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Finite part representations of hyper singular integral equation of acoustic scattering and radiation by open smooth surfaces

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Abstract. The Green's function solution of the Helmholtz's equation for acoustic scattering by hard surfaces and radiation by vibrating surfaces, lead in both the cases, to a hyper singular surface boundary integral equation. Considering a general open surface, a simple proof has been given to show that the integral is to be interpreted like the Hadmard finite part of a divergent integral in one variable. The equation is reformulated as a Cauchy principal value integral equation, but also containing the potential at the control point. It is amenable to numerical treatment by conventional methods. An alternative formulation in the better known form, containing the tangential derivative of the potential is also given. The two dimensional problem for an open arc is separately treated for its simpler feature.

Keywords. Finite part; hyper singular integral; integral equation; Cauchy principal value; acoustic scattering; acoustic radiation; open smooth surfaces.

1. Introduction

It has been recognized in recent years that the Green's function formulation of scattering and radiation of acoustic or elastic waves by surfaces on which Neumann boundary condition holds, leads to hyper singular integral equations. In the acoustic case, the surfaces are non-soft, that is, hard or partially absorbing, open or closed. References to and regularization of the hyper singular nature by conversion to Cauchy principal value (CPV) integral equations involving (tangential) derivatives of the unknown potential or integro-differential equations have been made in diverse literature (Burton and Miller [1], Meyer *et al* [9], Terai [11], Martin and Rizzo [7], Krishnasamy *et al* [5]). In the context of vector elastic waves, the surfaces are usually stress-free crack surfaces in the solid and considerable literature exists generally without direct reference to hyper singularity. References to the few which treat hyper singularity may be found in Martin and Rizzo [7] and Krishnasamy *et al* [5].

The hyper singularity in the boundary integral equation arises when the normal derivative implied by the boundary conditions is carried into the Cauchy principal value integral containing normal derivative of Green's function. Validation of the hyper singular integrals in different contexts as Hadamard finite part (HFP) [2] is generally obtainable (Krishnasamy *et al* [5]). In view of the straightforward formulation, there have been several suggestions following Iokimidis [3], to treat these numerically, using Gaussian quadrature formula developed by Kutt [6] for integrals in one dimension.

Herein, we treat the hyper singular integral equation that arises in the context of acoustic scattering or radiation by a hard open surface (three as well as two dimensional, that is, an arc). By adopting a definition of finite part of the hyper singular surface integral as a generalization of the usual HFP of a curvilinear integral stated in

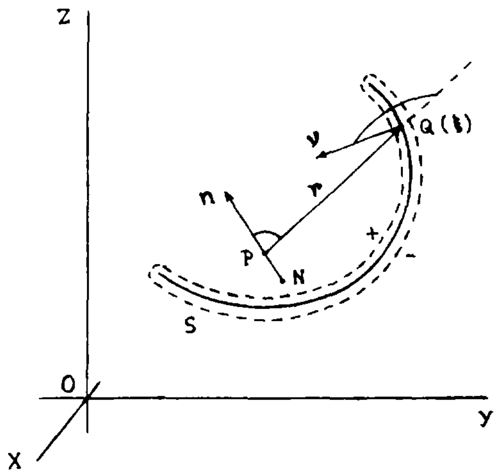

Figure 1. The geometry of the problem. N is the foot of the perpendicular from P on the surface S.

Martin and Rizzo [7], we present a short validation of the equation. The proof of Krishnasamy *et al* [5] based on the original definition of Hadamard for one dimensional integral is somewhat longer. Next the equation is transformed in a straightforward way to derive CPV linear integral equations, which can be numerically treated in conventional ways. Familiar type, containing the (tangential) derivative of the potential is also derived. The case of two dimensional arc is separately treated for its simplicity and elucidation.

