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Degree of approximation of functions associated with Hardy–Littlewood series in the Hölder metric by Euler means

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Abstract. After establishing the Fourier character of the Hardy–Littlewood series the authors have studied the degree of approximation of functions associated with the same series in the Hölder metric using Euler means.

Keywords. Hardy–Littlewood series; Banach space; Hölder metric; Euler means.

1. Definition

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic continuous functions defined on $[-\pi,\pi]$ under sup-norm. For $0<\alpha\leqslant 1$ and some positive constant K, the function space H_α is given by

$$H_\alpha=\{f\in C_{2\pi}\colon |f(x)-f(y)|\leqslant K|x-y|^\alpha\}. \tag{1.1}$$

The space H_α is a Banach space (6) with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha=\|f\|_c+\sup_{x,y}\{\Delta^\alpha f(x,y)\} \tag{1.2}$$

where

$$\|f\|_c=\sup_{-\pi\leqslant x\leqslant\pi}|f(x)|$$

and

$$\Delta^\alpha f(x,y)=|f(x)-f(y)|\,|x-y|^{-\alpha},\ x\neq y. \tag{1.3}$$

We shall use the convention that $\Delta^0 f(x,y)=0$. The metric induced by norm (1.2) on H_α is called Hölder metric.

Let f be a periodic function of period 2π and integrable in the Lebesgue sense over $[-\pi,\pi]$. Let the Fourier series associated with f at x be

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}A_n(x). \tag{1.4}$$

Let us write

$$\Phi_x(u)=\tfrac{1}{2}\{f(x+u)+f(x-u)-2f(x)\}, \tag{1.5}$$

$$\chi_x(u)=\int_u^\pi \Phi_x(w)\tfrac{1}{2}\cot w/2\,\mathrm{d}w. \tag{1.6}$$

Let $S_n(x)$ and $S_n^*(x)$ respectively denote the partial sum and the modified partial sum of (1.4), i.e.,

$$S_n(x) = \sum_{k=0}^{n} A_k(x),\ S_n^*(x) = \sum_{k=0}^{n-1} A_k(x) + \tfrac{1}{2} A_n(x).$$

It is known ([7], p. 50) that

$$S_n^*(x) - f(x) = \frac{2}{\pi} \int_0^{\pi} \frac{\Phi_x(u) \sin nu\, du}{2 \tan u/2}. \tag{1.7}$$

Given any sequence $\{t_n\}$ its (E, q), $(q > 0)$ transformation is defined by ([4], p. 180)

$$E_n^q(t) = (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} t_k. \tag{1.8}$$

2. Introduction

By writing Hardy–Littlewood series or in short HL-series, we mean the series

$$\sum_{n=1}^{\infty} \frac{S_n(x) - f(x)}{n}. \tag{2.1}$$

We take this opportunity to acknowledge the fact that this nomenclature for the series (2.1) is first due to Prof. R Mohanty (see [5]).

Hardy and Littlewood [2] have shown that (2.1) is summable $(C, 1)$ to the value

$$\frac{1}{\pi} \int_{0+}^{\pi} \left\{ \left(\frac{\pi - u}{2} \right) \cot u/2 - \log(2 \sin u/2) \right\} \Phi_x(u) du,$$

whenever the integral

$$\int_{0+}^{\pi} \Phi_x(u) \tfrac{1}{2} \cot u/2\, du \tag{2.2}$$

exists. Further ([2], see also [7], p. 122) if

$$\int_0^t |\Phi(u)| du = o(t) \text{ as } t \to 0+, \tag{2.3}$$

then (2.1) converges if and only if (2.2) exists. The interest of the HL-series lies in its relation to the integral (2.2), these relations being very similar to those between the conjugate series $\Sigma\, B_n(x)$ and the integral

$$\int_{0+}^{\pi} \frac{\psi_x(u)}{u} du, \tag{2.4}$$

where $\psi_x(u) = \frac{1}{2}\{f(x+u) - f(x-u)\}$. It is known [7] that if $f \in L$ then (2.4) exists almost everywhere. On the other hand there exists a continuous function f fo• which the integral (2.2) diverges for almost all x [2].

