Proc. Indian Acad. Sci. (Math. Sci.), Vol. 106, No. 4, November 1996, pp. 369–377.

A theorem of the Wiener–Tauberian type for $L^1(H^n)$

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MS received 12 September 1995; revised 15 January 1996

Abstract. The Heisenberg motion group $HM(n)$, which is a semi-direct product of the Heisenberg group H^n and the unitary group $U(n)$, acts on H^n in a natural way. Here we prove a Wiener–Tauberian theorem for $L^1(H^n)$ with this $HM(n)$-action on H^n i.e. we give conditions on the "group theoretic" Fourier transform of a function f in $L^1(H^n)$ in order that the linear span of $\{{}^g f: g \in HM(n)\}$ is dense in $L^1(H^n)$, where ${}^g f(z,t) = f(g\cdot(z,t))$, for $g \in HM(n)$, $(z,t) \in H^n$.

Keywords. Heisenberg group; Gelfand pairs; class-1 representations; elementary spherical functions.

1. Introduction

The celebrated Wiener–Tauberian theorem (see theorem 9·4 in [13]) asserts that for $f \in L^1(\mathbb{R}^n)$, the closed subspace generated by the translates of f is all of $L^1(\mathbb{R}^n)$ if and only if $\hat{f}$, the Euclidean Fourier transform of f, is nowhere vanishing. However there is also a larger group, i.e., the group of rigid motions $M(n)$ acting on $\mathbb{R}^n$. Therefore we can ask for conditions on $\hat{f}$ under which the rigid motion translates of f, i.e., ${}^\sigma f$, $\sigma \in M(n)$, (where ${}^\sigma f(y) = f(\sigma\cdot y)$), generate a dense subspace of $L^1(\mathbb{R}^n)$. Using a slightly stronger version of Wiener's theorem it can be shown easily that $\overline{\operatorname{Span}\{{}^\sigma f:\ \sigma \in M(n)\}} = L^1(\mathbb{R}^n)$ if and only if $\hat{f}(0) \neq 0$ and $\hat{f}$ is not identically zero on each C_α, $\alpha > 0$, where $C_\alpha = \{v \in \mathbb{R}^n: \|v\| = \alpha\}$ (see for example [6], [12]).

In this paper we are interested in the corresponding question when we replace $\mathbb{R}^n$ by the Heisenberg group H^n and $M(n)$ by the "Heisenberg motion group" $HM(n)$ (see § 3). For $f \in L^1(H^n)$, we have the notion of the group-theoretic Fourier transform (§ 4). So we would like to get conditions on the group-theoretic Fourier transform of f which guarantee that $\overline{\operatorname{Span}\{{}^g f: g \in HM(n)\}} = L^1(H^n)$.

In order to answer this question, we make crucial use of a theorem of Hulanicki–Ricci [10] about the ideals in the commutative Banach algebra of "radial" L^1-functions on H^n.

Finally we should mention that the analogue of Wiener's theorem for the two sided action of H^n on itself has been known for sometime – see for example [11] and [16].

This paper is organized as follows: In § 2 we collect the relevant facts about H^n, its representation theory, special Hermite functions and twisted convolution. In § 3, we discuss the class-1 representations of the Gelfand pair $(HM(n), U(n))$, the corresponding elementary spherical functions and the connection between these representations and the representations of H^n. In § 4 we state the Wiener–Tauberian theorem of Hulanicki and Ricci precisely and prove the analogue of the Wiener–Tauberian theorem in our set up.

2. H^n, its representations, special Hermite functions and twisted convolution

Let $H^n = \mathbb{C}^n \times \mathbb{R}$ denote the n-dimensional Heisenberg group endowed with the group law

$$(z,t)(w,s) = (z + w, t + s + \tfrac{1}{2}\operatorname{Im} z.\bar{w}).$$

Here $z.\bar{w} = \Sigma_{j=1}^n z_j.\bar{w}_j$, for $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n)$.

For each $\lambda \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$, we have an irreducible unitary representation π_λ of H^n realized on $L^2(\mathbb{R}^n)$, the action being

$$\pi_\lambda(z,t)\phi(\xi) = e^{i\lambda t} e^{i\lambda((x.y/2) + x.\xi)} \phi(\xi + y),$$

for $z = x + iy$, $\phi \in L^2(\mathbb{R}^n)$, $\xi \in \mathbb{R}^n$. Up to unitary equivalence these π_λ give all the infinite dimensional irreducible unitary representations of H^n (see [7]). We also have another family of one-dimensional irreducible unitary representations χ_w, $w \in \mathbb{C}^n$, given by

$$\chi_w(z,t) = e^{i\operatorname{Re} w.\bar{z}}, \quad (z,t) \in H^n.$$

The representations π_λ for $\lambda \in \mathbb{R}^*$ together with χ_w for $w \in \mathbb{C}^n$ exhaust all the irreducible, pairwise inequivalent, unitary representations of H^n.