2. The hyper singular integral equation

The acoustic field potential ϕ satisfies Helmholtz's equation

$$\nabla^2\phi + k^2\phi = 0, \tag{1}$$

where k is the wave number ω/c, $\omega/2\pi$ = frequency and c = wave velocity. By Green's identity (cf. figure 1), the solution of (1) can be written as

$$\phi = \frac{1}{\kappa}\int_S \left[\phi\frac{\partial G}{\partial \nu} - G\frac{\partial \phi}{\partial \nu}\right]_-^+ \mathrm{d}S, \tag{2}$$

where $\kappa = -2\pi$ or 4π with regard to the (open) surface S being two or three dimensional, and G is the free space Green's function $(i\pi/2)H_0^{(2)}(kr)$ or e^{-ikr}/r, $H_0^2(\cdot)$ being the Hankel function of second kind and zero order. For an acoustically hard surface scattering problem, the normal component of velocity at the variable point Q, namely $\partial\phi/\partial\nu$ is zero while it is prescribed $(-i\rho\omega v(Q))$ and is continuous across S for the radiation problem. Hence (2) simplifies to

$$\phi = \frac{1}{\kappa}\int_S \Phi\frac{\partial G}{\partial \nu}\mathrm{d}S, \quad \Phi(Q) = \phi^+ - \phi^-. \tag{3}$$

Boundary integral equation for Φ, completing the solution of (3), is obtained by again invoking the boundary condition on S. For the radiation problem $\partial\phi/\partial n = v(N)$ as the control point P tends to N where $v(N)$ is the amplitude of the normal component of

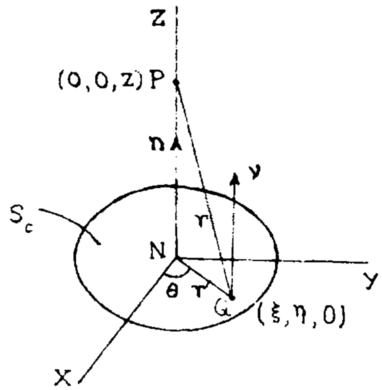

Figure 2. Local coordinates for circular region S_ε.

displacement. We thus get

$$\frac{\partial}{\partial n}\int_S \Phi(Q)\frac{\partial G}{\partial v}\mathrm{d}S = \Psi(N), \quad \text{as } P \to N, \tag{4}$$

where $\Psi = \kappa v(N)$. In the scattering problem, ϕ can be considered as the scattered field in a total field ϕ^{tot} and incident field ϕ^{inc}. For acoustically hard surface $\partial\phi^{\text{tot}}/\partial n = 0$ as P tends to N and we again have eq. (4), where $\Psi(N) = -\kappa\partial\phi^{\text{inc}}(N)/\partial n$. In the surface integral of eq. (4) as P tends normally to the point N on $S, \partial G/\partial v\,\mathrm{d}S$ becomes $O(1/r)$ singular. Hence the CPV of the integral denoted by $⨍$ has to be taken. If further, the derivative $\partial/\partial n$ is taken within the integral sign, there results

$$⨎_S \Phi(Q)\frac{\partial^2 G}{\partial n \partial v}\mathrm{d}S = \Psi(P), \quad \text{as } P \to N. \tag{5}$$

Now $\partial^2 G/\partial n\partial v\,\mathrm{d}S$ becomes $O(1/r^2)$ singular as $P \to N$ and the integral is divergent. Justification and meaning in the sense of Hadamard finite part (HFP) denoted by $⨎$ has been provided by Krishnasamy *et al* [5]. In the following we give a shorter justification.

Lemma 1. *For infinitesimal region S_0 with centroid N*

$$\int_{S_0} \frac{\partial^2 G}{\partial n \partial v}\mathrm{d}S \to 0, \quad \textit{as } S_0 \textit{ shrinks to } N.$$

Let 2ε be the diameter of the smallest circular region with centre N containing S_0. Then the integral in magnitude, is less than or equal to that over S_ε. Now introducing local coordinates with centre N as origin (figure 2).

$$\frac{\partial^2 G}{\partial n \partial v} = \frac{\partial^2}{\partial z^2}\left(\frac{\mathrm{e}^{-ikr}}{r}\right).$$

