Proc. Indian Acad. Sci. (Math. Sci.), Vol. 106, No. 3, August 1996, pp. 281–287.

Eigenvalue bounds for Orr–Sommerfeld equation 'No backward wave' theorem

MIHIR B BANERJEE, R G SHANDIL and BALRAJ SINGH BANDRAL

Department of Mathematics, Himachal Pradesh University, Shimla 171 005, India

MS received 18 January 1996

Abstract. Theoretical estimates of the phase velocity C_r of an arbitrary unstable, marginally stable or stable wave derived on the basis of the classical Orr–Sommerfeld eigenvalue problem governing the linear instability of plane Poiseuille flow ($U(z) = 1 - z^2$, $-1 \leqslant z \leqslant +1$), leave open the possibility of these phase velocities lying outside the range $U_{\min} < C_r < U_{\max}$, but not a single experimental or numerical investigation in this regard, which are concerned with unstable or marginally stable waves, has supported such a possibility as yet, $U_{\min}$ and $U_{\max}$ being respectively the minimum and the maximum value of $U(z)$ for $z \in [-1, +1]$. This gap between the theory on one side and the experiment and computation on the other has remained unexplained ever since Joseph derived these estimates, first, in 1968, and has even led to the speculation of a negative phase velocity (or rather, $C_r < U_{\min} = 0$) and hence the possibility of a 'backward' wave as in the case of the Jeffery–Hamel flow in a diverging channel with back flow ([1]). A simple mathematical proof of the non-existence of such a possibility is given herein by showing that the phase velocity C_r of an arbitrary unstable or marginally stable wave must satisfy the inequality $U_{\min} < C_r < U_{\max}$. It follows as a consequence stated here in this explicit form for the first time to the best of our knowledge, that 'overstability' and not the 'principle of exchange of stabilities' is valid for the problem of plane Poiseuille flow.

Keywords. Bounds; Orr–Sommerfeld equation.

1. Introduction

In the linear instability problem of parallel shear flow of a nonviscous fluid, Rayleigh showed that the phase velocity C_r of an arbitrary unstable wave must lie in the range $U_{\min} < C_r < U_{\max}$ and since then the problem of generalizing this result with the inclusion of the effect of viscosity of the fluid, which results in the basic flow being a Poiseuille one, has been much sought after. Joseph [3] discovered in the context of plane Poiseuille flow a lower and an upper bound of C_r of an arbitrary unstable, marginally stable or stable wave in the form

$$\frac{U_{\min} + 2\left[\dfrac{d^2U}{dz^2}\right]_{\min}}{\pi^2 + 4\alpha^2} = \frac{-4}{\pi^2 + 4\alpha^2} < C_r < U_{\max} = 1, \tag{1}$$

which leave open the possibility of these phase velocities lying outside the range $U_{\min} < C_r < U_{\max}$, but not a single experimental or numerical investigation in this regard, which are concerned with unstable or marginally stable waves has supported such a possibility as yet. This gap between the theory on one side and the experiment and the computation on the other has remained unexplained ever since Joseph derived these estimates and has even led to the speculation of a negative phase velocity (or, rather, $C_r < U_{\min} = 0$) and hence the possibility of a 'backward' wave as in the case of the Jeffery–Hamel flow in a diverging channel with back flow. A simple mathematical proof of the non-existence of such a possibility is given herein by showing that the phase velocity C_r of an arbitrary unstable or marginally stable wave must satisfy the inequality $U_{min} < C_r < U_{max}$. It follows as a consequence, stated here in this explicit form for the first time to the best of our knowledge, that 'overstability' and not the 'principle of exchange of stabilities', is valid for the problem of plane Poiseuille flow.

2. Mathematical analysis

The classical Orr–Sommerfeld eigenvalue problem governing the linear instability of plane Poiseuille flow against two-dimensional perturbations is given by

$$\frac{1}{i\alpha R}(D^2-\alpha^2)^2\phi=(U-C)(D^2-\alpha^2)\phi-\frac{d^2U}{dz^2}\phi \tag{2}$$

and

$$\phi=0=D\phi \quad \text{at} \quad z=-1 \quad \text{and} \quad z=+1, \tag{3}$$

where z is the real independent variable such that $-1\leqslant z\leqslant +1$ and $D\equiv d/dz$; α is the wave number of the perturbation and is real; $R>0$ is the Reynold number of the flow; $C=C_r+iC_i$ is the complex wave velocity of the perturbation, C_r and C_i being respectively the phase velocity and the amplification factor; $\phi(z)$ is the amplitude of the stream function perturbation in the form $\phi(z)e^{i\alpha(x-ct)}$ and is a complex valued function of the real variable z while $U(z)=1-z^2$ is the basic background flow. Equations (2) and (3), thus define an eigenvalue problem for C for given values of α and R, and a perturbation is said to be unstable if $\alpha C_i>0$, marginally stable if $\alpha C_i=0$ for some values of α and R with the further condition that $\alpha C_i>0$ for any neighbouring values of α and R, and stable if $\alpha C_i\leqslant 0$.