Throughout this paper we follow the convention that $\mathbb{N}$, the set of natural numbers, also includes zero. Let us take the orthonormal basis $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ of $L^2(\mathbb{R}^n)$ consisting of normalized Hermite functions. These Hermite functions can be given explicitly as follows: $\Phi_\alpha(x) = \Pi_{j=1}^n h_{\alpha_j}(x_j)$, for $x = (x_1, \dots, x_n)$, $\alpha = (\alpha_1, \dots, \alpha_n)$ where $h_k(y) = (2^k k! \sqrt{\pi})^{-1/2} (-1)^k \mathrm{d}^k/\mathrm{d}y^k (e^{-y^2}) e^{y^2/2}$, $y \in \mathbb{R}$, $k = 0, 1, 2, \dots$. Moreover Φ_α is an eigenfunction for the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$ with the eigenvalue $(2|\alpha| + n)$. Here $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The special Hermite functions $\Phi_{\alpha\beta}$ are defined as follows:

$$\Phi_{\alpha\beta}(z) = (2\pi)^{-n/2} \langle \pi(z)\Phi_\alpha, \Phi_\beta \rangle_{L^2(\mathbb{R}^n)},$$

where $\pi(z) = \pi_1(z, 0)$. The system $\{\Phi_{\alpha\beta}\}_{\alpha,\beta}$ forms an orthonormal basis for $L^2(\mathbb{C}^n)$. The functions $\Phi_{\alpha\beta}$ can be expressed in terms of Laguerre polynomials (see [15]) from which it follows that $\Phi_{\mu\mu}(z_1, \dots, z_n) = \Phi_{\mu\mu}(|z_1|, \dots, |z_n|)$. Hence each $\Phi_{\mu\mu}$ is real-valued. Further let

$$\phi_k(z) = L_k^{n-1}(\tfrac{1}{2}|z|^2) e^{-1/4|z|^2}$$

denote the kth Laguerre function, where L_k^{n-1} denotes the kth Laguerre polynomial of type $n - 1$. Then

$$\phi_k(z) = (2\pi)^{n/2} \sum_{|\alpha| = k} \Phi_{\alpha\alpha}(z).$$

If $F_1, F_2 \in L^1(\mathbb{C}^n)$ and $\lambda \in \mathbb{R}^*$, we define $F_1 *_\lambda F_2$, the λ- twisted convolution of F_1 and F_2 by

$$F_1 *_\lambda F_2(z) = \int_{\mathbb{C}^n} F_1(z - w) F_2(w) e^{i(\lambda/2)\operatorname{Im} z \cdot \bar{w}} \mathrm{d}w.$$

Then it can be seen that

$$\Phi_{\alpha\beta}^\lambda *_\lambda \Phi_{\mu\nu}^\lambda = (2\pi)^{n/2} \delta_{\beta\mu} \Phi_{\alpha\nu}^\lambda,$$

where $\Phi_{\alpha\beta}^\lambda(z) = |\lambda|^{n/2} \Phi_{\alpha\beta}(|\lambda|^{1/2} z)$.

A reference for results quoted for Hermite functions, special Hermite functions, twisted convolutions etc is [15].

3. The Gelfand pair $(HM(n), U(n))$

The compact group $U(n)$, of $n \times n$ unitary matrices with entires in $\mathbb{C}$, acts on H^n via the automorphism

$$\sigma(z,t) = (\sigma z, t), \quad \sigma \in U(n), \quad (z,t) \in H^n.$$

Therefore we can form the Heisenberg motion group $HM(n) = H^n \rtimes U(n)$, as a semi-direct product of H^n and $U(n)$. The group law in $HM(n)$ is given by

$$(\sigma, z, t)(\tau, w, s) = (\sigma\tau, \sigma w + z, s + t + \tfrac{1}{2}\operatorname{Im}\sigma w \cdot \bar{z})$$

for (σ, z, t), $(\tau, w, s) \in HM(n)$. The group $HM(n)$ acts on H^n in the following way

$$(\sigma, z, t)(w, s) = (\sigma w + z, s + t + \tfrac{1}{2}\operatorname{Im}\sigma w \cdot \bar{z}).$$

The group $U(n)$ is a maximal compact subgroup of $HM(n)$.

Henceforth we also write G for $HM(n)$ and K for $U(n)$.

Let $L^1(H^n)^\#$ be the closed subalgebra of K-invariant functions in $L^1(H^n)$. As shown in [1], $L^1(H^n)^\#$ is a *commutative* Banach $*$-algebra with respect to the usual convolution on H^n. (K-invariant functions on H^n are sometimes referred to as "radial functions".) Note that functions on H^n can be identified with functions on G that are right K-invariant. Thus $L^1(H^n)^\#$ can be identified with $L^1(K\backslash G/K)$, the subalgebra of $L^1(G)$ consisting of all K-bi-invariant functions on G. Further for f, $g \in L^1(H^n)^\# = L^1(K\backslash G/K)$, $f * g = f *_G g$ where $*$, $*_G$ denote the convolutions in H^n and G respectively. Hence $L^1(K\backslash G/K)$ is also a commutative Banach $*$-algebra and therefore (G, K) is a Gelfand pair. (See [9] for details about Gelfand pairs in general and [2], [3] and [4] for the Gelfand pairs associated with the Heisenberg group in particular.)