Hence

$$\int_{S_\varepsilon} \frac{\partial G}{\partial n \partial v}\mathrm{d}S = \int_{\theta=0}^{2\pi}\mathrm{d}\theta\int_{r'=0}^{\varepsilon} \mathrm{e}^{-ikr}\left[\left(\frac{3}{r^3}+\frac{3ik}{r^2}-\frac{k^2}{r}\right)\frac{z^2}{r^2}-\frac{1}{r^3}-\frac{ik}{r^2}\right]r'\mathrm{d}r'$$

$$= -\int_{\theta=0}^{2\pi}\int_{z}^{(\varepsilon^2+z^2)^{1/2}} \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{e}^{-ikr}}{r}\left\{(1+ikr)\frac{z^2}{r^2}-1\right\}\right]\mathrm{d}r$$

$$= -2\pi\left[\frac{e^{-ik(\varepsilon^2+z^2)^{1/2}}}{\sqrt{\varepsilon^2+z^2}}\left\{\frac{ikz^2}{\sqrt{\varepsilon^2+z^2}} - \frac{\varepsilon^2}{\varepsilon^2+z^2}\right\} - ike^{-ikz}\right]$$

$$\to 0 \text{ as } \varepsilon \to 0.$$

(Terai [11]).

Reverting to the hyper singular integral, we define its finite part in the manner of the integral in one variable (eq. (22))

$$\oint_S \Phi(Q)\frac{\partial^2 G}{\partial n \partial v}\mathrm{d}S \overset{\text{def}}{=} \int_{S-S_0} \Phi(Q)\frac{\partial^2 G}{\partial n \partial v}\mathrm{d}S + \Phi(N)\int_{S_0}\frac{\partial^2 G}{\partial n \partial v}\mathrm{d}S, \quad \text{as } S_0 \to N$$

$$= \int_{S-S_0} \Phi(Q)\frac{\partial^2 G}{\partial n \partial v}\mathrm{d}S, \quad \text{as } S_0 \to N$$

by lemma 1. Hence,

$$\frac{\partial}{\partial n}\oint_S \Phi(Q)\frac{\partial G}{\partial v}\mathrm{d}S = \frac{\partial}{\partial n}\int_{S-S_0}\Phi(Q)\frac{\partial G}{\partial v}\mathrm{d}S, \quad \text{as } S_0 \to N$$

$$= \int_{S-S_0}\Phi(Q)\frac{\partial^2 G}{\partial n \partial v}\mathrm{d}S, \quad \text{as } S_0 \to N$$

$$= \oint_S \Phi(Q)\frac{\partial^2 G}{\partial n \partial v}\mathrm{d}S. \tag{6}$$

In the case of an open arc in two dimensions, the above does not hold. It can however be proved more easily as indicated in Remark of § 6.

Referring to figure 1, the hyper singular kernel can be written in view of

$$G = G(r) \quad \text{and} \quad \frac{\partial r}{\partial v} = \cos(\mathbf{r}, \mathbf{v}),$$

$$\frac{\partial^2 r}{\partial n \partial v} = -\frac{\cos(\mathbf{n}, \mathbf{v})}{r} + \frac{1}{r}\cos(\mathbf{r}, \mathbf{n})\cos(\mathbf{r}, \mathbf{v})$$

as

$$\frac{\partial^2 G}{\partial n \partial v} = -\left(\frac{\partial^2 G}{\partial r^2} - \frac{1}{r}\frac{\partial G}{\partial r}\right)\cos(\mathbf{r}, \mathbf{n})\cos(\mathbf{r}, \mathbf{v}) - \frac{1}{r}\frac{\partial G}{\partial r}\cos(\mathbf{n}, \mathbf{v}). \tag{7}$$

In the following we consider reduction of the finite part integral of eq. (5) to CPV and ordinary integrals. We need separate treatment for three- and two-dimensional cases.