We now prove the following theorems:

Theorem 1. *If (ϕ, C) with $\alpha C_i\neq 0$ is a solution of the Orr–Sommerfeld eigenvalue problem described by eqs* (2) *and* (3) *for prescribed values of α and R then the integral relation*

$$\begin{aligned}&2C_i\int_{-1}^{+1}(U-C_r)[|DF|^2+\alpha^2|F|^2]dz\\&+\frac{1}{\alpha R}\int_{-1}^{+1}(U-C_r)[|D^2F|^2+2\alpha^2|DF|^2+\alpha^4|F|^2]dz\\&-\frac{1}{\alpha R}\int_{-1}^{+1}\frac{d^2U}{dz^2}[2|DF|^2+\alpha^2|F|^2]dz+\frac{1}{2\alpha R}\int_{-1}^{+1}\frac{d^4U}{dz^4}|F|^2dz=0,\end{aligned} \tag{4}$$

with $F=\phi/(U-C)$, is true.

Proof. We apply the transformation $\phi=(U-C)F$ which remains valid for all values of $z\in[-1, +1]$, since $\alpha C_i\neq 0$. Equations (2) and (3) then transform into

$$\frac{1}{i\alpha R}(D^2-\alpha^2)^2[(U-C)F]=D[(U-C)^2DF]-\alpha^2(U-C)^2F, \tag{5}$$

and

$$F=0=DF \quad \text{at} \quad z=-1 \quad \text{and} \quad +1. \tag{6}$$

Multiplying eq. (5) throughout by F^* (the complex conjugate of F) and integrating the resulting equation over the range of z, we get

$$\frac{1}{i\alpha R}\int_{-1}^{+1} F^*(D^2-\alpha^2)^2[(U-C)F]\mathrm{d}z$$
$$=\int_{-1}^{+1} F^*D[(U-C)^2DF]\mathrm{d}z-\alpha^2\int_{-1}^{+1}(U-C)^2|F|^2\mathrm{d}z. \tag{7}$$

Equating the imaginary parts of both sides of (7), we have

$$-\frac{1}{\alpha R}\mathrm{Re}\int_{-1}^{+1} F^*(D^2-\alpha^2)^2[(U-C)F]\mathrm{d}z$$
$$=\mathrm{Im}\int_{-1}^{+1} F^*D[(U-C)^2DF]\mathrm{d}z-\alpha^2\,\mathrm{Im}\int_{-1}^{+1}(U-C)^2|F|^2\mathrm{d}z, \tag{8}$$

where the symbols Re and Im respectively denote the real and imaginary parts of the quantities that succeed them.

Now

$$\mathrm{Re}\int_{-1}^{+1} F^*(D^2-\alpha^2)^2[(U-C)F]\mathrm{d}z$$
$$=\mathrm{Re}\int_{-1}^{+1} F^*(D^4-2\alpha^2D^2+\alpha^4)[(U-C)F]\mathrm{d}z$$
$$=\mathrm{Re}\int_{-1}^{+1} F^*D^4[(U-C)F]\mathrm{d}z$$
$$-2\alpha^2\,\mathrm{Re}\int_{-1}^{+1} F^*D^2[(U-C)F]\mathrm{d}z+\alpha^4\,\mathrm{Re}\int_{-1}^{+1}(U-C)|F|^2\mathrm{d}z,$$

which upon integrating the first integral twice and the second integral once by parts and making use of the boundary conditions (6) yields

$$\mathrm{Re}\int_{-1}^{+1} D^2F^*[(U-C)D^2F+2\frac{\mathrm{d}U}{\mathrm{d}z}DF+\frac{\mathrm{d}^2U}{\mathrm{d}z^2}F]\mathrm{d}z$$
$$+2\alpha^2\mathrm{Re}\int_{-1}^{+1} DF^*[(U-C)DF+\frac{\mathrm{d}U}{\mathrm{d}z}F]\mathrm{d}z+\alpha^4\int_{-1}^{+1}(U-C_r)|F|^2\mathrm{d}z,$$

which upon rearranging yields

$$\int_{-1}^{+1}(U-C_r)[|D^2F|^2+2\alpha^2|DF|^2+\alpha^4|F|^2]\mathrm{d}z$$
$$+\mathrm{Re}\int_{-1}^{+1} D^2F^*\left[2\frac{\mathrm{d}U}{\mathrm{d}z}DF+\frac{\mathrm{d}^2U}{\mathrm{d}z^2}F\right]\mathrm{d}z$$
$$+2\alpha^2\,\mathrm{Re}\int_{-1}^{+1} DF^*\frac{\mathrm{d}U}{\mathrm{d}z}F\mathrm{d}z. \tag{9}$$