Let N be any locally compact topological group and K_0 be a compact subgroup of N. Let $\pi: N \mapsto \mathscr{U}(\mathscr{H})$ be an irreducible unitary representation of N on a Hilbert space $\mathscr{H}$. We say that π is a class-1 representation for the pair (N, K_0) if the restriction of π to K_0 contains the trivial representation of K_0, i.e., the space $H_0 = \{v \in \mathscr{H}: \pi(k)v = v, \forall k \in K_0\} \neq (0)$.

In case (N, K_0) is a Gelfand pair, i.e., if the algebra $\{f \in L^1(N): f(k_1 x k_2) = f(x), k_1, k_2 \in K_0, x \in N\}$, is commutative with respect to usual convolution on N, it is known that, for π, $\mathscr{H}$, H_0 as above, dim $H_0 = 1$. The function $x \mapsto \langle \pi(x)v_0, v_0 \rangle$, $x \in N$ where $v_0 \in H_0$ is such that $\|v_0\| = 1$ is called the elementary spherical function corresponding to π. For more details on Gelfand pairs, elementary spherical functions etc. see [8], [9].

A family $\{\rho_{\lambda,k}\}_{\lambda \in \mathbb{R}^*, k \in \mathbb{N}}$ of class-1 representations for the pair (G, K) (see[14]) is defined as follows:

For $\lambda \in \mathbb{R}^*$ and $k \in \mathbb{N}$, define

$$\tilde{H}_{\lambda,k} = \left\{ f: H^n \to \mathbb{C} \text{ smooth: } \mathscr{L} f = |\lambda|(2k+n)f,\ Tf = i\lambda f,\ \int_{\mathbb{C}^n} |f(z,0)|^2 \mathrm{d}z < \infty \right\},$$

where $\mathscr{L}$ is the Heisenberg sublaplacian and $T = \partial/\partial t$ (see [14]). An inner product $(.,.)$

on $\tilde{H}_{\lambda,k}$ is given as follows

$$(f,g)=(2\pi)^{-n}|\lambda|^n\int_{\mathbb{C}^n} f(z,0)\overline{g(z,0)}\mathrm{d}z.$$

Let $H_{\lambda,k}$ be the completion of $\tilde{H}_{\lambda,k}$ with respect to $(.,.)$. Let $\Phi_\alpha^\lambda(x)=|\lambda|^{n/4}\Phi_\alpha(|\lambda|^{1/2}x)$, $x\in\mathbb{R}^n$. Then the functions $E_{\alpha\beta}^\lambda(z,t)=\langle\pi_\lambda(z,t)\Phi_\alpha^\lambda,\Phi_\beta^\lambda\rangle$, $\alpha,\beta\in\mathbb{N}^n$, with $|\beta|=k$, form an orthonormal basis for $H_{\lambda,k}$. Define

$$\rho_{\lambda,k}(\sigma,z,t)f(w,s)=f((\sigma,z,t)^{-1}(w,s)),$$

for $(\sigma,z,t)\in G$, $f\in H_{\lambda,k}$, $(w,s)\in H^n$. Then $\rho_{\lambda,k}$ is a unitary representation of G. The following can be essentially found in [14]:

Theorem 3.1. *The representation $\rho_{\lambda,k}$ defined above is an irreducible unitary class-1 representation of G. The corresponding bounded elementary spherical function $e_{\lambda,k}$ is given as*

$$e_{\lambda,k}(\sigma,z,t)=\frac{k!(n-1)!}{(k+n-1)!}\mathrm{e}^{-i\lambda t}\phi_k(|\lambda|^{1/2}z),$$

$(\sigma,z,t)\in G$. The restriction of $\rho_{\lambda,k}$ to H^n breaks up as the sum of $(k+n-1)!/k!(n-1)!$ irreducible representations, each of which is equivalent to the representation π_λ of H^n. Moreover, for $\lambda,\lambda_1\in\mathbb{R}^*$, $k,k_1\in\mathbb{N}$, $\rho_{\lambda,k}$ is equivalent to ρ_{λ_1,k_1} if and only if $\lambda=\lambda_1$, $k=k_1$.