3. Three dimensional case

In this case, $G = e^{-ikr}/r$ and eq. (7) becomes

$$\frac{\partial^2 G}{\partial n \partial v} = -e^{-ikr}\left(\frac{3}{r^3} + \frac{3ik}{r^2} - \frac{k^2}{r}\right)\cos(\mathbf{r}, \mathbf{n})\cos(\mathbf{r}, \mathbf{v}) + e^{-ikr}\left(\frac{1}{r^3} + \frac{ik}{r^2}\right)\cos(\mathbf{n}, \mathbf{v}). \tag{8}$$

The $O(r^{-3})$ terms give rise to finite part hyper singular integrals in eq. (5), while $O(r^{-2})$ and $O(r^{-1})$ terms yield CPV singular and regular integrals respectively. For the left hand side of the equation we can write

$$\oint_S \Phi(Q)\frac{\partial^2 G}{\partial n \partial v} dS = \oint_S [\Phi(Q) - \Phi(N)]\frac{\partial^2 G}{\partial n \partial v} dS + \Phi(N)\oint_S \frac{\partial^2 G}{\partial n \partial v} dS. \tag{9}$$

Significantly, by Lemma 1, the singularity in the last integral can in fact be ignored and it is possible to transform the integral by Stokes theorem. Noting that $G = G(r)$ with $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$, $\nabla_x G = -\nabla G$, where ∇_x is the divergence operator in the space of $P(x, y, z)$ and ∇ is the same operator in the space of the variable point $Q(\xi, \eta, \zeta)$ of S. Hence

$$\frac{\partial^2 G}{\partial n \partial v} = \mathbf{n}\cdot\nabla_x(\mathbf{v}\cdot\nabla G)$$

$$= \left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right)\left(\lambda\frac{\partial G}{\partial \xi} + \mu\frac{\partial G}{\partial \eta} + v\frac{\partial G}{\partial \zeta}\right),$$

where (l, m, n) and (λ, μ, v) are respectively the projections of $\mathbf{n}$ and v on the coordinate axes. Since $\nabla_x(\partial G/\partial\xi) = -\nabla(\partial G/\partial\xi)\ldots$, we can rearrange to obtain

$$\frac{\partial^2 G}{\partial n \partial v} = -(\mathbf{n}\cdot\mathbf{v})\nabla^2 G + \mathbf{v}\cdot\text{curl}[\mathbf{n}\times\nabla G]. \tag{10}$$

Thus, noting that $\nabla^2 G = -k^2 G$ and applying Stokes theorem to the second term

$$\oint_S \frac{\partial^2 G}{\partial n \partial v} dS = k^2 \int_S (\mathbf{n}\cdot\mathbf{v}) G dS + \int_C [\mathbf{n}\times\nabla G]\cdot d\boldsymbol{\xi}, \tag{11}$$

the right hand side of which consists of regular integrals.

In the first integral on the right hand side of eq. (10), there is a large part in the integrand which yields regular integrals. To separate the Cauchy principal value integral in it, we introduce the potential $G_0 = 1/r$ for which

$$\frac{\partial^2 G_0}{\partial n \partial v} = -\frac{3}{r^3}\cos(\mathbf{r}, \mathbf{n})\cos(\mathbf{r}, \mathbf{v}) + \frac{1}{r^3}\cos(\mathbf{n}, \mathbf{v}). \tag{12}$$

We can then write

$$\oint_S [\Phi(Q) - \Phi(N)]\frac{\partial^2 G}{\partial n \partial v} dS = \int_S [\Phi(Q) - \Phi(N)]\frac{\partial^2 (G - G_0)}{\partial n \partial v} dS + \oint_S [\Phi(Q) - \Phi(N)]\frac{\partial^2 G_0}{\partial n \partial v} dS. \tag{13}$$

