^4}{e}; q^2\right)_m} \rho_m(q) = -\frac{(a,b;q^2)_\infty}{\left(\frac{e}{q^2}, \frac{abq^2}{e}; q^2\right)_\infty} \rho(q)$$
$$+ \frac{(q^2;q^2)_\infty}{(q\omega, q\omega^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(a,b,q\omega,q\omega^2;q^2)_m q^{2m}}{\left(\frac{e}{q^2}, \frac{abq^2}{e}, q^2; q^2\right)_m}. \tag{6.6}$$

(vii) Letting $\alpha_r = (-\omega, -\omega^2 q)_r q^r/(q)_r$ in (5.1), we get

$$\frac{(aq-e)(e-bq)}{(q-e)(e-abq)} \sum_{m=0}^{\infty} \frac{(a,b)_m q^m}{\left(e, \frac{abq^2}{e}\right)_m} \bar{\chi}_m(q) = -\frac{(a,b)_\infty}{\left(\frac{e}{q}, \frac{abq}{e}\right)_\infty} \bar{\chi}(q)$$
$$+ \frac{\omega(q)_\infty}{(-\omega q, -\omega^2 q)_\infty} \sum_{m=0}^{\infty} \frac{(a,b,-\omega,-\omega^2 q)_m q^m}{\left(\frac{e}{q}, \frac{abq}{e}, q\right)_m} \tag{6.7}$$

where $(1+\omega) - \chi_m(q) = \bar{\chi}_m(q)$, $\quad (1+\omega) - \chi(q) = \bar{\chi}(q)$.

7. Special cases

In this section we mention some special cases of our results in §6.

(i) Putting $e = -q$, $a = b = 0$ in (6.1), we get

$$(-q)_\infty \sum_{m=0}^{\infty} \frac{q^m \overline{f_m(q)}}{(-q)_m} + \overline{f(q)} = \frac{2}{1+q}.$$

(ii) Putting $e = -q$, $a = q/b$ and then $b = -i$ in (6.1), we get

$$\frac{(1+i)(1+qi)}{2i(1+q)} \sum_{m=0}^{\infty} \frac{(-i,qi)_m q^m}{(-q,-q^2)_m} \overline{f_m(q)} + \frac{(-i,qi)_\infty}{(-1,-q)_\infty} \overline{f(q)} = \frac{(iq,-iq)_\infty}{i(-q)^2_\infty} \overline{\phi(q)}.$$

(iii) Putting $e = -iq$, $a = b = 0$ in (6.2), we get

$$\frac{i}{1+i} \sum_{m=0}^{\infty} \frac{q^m}{(-iq)_m} \overline{\phi_m(q)} + \frac{1}{(-i)_\infty} \overline{\phi(q)} = \frac{i}{(1-iq)(-iq)_\infty}.$$

(iv) Putting $e = -iq$, $a = q/b$, and then $b = -1$ in (6.2), we get

$$\frac{(q-i)(1-i)}{(1+i)(q+i)} \sum_{m=0}^{\infty} \frac{(-1,-q)_m q^m}{(-iq,iq^2)_m} \overline{\phi_m(q)} + \frac{(-1,-q)_\infty}{(-i,iq)_\infty} \overline{\phi(q)} = \frac{i(-q)^2_\infty}{(iq,-iq)_\infty} \overline{f(q)}.$$

(v) Putting $a = 0$, $b = -q^3$ and then $e = 0$ in (6.3), we get

$$-q^3 \sum_{m=0}^{\infty} (-q^3;q^2)_m q^{2m} \Psi_m(q) + (-q^3;q^2)_\infty \Psi(q) = \frac{q(-q^3;q^2)_\infty}{(1-q)}.$$

(vi) Putting $a = -q^3$, $b = 0$ and then $e = -q^4$ in (6.3), we get

$$\frac{q(1-q)}{1+q^2} \sum_{m=0}^{\infty} \frac{(-q^3;q^2)_m q^{2m}}{(-q^4;q^2)_m} \Psi_m(q) + \frac{(-q^3;q^2)_\infty}{(-q^2;q^2)_\infty} \Psi(q) = \frac{q(-q^3;q^2)_\infty}{(-q^2;q^2)_\infty} \gamma(-q).$$

(vii) Putting $a = -q^2$, $b = 0$ and then $e = 0$ in (6.4), we get

$$-q^2 \sum_{m=0}^{\infty} (-q^2; q^2)_m q^{2m} \gamma_m(q) + (-q^2; q^2)_\infty \gamma(q) = \frac{(-q^2;q^2)_\infty}{(1+q)}.$$