The irreducibility and pairwise inequivalence of $\rho_{\lambda,k}$'s are proved in [14]. Also the fact that the restriction of $\rho_{\lambda,k}$ to H^n breaks up as the sum of $(k+n-1)!/k!(n-1)!$ irreducible representations, each of which is equivalent to the representation π_λ of H^n has been observed in [14]. To see that $\rho_{\lambda,k}$ is class-1 for each $\lambda\in\mathbb{R}^*$, $k\in\mathbb{N}$, note that the function

$$\begin{aligned}E_k^\lambda(z,t)&=N_k^{-1/2}\sum_{|\beta|=k}E_{\beta\beta}^\lambda(z,t),\quad\text{where } N_k=\frac{(k+n-1)!}{k!(n-1)!}\\&=N_k^{-1/2}\mathrm{e}^{i\lambda t}(2\pi)^{n/2}\sum_{\beta=k}\Phi_{\beta\beta}(|\lambda|^{1/2}z)\\&=N_k^{-1/2}\mathrm{e}^{i\lambda t}\phi_k(|\lambda|^{1/2}z)\\&=N_k^{-1/2}\mathrm{e}^{i\lambda t}L_k^{n-1}(\tfrac{1}{2}|\lambda||z|^2)\mathrm{e}^{-1/4|\lambda||z|^2}\end{aligned}$$

(using results quoted in §2) is the essentially unique K-fixed vector in $H_{\lambda,k}$. The corresponding elementary spherical function $e_{\lambda,k}$ is therefore given by

$$\begin{aligned}\mathrm{e}_{\lambda,k}(\sigma,z,t)&=\langle\rho_{\lambda,k}(\sigma,z,t)E_k^\lambda,E_k^\lambda\rangle_{H_{\lambda,k}}\\&=\langle\rho_{\lambda,k}(e,z,t)E_k^\lambda,E_k^\lambda\rangle_{H_{\lambda,k}},\end{aligned}$$

where e is the identity element in $U(n)$. Hence the above expression becomes

$$\begin{aligned}&(2\pi)^{-n}|\lambda|^n\int_{\mathbb{C}^n}E_k^\lambda((e,z,t)^{-1}(w,0))\overline{E_k^\lambda(w,o)}\mathrm{d}w\\&=(2\pi)^{-n}|\lambda|^n\int_{\mathbb{C}^n}E_k^\lambda((w,0)(z,t)^{-1})\overline{E_k^\lambda(w,o)}\mathrm{d}w\end{aligned}$$

$$= (2\pi)^{-n}|\lambda|^n N_k^{-1} \int_{\mathbb{C}^n} \sum_{|\alpha|=k} \langle \pi_\lambda((w,o)(z,t)^{-1})\Phi_\alpha^\lambda, \Phi_\alpha^\lambda \rangle$$

$$\times \sum_{|\beta|=k} \overline{\langle \pi_\lambda(w,0)\Phi_\beta^\lambda, \Phi_\beta^\lambda \rangle} \mathrm{d}w.$$

$$= N_k^{-1} \mathrm{e}^{-i\lambda t} \sum_{|\alpha|=k=|\beta|} \int_{\mathbb{C}^n} \mathrm{e}^{i(\lambda/2)\mathrm{Im} z.\bar{w}} \Phi_{\alpha\alpha}^\lambda(w-z)\overline{\Phi_{\beta\beta}^\lambda(w)}\mathrm{d}w.$$

Since $\Phi_{\mu\mu}^\lambda(z) = \Phi_{\mu\mu}^\lambda(-z)$ and $\Phi_{\mu\mu}^\lambda$ is real-valued, the above expression is equal to

$$N_k^{-1} \mathrm{e}^{-i\lambda t} \sum_{|\alpha|=k=|\beta|} \int_{\mathbb{C}^n} \mathrm{e}^{i(\lambda/2)\mathrm{Im} z.\bar{w}} \Phi_{\alpha\alpha}^\lambda(z-w)\Phi_{\beta\beta}^\lambda(w)\mathrm{d}w.$$

$$= N_k^{-1} \mathrm{e}^{-i\lambda t} \sum_{|\alpha|=k=|\beta|} \Phi_{\alpha\alpha}^\lambda *_\lambda \Phi_{\beta\beta}^\lambda(z)$$

$$= N_k^{-1} \mathrm{e}^{-i\lambda t} \sum_{|\alpha|=k=|\beta|} (2\pi)^{n/2} \delta_{\alpha\beta} \Phi_{\alpha\beta}^\lambda(z)$$

$$= N_k^{-1} \mathrm{e}^{-i\lambda t} (2\pi)^{n/2} \sum_{|\alpha|=k} \Phi_{\alpha\alpha}^\lambda(z)$$

$$= N_k^{-1} \mathrm{e}^{-i\lambda t} \phi_k(|\lambda|^{1/2} z).$$

Since $e_{\lambda,k}(\sigma,z,t)$ is independent of the choice of σ, we also write $e_{\lambda,k}(z,t)$ for $e_{\lambda,k}(\sigma,z,t)$ for any $\sigma \in U(n)$.