The full integrand of the first integral on the right hand side of eq. (12) can be explicitly written down from eqs (8) and (13). For the second we use identity (10) with $\nabla^2 G_0 = 0$ and ∇G_0 calculable explicitly, we get

$$\begin{aligned}\frac{\partial^2 G_0}{\partial n \partial v} = -2\frac{\mathbf{n}\cdot\mathbf{v}}{r^3} + \frac{3}{r^3}\Bigg[&\left(l\frac{\eta - y}{r} - m\frac{\xi - x}{r}\right)\left(\lambda\frac{\eta - y}{r} - \mu\frac{\xi - x}{r}\right) \\ &+ \left(m\frac{\zeta - z}{r} - n\frac{\eta - y}{r}\right)\left(\mu\frac{\zeta - z}{r} - v\frac{\eta - y}{r}\right) \\ &+ \left(n\frac{\xi - x}{r} - l\frac{\zeta - z}{r}\right)\left(v\frac{\xi - x}{r} - \lambda\frac{\zeta - z}{r}\right)\Bigg].\end{aligned} \tag{14}$$

The first term on the right hand side leads to CPV integral while the second leads to a regular integral since

$$(x-\xi)/r:(y-\eta)/r:(z-\zeta)/r \to \lambda:\mu:\nu: \to l:m:n$$

as $Q \to N$, $P \to N$.

With the above procedure (eqs (10)–(14)), eq. (9) in the limit $P \to N$, yields for the original hyper singular integral equation (5), the linear CPV singular integral equation

$$\begin{aligned}
&\Phi(P)\left[k^2 \int_S (\mathbf{n}\cdot\mathbf{v})\frac{e^{-ikr}}{r}dS + \int_C \left[\mathbf{n}\times\nabla\left(\frac{e^{-ikr}}{r}\right)\right]\cdot d\boldsymbol{\xi}\right.\\
&+ \int_S [\Phi(Q)-\Phi(P)]\left[-\left\{(e^{-ikr}-1)\frac{3}{r^3}+e^{-ikr}\left(\frac{3ik}{r^2}-\frac{k^2}{r}\right)\right\}\cos(\mathbf{r},\mathbf{n})\cos(\mathbf{r},\mathbf{v})\right.\\
&\left.+\left\{(e^{-ikr}-1)\frac{1}{r^3}+e^{-ikr}\frac{ik}{r^2}\right\}\cos(\mathbf{n},\mathbf{v})\right]dS\\
&+3\int_S [\Phi(Q)-\Phi(P)][\{l(\eta-y)-m(\xi-x)\}\{\lambda(\eta-y)-\mu(\xi-x)\}\\
&+\{m(\zeta-z)-n(\eta-y)\}\{\mu(\zeta-z)-\nu(\eta-y)\}+\{n(\xi-x)-l(\zeta-z)\}\\
&\{\nu(\xi-x)-\lambda(\zeta-z)\}\Big]\frac{dS}{r^4}-2\,⨍_S [\Phi(Q)-\Phi(P)]\frac{\mathbf{n}\cdot\mathbf{v}}{r^3}dS = \Psi(P), \quad P\to N.
\end{aligned} \tag{15}$$

A feature to note, is the occurence of $\Phi(P)$ inside integrals of the equation. This does not pose much numerical difficulty when the integrals are replaced by quadrature formulae.

4. A simple integral equation in tangential derivative

Integral equations of this type were first obtained by Maue [8] for the scattering problem. Using eq. (11)

$$\begin{aligned}
⨍_S \Phi(Q)\frac{\partial^2 G}{\partial n \partial \nu}dS &= k^2 \int_S (\mathbf{n}\cdot\mathbf{v})G\Phi(Q)dS\\
&+ ⨍_S \mathbf{v}\cdot\{\mathrm{curl}[\Phi(Q)(\mathbf{n}\times\nabla G)] - \nabla\Phi(Q)\times(\mathbf{n}\times\nabla G)\}dS.
\end{aligned}$$

By Stokes theorem, the second integral on the right hand side is equal to

$$\int_C \Phi(Q)(\mathbf{n}\times\nabla G)\cdot d\boldsymbol{\xi} = 0,$$

since $\Phi(Q) = \phi^+ - \phi^- = 0$ on the rim C. Hence eq. (5) becomes

$$\begin{aligned}
&⨍_S \nabla\Phi(Q)\times(\mathbf{n}\times\nabla r)e^{-ikr}\left(\frac{1}{r^2}+\frac{ik}{r}\right)dS\\
&+k^2 \int_S (\mathbf{n}\cdot\mathbf{v})\Phi(Q)\frac{e^{-ikr}}{r}dS = \Psi(P), \quad P\to N.
\end{aligned} \tag{16}$$