Integrating $2\int_{-1}^{+1}D^2F^*(\mathrm{d}U/\mathrm{d}z)DF\mathrm{d}z$, $\int_{-1}^{+1}D^2F^*(\mathrm{d}^2U/\mathrm{d}z^2)F\mathrm{d}z$ and $\int_{-1}^{+1}DF^*(\mathrm{d}U/\mathrm{d}z)F\mathrm{d}z$ by parts once, twice and once respectively and making use of the boundary conditions

(5), we derive

$$\mathrm{Re}\,2\int_{-1}^{+1} D^2F^*\frac{\mathrm{d}U}{\mathrm{d}z}DF\mathrm{d}z = -\int_{-1}^{+1}\frac{\mathrm{d}^2U}{\mathrm{d}z^2}|DF|^2\mathrm{d}z, \tag{10}$$

$$\mathrm{Re}\int_{-1}^{+1} D^2F^*\frac{\mathrm{d}^2U}{\mathrm{d}z^2}F\mathrm{d}z = -\int_{-1}^{+1}\frac{\mathrm{d}^2U}{\mathrm{d}z^2}|DF|^2\mathrm{d}z + \frac{1}{2}\int_{-1}^{+1}\frac{\mathrm{d}^4U}{\mathrm{d}z^4}|F|^2\mathrm{d}z, \tag{11}$$

and

$$\mathrm{Re}\int_{-1}^{+1} DF^*\frac{\mathrm{d}U}{\mathrm{d}z}F\mathrm{d}z = -\frac{1}{2}\int_{-1}^{+1}\frac{\mathrm{d}^2U}{\mathrm{d}z^2}|F|^2\mathrm{d}z. \tag{12}$$

It then follows from eqs (9), (10), (11) and (12) that

$$\begin{aligned}\mathrm{Re}&\int_{-1}^{+1} F^*(D^2-\alpha^2)^2[(U-C)F]\mathrm{d}z\\ &= \int_{-1}^{+1}(U-C_r)[|D^2F|^2+2\alpha^2|DF|^2+\alpha^4|F|^2]\mathrm{d}z\\ &\quad -\int_{-1}^{+1}\left(\frac{\mathrm{d}^2U}{\mathrm{d}z^2}\right)[2|DF|^2+\alpha^2|F|^2]\mathrm{d}z + \frac{1}{2}\int_{-1}^{+1}\frac{\mathrm{d}^4U}{\mathrm{d}z^4}|F|^2\mathrm{d}z.\end{aligned} \tag{13}$$

Further

$$\begin{aligned}\mathrm{Im}&\int_{-1}^{+1} F^*D[(U-C)^2DF]\mathrm{d}z - \alpha^2\,\mathrm{Im}\int_{-1}^{+1}(U-C)^2|F|^2\mathrm{d}z\\ &= -\mathrm{Im}\int_{-1}^{+1}(U-C)^2[|DF|^2+\alpha^2|F|^2]\mathrm{d}z\\ &= 2C_i\int_{-1}^{+1}(U-C_r)[|DF|^2+\alpha^2|F|^2]\mathrm{d}z,\end{aligned} \tag{14}$$

which follows by integrating the first integral by parts once and making use of the boundary conditions (5).

Combinding eqs (8), (13) and (14), we obtain the integral relation

$$\begin{aligned}2C_i&\int_{-1}^{+1}(U-C_r)[|DF|^2+\alpha^2|F|^2]\mathrm{d}z\\ &+\frac{1}{\alpha R}\int_{-1}^{+1}(U-C_r)[|D^2F|^2+2\alpha^2|DF|^2+\alpha^4|F|^2]\mathrm{d}z\\ &-\frac{1}{\alpha R}\int_{-1}^{+1}\frac{\mathrm{d}^2U}{\mathrm{d}z^2}[2|DF|^2+\alpha^2|F|^2]\mathrm{d}z + \frac{1}{2\alpha R}\int_{-1}^{+1}\frac{\mathrm{d}^4U}{\mathrm{d}z^4}|F|^2\mathrm{d}z = 0,\end{aligned} \tag{15}$$

and hence the theorem.