At this stage, we remark that the above results on HL-series remain unaltered if we replace the HL-series by

$$\tfrac{1}{2} c_0 + \sum_{n=1}^{\infty} \frac{S_n^*(x) - f(x)}{n}, \tag{2.5}$$

where

$$c_0 = \frac{2}{\pi}\int_0^{\pi} \Phi_x(u)\frac{u}{2}\cot u/2\mathrm{d}u.$$

The series (2.5) is summable $(C, 1)$ to the value

$$\int_{0+}^{\pi} \Phi_x(u)\tfrac{1}{2}\cot u/2\mathrm{d}u$$

whenever this integral exists. Thus the convergence or summability problem of (2.5) is same as that of (2.1) though their sums are different and hence we may term (2.5) as HL-series.

Let $T_n(x)$ denote the nth partial sum of the series (2.5), i.e.,

$$T_n(x) = \tfrac{1}{2}c_0 + \sum_{k=1}^{n} \frac{S_k^*(u) - f(x)}{k}. \tag{2.6}$$

3. Main results

Chandra [1] has studied the degree of approximation problem for Fourier series by Euler's means. The object of the present paper is to determine the degree of approximation of the series (2.5) by means of Euler's transformation in the Hölder metric. We prove the following theorem.

Theorem. *If* $0 \leqslant \beta < \alpha \leqslant 1$ *and* $f \in H_\alpha$ *then*

$$\|E_n^q(T,.) - \chi(N)\|_\beta = O(n^{\beta-\alpha}(\log n)^{1+\beta/\alpha}) \tag{3.1}$$

where

$$N = \frac{\pi(1+q)}{n}$$

and $E_n^q(T,x)$ *is the Euler's transformation of* $T_n(x)$.

Fourier character of HL-series (2.5)

Let

$$\chi(u) = \chi_x(u) = \int_u^{\pi} \Phi_x(w)\tfrac{1}{2}\cot w/2\mathrm{d}w. \tag{3.2}$$

It is known [3] that χ is even and Lebesgue integrable.

Let

$$\chi \sim \tfrac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos nt. \tag{3.3}$$

We have

$$c_0 = \frac{2}{\pi}\int_0^{\pi} \chi_x(t)\mathrm{d}t = \frac{2}{\pi}\int_0^{\pi}\left(\int_t^{\pi} \Phi(u)\tfrac{1}{2}\cot u/2\mathrm{d}u\right)\mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^{\pi} \Phi(u)\tfrac{1}{2}\cot u/2\mathrm{d}u \int_0^{u}\mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^{\pi} \Phi(u)\frac{u}{2}\cot u/2\mathrm{d}u \tag{3.4}$$

and for $n \geqslant 1$

$$\begin{aligned} c_n &= \frac{2}{\pi}\int_0^\pi \chi_x(t)\cos nt\,dt \\ &= \frac{2}{\pi}\int_0^\pi \cos nt\left(\int_t^\pi \Phi(u)\tfrac{1}{2}\cot u/2\,du\right)dt \\ &= \frac{2}{\pi}\int_0^\pi \Phi(u)\tfrac{1}{2}\cot u/2\,du \int_0^u \cos nt\,dt \\ &= \frac{2}{\pi n}\int_0^\pi \frac{\Phi(u)\sin nu\,du}{2\tan u/2} \\ &= \frac{S_n^*(x) - f(x)}{n}. \end{aligned} \tag{3.5}$$

Thus, we have the following.