We now describe the other set of class-1 representations of (G,K). Consider the one dimensional representation $\chi_w(z,t) = \mathrm{e}^{i\mathrm{Re} w.\bar{z}}$, $w \in \mathbb{C}^n \setminus \{0\}$, $(z,t) \in H^n$, of H^n. Let $K_0 = \{k \in K : k.w = w\}$. Then K_0 is a closed subgroup of K. Let ρ_w be the induced representation obtained by inducing $\chi_w \otimes 1$ from $H^n \rtimes K_0$ to $H^n \rtimes K = G$. Here 1 denotes the trivial representation of K_0. The representation space of ρ_w is described as follows: Let

$$\tilde{H}_w = \{f : G \to \mathbb{C} \text{ continuous}: f(g_0 g) = (\chi_w \otimes 1)(g_0) f(g),\ g_0 \in H^n \rtimes K_0,\ g \in G\}.$$

Therefore for $f \in \tilde{H}_w$, $(\sigma,z,t) \in G$,

$$\begin{aligned} f(\sigma,z,t) &= f((e,z,t)(\sigma,0,0)) \\ &= (\chi_w \otimes 1)(e,z,t) f(\sigma,0,0) \\ &= \mathrm{e}^{i\mathrm{Re} w.\bar{z}} f(\sigma,0,0), \end{aligned}$$

and hence f can be viewed as a function on K. Let H_w be the completion of $\tilde{H}_w$ with respect to the inner product

$$(f,g) = \int_K f(k)\overline{g(k)}\mathrm{d}k,\ f,g \in \tilde{H}_w.$$

The induced representation ρ_w is given by

$$\rho_w(\sigma,z,t)f(\tau,w,s) = f((\tau,w,s)(\sigma,z,t)), \quad f \in H_w, (\sigma,z,t),(\tau,w,s) \in G.$$

Then ρ_w is an irreducible unitary representation of G, with $f_0(\sigma,z,t) = \Omega_{2n-1}^{-1/2} \mathrm{e}^{i\mathrm{Re} w.\bar{z}}$ as the essentially unique K-fixed vector. Here Ω_{2n-1} is the total surface measure of the unit sphere in $\mathbb{R}^{2n}$ (see [2] for details). The corresponding elementary spherical functions η_τ can be computed to be the following

$$\eta_\tau(\sigma, z, t) = \frac{2^{n-1}(n-1)!\,J_{n-1}(\tau|z|)}{(\tau|z|)^{n-1}},$$

where $\tau = |w| > 0$ and J_{n-1} is the Bessel function of order $n-1$. Also ρ_w is equivalent to $\rho_{w'}$ if and only if $|w| = |w'|$. For $w = 0$, χ_0, the trivial representation of G is clearly a class-1 representation with the elementary spherical function $\eta_0 \equiv 1$ on G.

We also write $\eta_\tau(z,t)$ for $\eta_\tau(\sigma, z, t)$ for any $\sigma \in U(n)$. Since we know that $e_{\lambda,k}$ with $\lambda \in \mathbb{R}^*$, $k \in \mathbb{N}$ and η_τ with $\tau \geqslant 0$ are all the bounded elementary spherical functions for the pair (G, K), (see for example [1]), the above discussion completes the description of these in terms of class-1 representations of (G, K). The connection between representations and elementary spherical functions for Gelfand pairs associated with solvable Lie groups has been studied in detail in [2].

4. Wiener–Tauberian theorem for $L^1(H^n)$ with the $HM(n)$ action

We first state the Wiener–Tauberian theorem for $L^1(H^n)^\#$ due to Hulanicki and Ricci [10].

Theorem 4.1. *(Hulanicki and Ricci) Let J be a closed ideal in $L^1(H^n)^\#$ and suppose that*

(1) *For any $\lambda \in \mathbb{R}^*$, $k \in \mathbb{N}$, there exists some $f \in J$ such that*

$$\int f(z,t)\,e_{\lambda,k}(z,t)\,\mathrm{d}z\,\mathrm{d}t \neq 0.$$

(2) *For any $\tau \geqslant 0$, there exists some $f \in J$ such that*

$$\int f(z,t)\,\eta_\tau(z,t)\,\mathrm{d}z\,\mathrm{d}t \neq 0.$$

Then $J = L^1(H^n)^\#$.

To state the analogue of the Wiener–Tauberian theorem for the action of G on H^n, we set up some notation.

Let $\hat{H}^n$ denote the equivalence classes of irreducible unitary representations of H^n. For $h \in L^1(H^n)$, we define the "group theoretic" Fourier transform on $\hat{H}^n$ as follows: Let π be in $\hat{H}^n$ with $\mathscr{H}_\pi$ as the corresponding representation space. Then $\pi(h)$ is the bounded operator defined by

$$\pi(h) = \int_{H^n} h((z,t)^{-1})\,\pi(z,t)\,\mathrm{d}z\,\mathrm{d}t,$$

where the integral is to be interpreted suitably. The assignment $\pi \mapsto \pi(h)$, defined on $\hat{H}^n$ is known as the "group theoretic" Fourier transform of h. Thus for each $\lambda \in \mathbb{R}^*$, $\pi_\lambda(h)$ acts on the Hilbert space $L^2(\mathbb{R}^n)$ and for each $w \in \mathbb{C}^n$, $\chi_w(h)$ (is a scalar and) acts on the 1-dimensional space $\mathbb{C}$.