Theorem 2. *If* (ϕ, C) *with* $\alpha C_i > 0$ *is a solution of the Orr-Sommerfeld eigenvalue problem described by eqs* (2) *and* (3) *for prescribed values of* α *and* R *then* C_r *must satisfy the inequality*

$$U_{\min} < C_r < U_{\max}. \tag{16}$$

Proof. We write eq. (15), which is valid under the present conditions as

$$2\alpha C_i \int_{-1}^{+1} (U - C_r)[|DF|^2 + \alpha^2 |F|^2] \mathrm{d}z$$
$$+ \frac{1}{R} \int_{-1}^{+1} (U - C_r)[|D^2F|^2 + 2\alpha^2 |DF|^2 + \alpha^4 |F|^2] \mathrm{d}z$$
$$- \frac{1}{R} \int_{-1}^{+1} \frac{\mathrm{d}^2 U}{\mathrm{d}z^2} [2|DF|^2 + \alpha^2 |F|^2] \mathrm{d}z + \frac{1}{2R} \int_{-1}^{+1} \frac{\mathrm{d}^4 U}{\mathrm{d}z^4} |F|^2 \mathrm{d}z = 0. \tag{17}$$

Now, for $U(z) = 1 - z^2$, we have $(\mathrm{d}^2U/\mathrm{d}z^2) = -2$ and $(d^4U/\mathrm{d}z^4) = 0$ for all values of $z \in [-1, +1]$ and further since $\alpha C_i > 0$, it follows from eq. (17) that

$$\int_{-1}^{+1} (U - C_r)\left[2\alpha C_i(|DF|^2 + \alpha^2 |F|^2) + \frac{1}{R}(|D^2F|^2 + 2\alpha^2 |DF|^2 + \alpha^4 |F|^2)\right] \mathrm{d}z$$
$$+ \frac{2}{R} \int_{-1}^{+1} (2|DF|^2 + \alpha^2 |F|^2) \mathrm{d}z = 0. \tag{18}$$

The quantity within the square brackets under the first integral sign is a positive definite and therefore, for the validity of eq. (18), we must have for some z_s

$$U(z_s) - C_r < 0, \quad z_s \in [-1, +1], \tag{19}$$

which implies that

$$C_r > U_{\min}. \tag{20}$$

Combining inequality (20) with Joseph's inequality given by (1) which holds good for $\alpha C_i \gtreqless 0$, we derive the result that

$$U_{\min} < C_r < U_{\max}, \tag{21}$$

and hence the theorem.

Theorem 3. *If (ϕ, C) with $\alpha C_i = 0$ ($\alpha \neq 0$ since $\alpha = 0$ corresponds to a trivial solution for ϕ) is a solution of the Orr–Sommerfeld eigenvalue problem described by eqs (2) and (3) for prescribed values of α and R then C_r must satisfy the inequality*

$$U_{\min} \leqslant C_r < U_{\max}. \tag{22}$$

Proof. Since $C_i = 0$, it follows that the behaviour of $U - C = U - C_r$ must fall into one of the three mutually exclusive classes namely

(i) $U - C_r > 0$ for all values of $z \in [-1, +1]$,
(ii) $U - C_r < 0$ for all values of $z \in [-1, +1]$, and
(iii) $U - C_r = 0$ for some value of $z = z_s \in [-1, +1)$.

If (i) is valid then under the present conditions the transformation $\phi = (U - C)F$ remains well defined for all values of $z \in [-1, +1]$ so that we derive from eq. (15) that

$$\int_{-1}^{+1} (U - C_r)[|D^2F|^2 + 2\alpha^2 |DF|^2 + \alpha^4 |F|^2] \mathrm{d}z$$
$$+ 2 \int_{-1}^{+1} (2|DF|^2 + \alpha^2 |F|^2) \mathrm{d}z = 0. \tag{23}$$

A necessary condition for the validity of eq. (23) is that

$$U - C_r < 0 \text{ for some value of } z \in [-1, +1],$$

which clearly contradicts the starting hypothesis, namely (i). Thus (i) cannot be valid.

If (ii) is valid, then we must have

$$U - C_r < 0 \quad \text{for all values of } z \in [-1, +1],$$

from which it follows that

$$C_r > U(0) = U_{\max},$$

which presents a contradiction, since by Joseph's estimate given by inequality (1) C_r satisfies $C_r < U_{\max}$. Thus (ii) also cannot be valid.