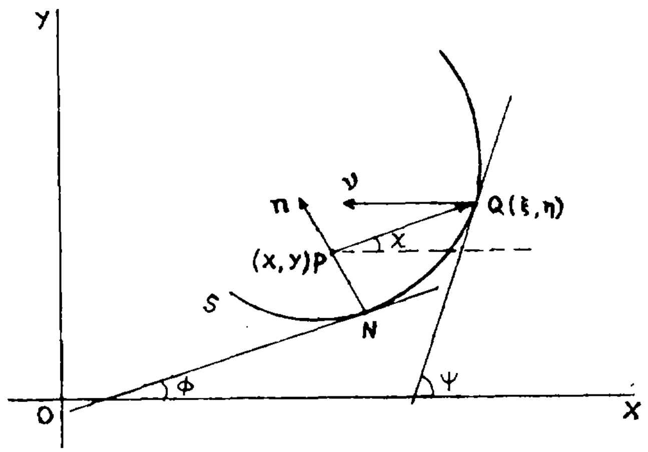

Figure 3. The geometry in two dimensions.

This equation does not contain $\Phi(P)$ inside the integrals.

5. Two dimensional case

In view of the fact that the surface integral degenerates into a line integral along a plane arc, the two dimensional case needs separate treatment. Denoting the inclinations of the tangents at N and Q to OX by ϕ, ψ and that of $\mathbf{r}$ by χ (figure 3),

$$\angle(\mathbf{r},\mathbf{n}) = \frac{\pi}{2} + \phi - \chi, \quad \angle(\mathbf{r},\mathbf{v}) = \frac{\pi}{2} + \psi - \chi, \quad \angle(\mathbf{n},\mathbf{v}) = \psi - \phi$$

in eq. (7). Also since

$$G = \frac{i\pi}{2} H_0^{(2)}(kr), \quad \frac{\partial G}{\partial r} = -\frac{i\pi k}{2} H_1^{(2)}(kr),$$

$$\frac{\partial^2 G}{\partial r^2} = \frac{i\pi k^2}{2}\left[\frac{1}{kr} H_1^{(2)}(kr) - H_0^{(2)}(kr)\right].$$

Thus, (7) takes the form

$$\frac{\partial^2 G}{\partial n \partial v} = \frac{i\pi k^2}{2}[H_0^{(2)}(Kr)\sin(\phi - \chi)\sin(\psi - \chi)$$

$$+ \frac{1}{kr} H_1^{(2)}(kr)\cos(\phi + \psi - 2\chi)]. \tag{17}$$

If we assume that the equation of S is given parametrically

$$x \to f(t), \quad y \to g(t); \quad \xi = f(\tau), \quad \eta = g(\tau), \quad a \leqslant t, \quad \tau \leqslant b \tag{18}$$

then the trigonometric expressions in the limit $P \to N$ can be written as

$$\sin(\phi - \chi)\sin(\psi - \chi) = \frac{K_1(t,\tau)}{\sqrt{x'^2 + y'^2}\sqrt{\xi'^2 + \eta'^2}} \tag{19}$$

$$\cos(\phi + \psi - 2\chi) = \frac{K_2(t,\tau)}{\sqrt{x'^2 + y'^2}\sqrt{\xi'^2 + \eta'^2}} \tag{19a}$$

where, $\tan\phi = y'/x'$, $\tan\psi = \eta'/\xi'$, primes denoting the derivatives and $\sin\chi = (\eta - y)/r$, $\cos\chi = (\xi - x)/r$. Also if $\phi < \pi/2$ and $\psi < \pi/2$,

$$K_1(t,\tau) = \frac{1}{r^2}[(\xi - x)^2\eta' y' + (\eta - y)^2\xi' x' - (\xi - x)(\eta - y)(\xi' y' + \eta' x')]. \tag{20}$$

$$K_2(t,\tau) = \frac{1}{r^2}[\{(\xi - x)^2 - (\eta - y)^2\}(\xi' x' - \eta' y') + 2(\xi - x)(\eta - y)(\xi' y' + \eta' x')]. \tag{20a}$$