(viii) Putting $e = q^5$, $a = -q^2$, $b = 0$ in (6.4), we get

$$-\frac{(1+q)q^2}{(1-q^3)} \sum_{m=0}^{\infty} \frac{(-q^2;q^2)_m q^{2m}}{(q^5;q^2)_m} \gamma_m(q) + \frac{(-q^2;q^2)_\infty}{(q^3;q^2)_\infty} \gamma(q) = -\frac{(-q^2;q^2)_\infty}{q(q^3;q^2)_\infty} \Psi(-q).$$

(ix) Putting $e = q^3$, $a = b = 0$ in (6.5), we get

$$-\frac{q}{(1-q)} \sum_{m=0}^{\infty} \frac{q^{2m}}{(q^3; q^2)_m} \omega_m(q) + \frac{1}{(q;q^2)_\infty} \omega(q) = \frac{1}{(1-q)(q;q^2)_\infty}.$$

(x) Putting $e = q^3$, $a = q^2/b$ and then $b = -1$ in (6.5), we get

$$-\frac{(1+q)^2}{(1-q)^2} \sum_{m=0}^{\infty} \frac{(-1,-q^2;q^2)_m q^{2m}}{(q^3;q^2)_m^2} \omega_m(q) + \frac{(-1,-q^2;q^2)_\infty}{(q;q^2)_\infty^2} \omega(q)$$
$$= \frac{(-q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \overline{f(q^2)}.$$

(xi) Putting $a = b = 0$, $e = q^3\omega^2$ in (6.6), we get

$$-\frac{q\omega^2}{(1-q\omega^2)} \sum_{m=0}^{\infty} \frac{q^{2m}}{(q^3\omega^2; q^2)_m} \rho_m(q) + \frac{1}{(q\omega^2;q^2)_\infty} \rho(q) = \frac{1}{(1-q\omega)(q\omega^2;q^2)_\infty}.$$

(xii) Putting $e = q^3\omega$, $a = q^2/b$ and then $b = -1$, in (6.6), we get

$$\frac{(q+\omega)(1+q\omega)}{(1-q\omega)(q-\omega)} \sum_{m=0}^{\infty} \frac{(-1,-q^2;q^2)_m q^{2m}}{(q^3\omega, q^3\omega^2;q^2)_m} \rho_m(q)$$
$$+ \frac{(-1,-q^2;q^2)_\infty}{(q\omega,q\omega^2;q^2)_\infty} \rho(q) = \frac{(-q^2;q^2)_\infty^2}{(q\omega, q\omega^2;q^2)_\infty} \overline{f(q^2)}.$$

(xiii) Putting $a = b = 0$, $e = -\omega q$ in (6.7), we get

$$\frac{\omega}{(1+\omega)} \sum_{m=0}^{\infty} \frac{q^m}{(-\omega q)_m} \overline{\chi_m}(q) + \frac{1}{(-\omega)_\infty} \overline{\chi(q)} = \frac{\omega}{(1+\omega^2 q)(-\omega q)_\infty}.$$

(xiv) Change q to q^2, then $a = \omega q$, $b = \omega^2 q$, $e = -\omega q^2$ in (6.7) and we get

$$\frac{\omega^2(1+q)(1+\omega q)}{(1+\omega)(\omega+q^2)} \sum_{m=0}^{\infty} \frac{(\omega q,\omega^2 q;q^2)_m q^{2m}}{(-\omega q^2,-q^4\omega^2;q^2)_m} \bar{\chi}_m(q^2)$$
$$+ \frac{(\omega q,\omega^2 q;q^2)_\infty}{(-\omega,-\omega^2q^2;q^2)_\infty} \bar{\chi}(q^2) + \frac{\omega(q\omega,q\omega^2;q^2)_\infty}{(-\omega q^2,-q^2\omega^2)_\infty} \rho(q).$$

8. The continued fractions, Padé approximants and their convergents

Agarwal [2] represented the continued fractions, Padé approximants and their convergents for ${}_2\Phi_1\left[\begin{matrix}\alpha, q; \\ \gamma\end{matrix}\; x\right]$ as follows:

$$
{}_2\Phi_1\left[\begin{matrix}\alpha,\ q; & \\ & x \\ \gamma & \end{matrix}\right]=\frac{1}{1+q\gamma_1}+\frac{1}{q\gamma_2}+\frac{1}{q\gamma_3}+\frac{1}{---}, \tag{8.1}
$$

where

$$
\gamma_{2n+1}=\frac{(1-\alpha q^n)(1-\gamma q^{n-1})q^n}{(1-\gamma q^{2n-1})(1-\gamma q^{2n})}
$$

and

$$
\gamma_{2n+2}=\frac{(\alpha-\gamma q^n)(1-q^{n+1})q^n}{(1-\gamma q^{2n})(1-\gamma q^{2n+1})}.
$$