Therefore (iii) must hold good so that we have

$$U - C_r = 0 \text{ for some value of } z = z_s \in [-1, +1],$$

which implies that

$$U(z_s) - C_r = 0,$$

from which it follows that

$$U_{\min} \leqslant C_r \leqslant U_{\max}. \tag{24}$$

Combining inequality (24) with inequality (1) established by Joseph, we derive that such values of C_r as given under conditions of Theorem 3 satisfies the inequality

$$U_{\min} \leqslant C_r < U_{\max}, \tag{25}$$

and hence the theorem.

Remarks. An arbitrary perturbation with $\alpha C_i = 0$ $(\alpha \neq 0)$ is a neutral perturbation and since $U_{\min} = 0$, Theorem 3 clearly shows that stationary (i.e. $C_r = 0$) as well as oscillatory (i.e. $C_r \neq 0$) neutral perturbations are both allowed by inequality (25). However, for such a neutral perturbation to be a marginal or marginally stable perturbation it is necessary that it lies on the stability boundary i.e. a boundary or boundaries in the (α, R) – plane on crossing which, C_i changes sign, and this requires a more detailed analysis of the eigenvalue problem. In the next theorem we shall prove that all stationary non-neutral perturbations must of necessity decay which rules out the possibility of stationary neutral perturbations lying on the stability boundary which exists after the rigorous mathematical validation of Heisenberg's [2] results by Krylov [4]. Thus, it is the oscillatory neutral perturbations and not the stationary neutral ones that constitute the stability boundary or equivalently, in the Poincaré–Eddington terminology it is 'overstability' and not the 'principle of exchange of stabilities' that is valid for the problem of plane Poiseuille flow.

Theorem 4. *If* (ϕ, C) *with* $\alpha C_i \neq 0$ *and* $C_r = 0$ *is a solution of the Orr–Sommerfeld eigenvalue problem described by eqs* (2) *and* (3) *for prescribed values of* α *and* R *then* $\alpha C_i < 0$.

Proof. From eq. (15), which is valid under the present conditions we derive that

$$2\alpha C_i \int_{-1}^{+1} U[|DF|^2 + \alpha^2 |F|^2] \mathrm{d}z + \frac{1}{R}\int_{-1}^{+1} U[|D^2F|^2 + 2\alpha^2|DF|^2 + \alpha^4|F|^2]\mathrm{d}z + \frac{2}{R}\int_{-1}^{+1} (2|DF|^2 + \alpha^2|F|^2)\mathrm{d}z = 0. \tag{26}$$

But, since $U(z) = 1 - z^2 \geqslant 0$ for all values of $z \in [-1, +1]$, we must have, for the validity of eq. (23), $\alpha C_i < 0$ and hence the theorem.

In view of the Remark mentioned above and Theorem 4 the following theorem is valid:

Theorem 5. *If* (ϕ, C) *with* $\alpha C_i = 0$ $(\alpha \neq 0)$ *and* $C_r \neq 0$ *is a solution of the Orr–Sommerfeld eigenvalue problem described by eqs* (2) *and* (3) *for prescribed values of* α *and* R *then* C_r *must satisfy the inequality*

$$U_{\min} < C_r < U_{\max}. \tag{27}$$

Theorem 2 and Theorem 5 show that the phase velocity of an arbitrary unstable or marginally stable wave must lie in the range $U_{\min} < C_r < U_{\max}$ while Theorem 2 and Theorem 3 show that the phase velocity of an arbitrary unstable or neutrally stable wave must lie in the range $U_{\min} \leqslant C_r < U_{\max}$. Thus, in both cases the possibility of C_r being, less than $U_{\min}$ and hence, negative is ruled out and therefore no 'backward' wave can exist in the instability of Poiseuille flow.

Acknowledgement

One of the authors (BSB) gratefully acknowledges the financial support of CSIR.

References

[1] Drazin P G and Reid W H, *Hydrodynamic Stability* (1981) (London: Cambridge University Press)

[2] Heisenberg W, Uber Stabilitat und Turbulenz von Flussigkeitsstromen, *Ann. Phys. Lpz.* **74** **(4)** (1924) 577–627

[3] Joseph D D, Eigenvalue bounds for Orr–Sommerfeld equation, *J. Fluid Mech.* **33** (1968) 617–621

[4] Krylov A L, The proof of the instability of a certain flow of viscous incompressible fluid, *Dokl. Akad. Nauk.* **153** **(4)** (1963) 787–789