If however, $\phi > \pi/2$ or $\psi > \pi/2, x', \xi'$ are to be replaced by $-x', -\xi'$. In the limiting case $P, Q \to N, r \to 0$ and $\psi \to \phi, \chi \to -((\pi/2) - \phi)$ and the left hand sides of (19) tend to 1 or -1 respectively. Hence, as $\tau \to t, K_1(t,\tau) = -K_2(t,\tau) \to x'^2 + y'^2$. With eqs (17), (19) and (20), eq. (5) becomes after a little rearrangement

$$\int_a^b \Phi(\tau)\left[H_0^{(2)}(kr)K_1(t,\tau) + \frac{1}{kr}\left\{H_1^{(2)}(kr) - \frac{2i}{\pi kr}\right\}K_2(t,\tau) + \frac{2i}{\pi k^2 r^2}K_2(t,\tau)\right]\mathrm{d}\tau = -\frac{2i}{\pi k^2}\sqrt{x'^2 + y'^2}\,\Psi(P), \quad P \to N. \tag{21}$$

The first term of the kernal within the square brackets is logarithmically singular and hence integrable; the second regular and the third hyper singular. In fact it can be shown from definition (20) that

$$\frac{K_2(t,\tau)}{r^2} \to -\tfrac{1}{2}(\tau - t)^{-2}, \quad \text{as } \tau \to t. \tag{22}$$

To extract the finite part of the hyper singular integral, we may use the following.

Lemma 2.

$$\int_a^b \frac{f(t,\tau)}{(\tau - t)^2}\mathrm{d}\tau = \int_a^b \frac{f(t,\tau) - f(t,t)}{(\tau - t)^2}\mathrm{d}\tau + f(t,t)\frac{b - a}{(t - a)(t - b)}.$$

From definition of the finite part of the hyper singular integral

$$\int_a^b \frac{g(\tau)}{(\tau - t)^2}\mathrm{d}\tau = \int_a^{t-\varepsilon} \frac{g(\tau)}{(\tau - t)^2} + \int_{t+\varepsilon}^b \frac{g(\tau)}{(\tau - t)^2}\mathrm{d}\tau + g(t)\int_{t-\varepsilon}^{t+\varepsilon} \frac{\mathrm{d}\tau}{(\tau - t)^2}, \quad \varepsilon \to 0. \tag{23}$$

The proof follows from the particular case $g(\tau) = 1$:

$$\int_a^b \frac{\mathrm{d}\tau}{(\tau - t)^2} = \frac{1}{t - b} - \frac{1}{t - a}$$

(Kaya and Erdogan [4]).

The integral equation can thus be rewritten as

$$\Phi(t) + \frac{(b - t)(t - a)}{b - a}\left[\pi k^2 \int_a^b \Phi(\tau)\left\{H_0^{(2)}(kr)K_1(t,\tau)\right.\right.$$

$$+\frac{1}{kr}\left(H_1^{(2)}(kr)-\frac{2i}{\pi kr}\right)K_2(t,\tau)\Bigg\}+2\,\,⨍_a^b\left\{\frac{\Phi(\tau)K_2(t,\tau)}{(\xi-x)^2+(\eta-y)^2}\right.$$

$$\left.+\frac{1}{2}\frac{\Phi(t)}{(\tau-t)^2}\right\}\mathrm{d}\tau\Bigg]$$

$$=-2\frac{(b-t)(t-a)}{b-a}\sqrt{x'^2+y'^2}\,\Psi(P),\quad P\to N. \tag{24}$$

As in the three dimensional case, $\Phi(t)$ occurs inside the (CPV) integral, in this formulation. From the equation we find that in the neighbourhood of the extremities of the arc $t\to a$ or $t\to b$, $\Phi(t)\to 0$, but the law of approach is not apparent.