The Padé approximants (n,n) and $(n,n+1)$ are given by

$$
R_{n,n}=\frac{P_{n,n}}{Q_{n,n}}, \tag{8.2}
$$

where

$$
P_{n,n}=\sum_{k=0}^{n}\frac{(\alpha)_k x^k}{(\gamma)_k}\sum_{s=0}^{n-k}\frac{(q^{-n})_s(q^{-n}/\alpha)_s}{(q)_s(q^{1-2n}/\gamma)_s}\left(\frac{x\alpha q}{\gamma}\right)^s
$$

and

$$
Q_{n,n}={}_2\Phi_1\left[q^{-n},q^{-n}/\alpha;\,q^{1-2n}/\gamma;\frac{x\alpha q}{\gamma}\right],
$$

and

$$
R_{n,n+1}=\frac{P_{n,n+1}}{Q_{n,n+1}}, \tag{8.3}
$$

where

$$
P_{n,n+1}=\frac{(-x)^h q^{3n(1-n)/2}(q^2;q)_n(q\alpha;q)_n(q\gamma;q)_n}{(q;q)_n(q\gamma;q)_{2n}}
$$

$$
\times\sum_{k=0}^{n}\frac{(q^{-n})_k(\alpha)_k(\gamma q^{n+1})_k q^k}{(\gamma)_k(\alpha q)_k(q^2)_k}\,{}_3\Phi_2\left[\begin{matrix}q^{-n+k};\ \gamma q^{n+k+1},\ q; & \\ & \dfrac{q}{x} \\ q^{2+k},\ \ \alpha q^{1+k} & \end{matrix}\right]
$$

and

$$
Q_{n,n+1}={}_2\Phi_1\left[\begin{matrix}q^{-n-1},\ q^{-n}/\alpha; & \\ & \dfrac{x\alpha q}{\gamma} \\ q^{-2n}/\gamma & \end{matrix}\right].
$$

From (7.1), (7.2) and (7.3) the continued fraction, Padé approximants and their covergents for the third order partial mock theta function $\Phi\left[\begin{matrix}-q^{1+m},\ q; & \\ & -q \\ -q & \end{matrix}\right]$ are as follows:

$$
\frac{(-q)_\infty^2(q)_m}{(q)_\infty(-q)_{m+1}(-q)_m}\{2-f_m(q)\}=\frac{1}{1+q\gamma_1}+\frac{1}{q\gamma_2}+\frac{1}{q\gamma_3}+\frac{1}{---},
$$

where

$$\gamma_{2n+1}=\frac{(1+q^{m+n+1})(1+q^n)q^n}{(1+q^{2n})(1+q^{2n+1})}$$

and

$$\gamma_{2n+2}=\frac{(-q^{1+m}+q^{1+n})(1-q^{n+1})q^n}{(1+q^{2n+1})(1+q^{2n+2})},$$

and if

$$R_{n,n}=\frac{P_{n,n}}{Q_{n,n}}$$

then

$$P_{n,n}=\sum_{k=0}^{n}\frac{(-q^{1+m})_k q^k}{(-q)_k}\sum_{s=0}^{n-k}\frac{(q^{-n})_s(-q^{-n-m-1})_s(-q^{2+m})^s}{(q)_s(-q^{-2n})_s}$$

and

$$Q_{n,n}={}_2\Phi_1\left[\begin{matrix}q^{-n}, & -q^{-n-m-1}; & \\ & & -q^{2+m}\\ -q^{-2n} & & \end{matrix}\right],$$

and

$$R_{n,n+1}=\frac{P_{n,n+1}}{Q_{n,n+1}},$$

where

$$P_{n,n+1}=\frac{q^n q^{3n(1-n)/2}(q^2)_n(-q^{2+m})_n(-q^2)_n}{(q)_n(-q^2)_{2n}}$$
$$\times\sum_{k=0}^{n}\frac{(q^{-n})_k(-q^{1+m})_k(-q^{2+n})_k q^k}{(-q)_k(-q^{2+m})_k(q^2)_k}\,{}_3\Phi_2\left[\begin{matrix}q^{-n+k}, & -q^{n+k+2}, & q; & \\ & & & -1\\ q^{2+k}, & -q^{m+k+2} & & \end{matrix}\right]$$

and

$$Q_{n,n+1}={}_2\Phi_1\left[\begin{matrix}q^{-n-1}, & -q^{-n-m-1}; & \\ & & -q^{m+2}\\ -q^{-2n-1} & & \end{matrix}\right].$$

Similarly, one can get continued fractions, Padé approximants and their convergents for the other six partial third order mock theta functions $\phi_N(q)$, $\Psi_N(q)$, $\chi_N(q)$, $\omega_N(q)$, $\gamma_N(q)$ and $\rho_N(q)$. The Padé approximants are useful in finding the bounds for the partial mock theta functions by taking simple values of n.

Acknowledgement

The author is grateful to Prof. R P Agarwal for his kind guidance during the preparation of this paper.

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