PROPOSITION

The Hardy–Littlewood series (2.5) is the Fourier series of the even function $\chi(u)$ at $u = 0$. In these circumstances

$$T_n(x) = \tfrac{1}{2}c_0 + \sum_{k=1}^{n} c_k = \frac{2}{\pi}\int_0^\pi \chi_x(u)D_n(u)\,du \tag{3.6}$$

where

$$D_n(u) = \frac{\sin(n+\frac{1}{2})u}{2\sin u/2}. \tag{3.7}$$

Throughout the paper we suppose $2\delta < \min[\pi/4,\ 1/q]$, $q > 0$. We further use the following notations

$$F_x(u) = \chi_x(u) - \chi_x(N) \tag{3.8}$$

$$G(u) = F_x(u) - F_y(u) \tag{3.9}$$

$$p_q^n(u) = (q+1)^{-n}(1 + q^2 + 2q\cos u)^{n/2} \tag{3.10}$$

$$g(u) = 1 + \frac{q\cos u}{(1 - q^2\sin^2 u)^{1/2}},\ qu < 1 \tag{3.11}$$

$$N = \frac{\pi(1+q)}{n} \tag{3.12}$$

$$b(y) = \tan^{-1}\left(\frac{\sin y}{q + \cos u}\right) \tag{3.13}$$

$$t_r = t_r(z) = z + \frac{r\pi}{n} + \sin^{-1}\left\{q\sin\left(z + \frac{r\pi}{n}\right)\right\} \text{ for } r = 0, 1, 2, \tag{3.14}$$

$$t_0(z) = t(z) = t \tag{3.15}$$

$$R_n^q(x) = \int_{1/n}^{\log n/(n)^{1/2}} t^{-1}\{F_x(t) - F_x(t_1)\}\,p_q^n(t)\,dz \tag{3.16}$$

$$P(n,u)=(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\cos ku \tag{3.17}$$

$$Q(n,u)=(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sin ku \tag{3.18}$$

$$K_n(F_x)=\int_{b(N)}^{b(\delta)}\left\{\left(\frac{1}{t}-\frac{1}{t_1}\right)F_x(t)p_q^n(t)g(z)\right.$$
$$\left.-\left(\frac{1}{t_1}-\frac{1}{t_2}\right)F_x(t_1)p_q^n(t_1)g(z+\pi/n)\right\}\sin nz\,dz \tag{3.19}$$

$$L_n(x)=E_n^q(T,x)-\chi_x(N). \tag{3.20}$$

4. Lemmas

To prove the theorem we will use the following lemmas.

Lemma 1. *If* $f\in H_\alpha$, $0<\alpha\leqslant 1$, *Then*

$$|\Phi_x(u)-\Phi_y(u)|=O(u^\alpha) \tag{4.1}$$

and also

$$|\Phi_x(u)-\Phi_y(u)|=O(|x-y|^\alpha). \tag{4.2}$$

The proof of the Lemma is an easy consequence of definition of $\Phi_x(u)$, $\Phi_y(u)$ and H_α.

Lemma 2. ([1], p. 101). *Let* $0\leqslant u\leqslant\pi$. *Then* $p_q^n(u)\leqslant e^{-Anu^2}$ (4.3)

where

$$A=2q(\pi(q+1))^{-2}$$

and

$$b(N)>n^{-1},\quad (n>4(q+1),\ q>0). \tag{4.4}$$

Lemma 3. ([1], p. 101). *Let* $0<z<\delta$. *Then*

$$t_r-t_{r-1}=O(n^{-1}),\quad (r=1,2) \tag{4.5}$$

and

$$2t_1-t-t_2=O(n^{-2})(z+\pi/n). \tag{4.6}$$

Lemma 4. ([1], p. 103). *Let* $0<z<\delta$. *Then*

$$p_q^n(t_1)g(z+\pi/n)-p_q^n(t)g(z)=O(n^{-1})\{z+\pi/n+n\sin t_1\}p_q^n(z). \tag{4.7}$$

Lemma 5. *Let* $\theta=\tan^{-1}\left(\dfrac{\sin u}{q+\cos u}\right)$ *and* $p_q^n(u)$ *be as in* (3.10), *then*

$$P(n,u)=p_q^n(u)\cos n\theta, \tag{4.8}$$

$$Q(n,u)=p_q^n(u)\sin n\theta, \tag{4.9}$$

$$P(n,u)=O(1), \tag{4.10}$$

$$Q(n,u)=O(nu). \tag{4.11}$$

Proof. By familiar computation

$$\begin{aligned} P(n,u) + iQ(n,u) &= (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} e^{iku} \\ &= (q+1)^{-n}(q+e^{iu})^n \\ &= p_q^n(u)(\cos n\theta + i \sin n\theta) \end{aligned}$$

which ensures (4.8) and (4.9). Since $|\cos ku| \leqslant 1, |\sin ku| \leqslant ku$ and $\Sigma_{k=0}^{n} \binom{n}{k} q^{n-k} = (q+1)^n$, estimates (4.10) and (4.11) follow at once.