6. Alternative formula containing derivative

Lemma 3.

$$⨍_a^b\frac{f(t,\tau)\Phi(\tau)}{(\tau-t)^2}\mathrm{d}\tau=⨍_a^b\frac{f(t,\tau)-f(t,t)}{(\tau-t)^2}\Phi(\tau)\mathrm{d}\tau$$

$$+f(t,t)\left[⨍_a^b\frac{\Phi'(\tau)}{\tau-t}\mathrm{d}\tau+\frac{\Phi(b)}{t-b}-\frac{\Phi(a)}{t-a}\right].$$

The proof follows from (Kaya and Erdogan [4])

$$⨍_a^b\frac{g(\tau)}{(\tau-t)^2}\mathrm{d}\tau=\frac{\mathrm{d}}{\mathrm{d}t}⨍_a^b\frac{g(\tau)}{\tau-t}\mathrm{d}\tau$$

$$=-\frac{g(a)}{t-a}+\frac{g(b)}{t-b}-\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b g'(\tau)\ln|\tau-t|\mathrm{d}\tau$$

$$=-\frac{g(a)}{t-a}+\frac{g(b)}{t-b}+⨍_a^b\frac{g'(\tau)}{\tau-t}\mathrm{d}\tau \tag{25}$$

where the second line is obtained by splitting (a,b) into $(a,t-\varepsilon)$ and $(t+\varepsilon,b)$, $\varepsilon\to 0$ and integration by parts.

Remark. Equation (25) justifies the validity of (5) in two dimensional case.

With the aid of the above lemma and eq. (21), the hyper singular integral in it can be written as

$$⨍_a^b\frac{K_2(t,\tau)}{r^2}\Phi(\tau)\mathrm{d}\tau=⨍_a^b\left[\frac{K_2(t,\tau)}{r^2}+\frac{1}{2(\tau-t)^2}\right]\Phi(\tau)\mathrm{d}\tau-\frac{1}{2}⨍_a^b\frac{\Phi(\tau)}{\tau-t}\mathrm{d}\tau. \tag{26}$$

The term contributed by $\Phi(a)$, $\Phi(b)$ drop out according to what has been stated at the end of the previous section. Equation (21) thus takes the form

$$⨍_a^b\frac{\Phi'(\tau)}{\tau-t}\mathrm{d}\tau-2⨍_a^b\Phi(\tau)\left[\frac{K_2(t,\tau)}{(\xi-x)^2+(\eta-y)^2}+\frac{1}{2(\tau-t)^2}\right]\mathrm{d}\tau$$

$$+i\pi k^2\int_a^b\Phi(\tau)\left[H_0^{(2)}(kr)K_1(t,\tau)+\frac{1}{kr}\left\{H_1^{(2)}(kr)-\frac{2i}{\pi kr}\right\}K_2(t,\tau)\right]\mathrm{d}\tau$$

$$= 2\sqrt{x'^2 + y'^2}\,\Psi(P), \quad P \to N. \tag{27}$$

This does not contain $\Phi(t)$ inside the integral.

The behaviour of $\Phi(t)$ at $t = a, b$ depends on the nature of solution of eq. (25) which is of the form

$$\int_a^b \frac{\Phi'(\tau)}{\tau - t} = F(t), \quad a \leqslant t \leqslant b. \tag{28}$$

If we think of $\Phi(t)$ as displacement discontinuity across S, $\Phi'(t)$ will represent a quantity proportional to stress and must become unbounded at $t = a, b$. Hence from the theory of singular integral equation of the above form (Sih [10]), the fundamental solution of (28) must be of the form

$$w(t) = \{(t-a)(b-t)\}^{-(1/2)}.$$

Thus, as $t \to a$ or b

$$\Phi'(t) = \{(t-a)(b-t)\}^{-(1/2)} \times \text{a bounded function}$$

or

$$\Phi(t) = \{(t-a)(b-t)\}^{1/2} \times \text{a bounded function},$$

which verifies a well-known fact.

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