For each $\lambda \in \mathbb{R}^*$ and $k \in N$, let $P_{\lambda,k}$ be the projection on the kth eigenspace $M_{\lambda,k} = \mathrm{Span}\{\Phi_\alpha^\lambda : |\alpha| = k\}$ of the scaled Hermite operator $H_\lambda = -\Delta + |\lambda|^2|x|^2$ on $\mathbb{R}^n$. Recall $\Phi_\alpha^\lambda(x) = |\lambda|^{n/4}\Phi_\alpha(|\lambda|^{1/2}x), x \in \mathbb{R}^n$. We remark that if we take the Fock space model for describing the infinite dimensional representations of H^n, then the λ-dilated Hermite function Φ_α^λ corresponds to a nonzero multiple of the polynomial $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Hence the subspace $M_{\lambda,k}$ of $L^2(\mathbb{R}^n)$ can be identified with the space of homogeneous polynomials of degree k in n-variables $z_1,\dots,z_n$. The natural action, $u{\cdot}p(z)=p(u^{-1}{\cdot}z)$, $u\in U(n)$, of $U(n)$ on this space is irreducible. Thus $L^2(\mathbb{R}^n)=\oplus M_{\lambda,k}$ can be thought of as the decomposition of the representation space of π_λ into irreducible subspaces for the K-action, after $L^2(\mathbb{R}^n)$ is identified with the Fock space model.

For a function h on H^n, define ${}^g h(z,t)=h(g{\cdot}(z,t))$, for $g\in G$, $(z,t)\in H^n$. We are now in a position to give conditions under which $\overline{\mathrm{Span}\{{}^g f:g\in G\}}=L^1(H^n)$, for a given function $f\in L^1(H^n)$. These conditions are purely in terms of the group theoretic Fourier transform of f.

Theorem 4.2. *Let $f\in L^1(H^n)$ be such that*

(1) $\pi_\lambda(f)\,P_{\lambda,k}\neq 0$ *for each* $\lambda\in\mathbb{R}^*$ *and* $k\in\mathbb{N}$.
(2) *For each* $r>0$, *there exists* $w\in\mathbb{C}^n$ *with* $|w|=r$ *such that* $\chi_w(f)\neq 0$.
(3) $1(f)\neq 0$, *where* 1 *is the trivial representation of* H^n.

Then $\overline{\mathrm{Span}\{{}^g f:\ g\in G\}}=L^1(H^n)$.

Remark. If we define $f_0(z)=\int f(z,t)\mathrm{d}t$, $z\in\mathbb{C}^n$ then the condition (2) above can be rewritten as follows: For each $r>0$, $\hat{f}_0$, the Euclidean Fourier transform of f_0, does not vanish identically on S_r, the sphere of radius r in $\mathbb{R}^{2n}$. Also condition (3) is equivalent to $\hat{f}_0(0)\neq 0$.

Proof. By our remark in § 3, the given function f on H^n can be thought of as a right K-invariant function on G via $f(\sigma,z,t)=f(z,t)$, $(\sigma,z,t)=G$. Define $f^*(\sigma,z,t)=\bar{f}(\sigma,z,t)^{-1}=f(-\sigma^{-1}z,t),(\sigma,z,t)\in G$. Then f^* is a left K-invariant function on G. Hence $f^*{*_G}f$ is a K-bi-invariant function on G. Equivalently, it can be viewed as a K-invariant function on H^n.

We claim that the closed ideal generated by $f^*{*_G}f$ in $L^1(H^n)^\#$ is the full algebra $L^1(H^n)^\#$. Note that once we establish the claim, the theorem follows from the observations

(a) $h*(f^*{*_G}f)\in\overline{\mathrm{Span}\{{}^g f:\ g\in G\}}$, for $h\in L^1(H^n)^\#$ and hence $L^1(H^n)^\#=L^1(K\backslash G/K)\subseteq\overline{\mathrm{Span}\{{}^g f:\ g\in G\}}\subseteq L^1(G/K)=L^1(H^n)$.
(b) The smallest closed subspace of $L^1(G/K)$ containing $L^1(K\backslash G/K)$ and invariant under the (left) G-action, is the full space $L^1(G/K)$.

To prove the claim consider

$$\begin{aligned}&\int(f^*{*_G}f)(z,t)e_{\lambda,k}(z,t)\mathrm{d}z\,\mathrm{d}t\\&=\int(f^*{*_G}f)(z,t)\langle\rho_{\lambda,k}(e,z,t)E^\lambda_k,E^\lambda_k\rangle\mathrm{d}z\,\mathrm{d}t\\&=\int(f^*{*_G}f)(z,t)\mathrm{e}^{-i\lambda t}N_k^{-1}\phi_k(|\lambda|^{1/2}z)\mathrm{d}z\,\mathrm{d}t\\&=\int(f^*{*_G}f)(z,t)\mathrm{e}^{-i\lambda t}N_k^{-1}(2\pi)^{n/2}\sum_{|\alpha|=k}\langle\pi_\lambda(z)\Phi^\lambda_\alpha,\Phi^\lambda_\alpha\rangle\mathrm{d}z\,\mathrm{d}t\\&=N_k^{-1}(2\pi)^{n/2}\sum_{|\alpha|=k}\langle\pi_\lambda(f^*{*_G}f)\Phi^\lambda_\alpha,\Phi^\lambda_\alpha\rangle.\end{aligned}$$

Again an easy computation shows that

$$\begin{aligned}\pi_\lambda(f^* *_G f) &= \int (f^* *_G f)(z,t)^{-1}\pi_\lambda(z,t)\mathrm{d}z\mathrm{d}t\\ &= \int_{H^n}\int_{U(n)} (f*f^*)(-z,-t)\pi_\lambda(u.z,t)\mathrm{d}u\mathrm{d}z\mathrm{d}t\\ &= \int_{U(n)} \pi_{\lambda,u}(f*f^*)\mathrm{d}u,\end{aligned}$$

where $\pi_{\lambda,u}(z,t) = \pi_\lambda(u.z,t)$, $(z,t)\in H^n$, $u\in U(n)$. Therefore,

$$\begin{aligned}&\int (f^* *_G f)(z,t)e_{\lambda,k}(z,t)\mathrm{d}z\mathrm{d}t\\ &= N_k^{-1}(2\pi)^{n/2}\sum_{|\alpha|=k}\int_{U(n)} \langle \pi_{\lambda,u}(f*f^*)\Phi_\alpha^\lambda, \Phi_\alpha^\lambda\rangle \mathrm{d}u\\ &= N_k^{-1}(2\pi)^{n/2}\sum_{|\alpha|=k}\int_{U(n)} \langle \pi_{\lambda,u}(f^*)\pi_{\lambda,u}(f)\Phi_\alpha^\lambda, \Phi_\alpha^\lambda\rangle \mathrm{d}u\\ &= N_k^{-1}(2\pi)^{n/2}\sum_{|\alpha|=k}\int_{U(n)} \|\pi_{\lambda,u}(f)\Phi_\alpha^\lambda\|^2\mathrm{d}u.\end{aligned}$$

Hence $\int (f^* *_G f)(z,t)e_{\lambda,k}(z,t)\mathrm{d}z\mathrm{d}t = 0 \Leftrightarrow \|\pi_{\lambda,u}(f)\Phi_\alpha^\lambda\| = 0$, for all α such that $|\alpha| = k$ and a.e. $u\in U(n)$. Now as the irreducible representation $\pi_{\lambda,u}$ has the same central character as π_λ, by the Stone-von Neumann theorem (see [7]), $\pi_{\lambda,u}$ is equivalent to π_λ. Also for each $u\in U(n)$, $\lambda\in\mathbb{R}^*$ we can choose an intertwining unitary operator $m_\lambda(u)$: $L^2(\mathbb{R}^n)\mapsto L^2(\mathbb{R}^n)$ such that $m_\lambda(u)\pi_\lambda(z,t) = \pi_{\lambda,u}(z,t)m_\lambda(u)$, for all $(z,t)\in H^n$ and $u\mapsto m_\lambda(u)$: $U(n)\mapsto \mathscr{U}(L^2(\mathbb{R}^n))$ is a continuous projective representation of $U(n)$. Therefore, the condition $\int (f^* *_G f)(z,t)e_{\lambda,k}(z,t)\mathrm{d}z\mathrm{d}t = 0$ is equivalent to $m_\lambda(u)\pi_\lambda(f)m_\lambda(u)^{-1}\Phi_\lambda^\alpha = 0$, for all $u\in U(n)$, α such that $|\alpha| = k$. As for each λ and k, $m_\lambda(u)$ sends $M_{\lambda,k}$ onto $M_{\lambda,k}$, the above is equivalent to $\pi_\lambda(f)P_{\lambda,k} = 0$. The condition (1) in the hypothesis implies that this is not the case. Hence $\int (f^* *_G f)(z,t)e_{\lambda,k}(z,t)\mathrm{d}z\mathrm{d}t \neq 0$, for any $\lambda\in\mathbb{R}^*$, $k\in\mathbb{N}$. Also for $\tau > 0$, we have

$$\int (f^* *_G f)(z,t)\eta_\tau(z,t)\mathrm{d}z\mathrm{d}t = \int_{H^n}\int_{U(n)} (f^* *_G f)(z,t)e^{i\mathrm{Re}((u.w).\bar{z})}\mathrm{d}u\mathrm{d}z\mathrm{d}t,$$

where $w\in\mathbb{C}^n$ is such that $|w| = \tau$. Again using the fact that $(f^* *_G f)(z,t) = \int_{U(n)} (f*f^*)(u.z,t)\mathrm{d}u$, we have $\int (f^* *_G f)(z,t)\eta_\tau(z,t)\mathrm{d}z\mathrm{d}t = \mathrm{const.}\int_{U(n)}|\chi_{u.w}(f)|^2\mathrm{d}u$. Therefore,

$$\int (f^* *_G f)(z,t)\eta_\tau(z,t)\mathrm{d}z\mathrm{d}t = 0$$

if and only if $\chi_{u.w}(f) = 0$, for all $u\in U(n)$. This in turn is equivalent to $\hat{f}_0 \equiv 0$ on the sphere S_τ of radius τ in $\mathbb{C}^n$. But the condition (2) in the hypothesis implies that this is not the case. Hence $\int (f^* *_G f)(z,t)\eta_\tau(z,t)\mathrm{d}z\mathrm{d}t \neq 0$, for $\tau > 0$. For $\tau = 0$, $\int (f^* *_G f)(z,t)\mathrm{d}z\mathrm{d}t = |\int f(z,t)\mathrm{d}z\mathrm{d}t|^2 \neq 0$. Hence the Wiener–Tauberian theorem holds for the closed ideal $\overline{\{h*(f^* *_G f)\colon h\in L^1(H^n)^\#\}}$ generated by $f^* *_G f$ in $L^1(H^n)^\#$, i.e., $\overline{\{h*(f^* *_G f)\colon h\in L^1(H^n)^\#\}} = L^1(H^n)^\#$.

Remark. The conditions in Theorem 4.2 are also necessary for $f \in L^1(H^n)$ to have the property that $\overline{\mathrm{Span}\{{}^g f:\ g \in G\}} = L^1(H^n)$. This can be seen, for example, by taking the function $h(z,t) = \mathrm{e}^{-|z|^2}\mathrm{e}^{-t^2}$, $(z,t) \in H^n$, which is in $L^1(H^n)$ but fails to be in $\overline{\mathrm{Span}\{{}^g f : g \in G\}}$ if any of the condition (1), (2) or (3) is violated.

5. Concluding remarks

One can also consider analogues of Wiener's theorem for other Gelfand pairs. The result referred to in the introduction is actually about the Gelfand pair $(M(n), SO(n))$. In another direction, one can consider the pair (G, K) where G is a connected semisimple Lie group of the noncompact type with finite centre and K is a maximal compact subgroup of G. There is a whole body of literature devoted to this set up. For some very recent results, see for example [5].

Acknowledgement

The author wishes to thank Prof. A Sitaram for suggesting this problem and for providing useful insight in solving it and Prof. S Thangavelu for several useful discussions. The author also thanks the referee for several useful suggestions regarding the literature.

References

[1] Agranovsky M, Berenstein C, Chang D C and Pascuas D, Injectivity of the Pompeiu transform in the Heisenberg group, *J. Anal. Math.* **63** (1994) 131–173
[2] Benson C, Jenkins J and Ratcliff G, On Gelfand pairs associated with solvable Lie groups, *Trans. Am. Math. Soc.* **321** (1990) 85–116
[3] Benson C, Jenkins J and Ratcliff G, Bounded K-spherical functions on Heisenberg groups, *J. Funct. Anal.* **105** (1992) 409–443
[4] Benson C, Jenkins J, Lipsman R L and Ratcliff G, The moment map for a multiplicity free action, *Bull. Am. Math. Soc.* **31** (1994) 185–190
[5] Ben Natan Y, Benyamini Y, Hedenmalm H and Weit Y, Wiener's Tauberian theorem in $L^1(G\|K)$ and harmonic functions in the unit disk, *Bull. Am. Math. Soc.* **32** (1995) 43–49
[6] Brown L, Schreiber B M and Taylor B A, Spectral synthesis and the Pompeiu problem, *Ann. Inst. Fourier (Grenoble)* **23** (1973) 125–154
[7] Folland G, Harmonic analysis in phase space, *Ann. Math. Stud.* (1989) (Princeton: Princeton Univ. Press)
[8] Gangolli R, Spherical functions on semisimple Lie groups, in: Symmetric Spaces (eds) W Boothby and G Weiss (1972) (New York: Marcel Dekker Inc.)
[9] Helgason S, *Groups and Geometric Analysis* (1984) (New York: Academic Press)
[10] Hulanicki A and Ricci F, A Tauberian theorem and tangential convergence for boundary harmonic functions on balls in $\mathbb{C}^n$, *Invent. Math.* **62** (1980) 325–331
[11] Leptin H, On group algebras of nilpotent Lie groups, *Stud. Math.* **47** (1973) 37–49
[12] Rawat R and Sitaram A, The injectivity of the Pompeiu transform and L^p-analogues of the Wiener–Tauberian theorem, *Isr. J. Math.* **91** (1995) 307–316
[13] Rudin W, *Functional Analysis* (1973) (New York: McGraw-Hill)
[14] Strichartz R, L^p harmonic analysis and radon transforms on the Heisenberg group, *J. Funct. Anal.* **96** (1991) 350–406
[15] Thangavelu S, Lectures on Hermite and Laguerre expansions, *Math. Notes* (1993) (Princeton: Princeton Univ. Press)
[16] Weit Y, On spectral analysis in locally compact motion groups, *J. Funct. Anal.* **40** (1981) 45–53