Lemma 6. *If* $f \in H_\alpha$ *and* $0 < \alpha \leqslant 1$, *then*

$$F_x(u) = O(1) \begin{cases} u^\alpha, & u > N \\ n^{-\alpha}, & u < N, \end{cases} \tag{4.12}$$

$$G(u) = O(1) \begin{cases} u^\alpha, & u > N \\ n^{-\alpha}, & u < N \end{cases} \tag{4.13}$$

and

$$G(u) = O(1)|x-y|^\alpha \begin{cases} \log n, & u > N \\ \log 1/u, & u < N. \end{cases} \tag{4.14}$$

Proof. As $f \in H_\alpha$, we have

$$\begin{aligned} F_x(u) &= \chi_x(u) - \chi_x(N) \\ &= \int_u^N \Phi_x(w) \tfrac{1}{2} \cot w/2 \mathrm{d}w \\ &= O(1) \int_u^N w^{\alpha-1} \mathrm{d}w \\ &= O(1) \begin{cases} u^\alpha, & u > N \\ n^{-\alpha}, & u < N. \end{cases} \end{aligned}$$

Again using Lemma 1, we get

$$\begin{aligned} G(u) &= \int_u^N \frac{\Phi_x(w) - \Phi_y(w)}{2\tan w/2} \mathrm{d}w \quad = O(1) \int_u^N w^{\alpha-1} \mathrm{d}w \\ &= O(1) \begin{cases} u^\alpha, & u > N \\ n^{-\alpha}, & u < N \end{cases} \end{aligned}$$

and

$$\begin{aligned} G(u) &= \int_u^N \frac{\{\Phi_x(w) - \Phi_y(w)\}}{2\tan w/2} \mathrm{d}w \\ &= O(1)|x-y|^\alpha \int_u^N \frac{\mathrm{d}w}{w} \\ &= O(1)|x-y|^\alpha \begin{cases} \log n, & u > N \\ \log 1/u, & u < N. \end{cases} \end{aligned}$$

This completes the proof.

Lemma 7. *If* $f \in \mathrm{Lip}\,\alpha$, $(0 < \alpha \leqslant 1)$ *then*

$$\|K_n(F)\| = O(n^{-\alpha}) + O(\|R_n^q\|), \tag{4.15}$$

where R_n^q *and* $K_n(F_x)$ *are defined in* (3.16) *and* (3.19) *respectively.*

Proof.

$$K_n(F_x) = \int_{b(N)}^{b(\delta)} \left\{ \left(\frac{1}{t} - \frac{1}{t_1}\right) F_x(t) p_q^n(t) g(z) - \left(\frac{1}{t_1} - \frac{1}{t_2}\right) F_x(t_1) p_q^n(t_1) g(z + \pi/n) \right\} \sin nz \, \mathrm{d}z$$

$$= \int_{b(N)}^{b(\delta)} \left[\left(\frac{t_2 - t_1}{t_1 t_2}\right) \{F_x(t) - F_x(t_1)\} p_q^n(t_1) g(z + \pi/n) \sin nz + \left(\frac{t_2 - t_1}{t t_1}\right) \{p_q^n(t) g(z) - p_q^n(t_1) g(z + \pi/n)\} F_x(t) \sin nz + \frac{(t_2 - t_1)(t_2 - t)}{t t_1 t_2} \{p_q^n(t_1) g(z + \pi/n)\} F_x(t) \sin nz + \left(\frac{2t_1 - t - t_2}{t t_1}\right) p_q^n(t) g(z) F_x(t) \sin nz \right] \mathrm{d}z$$

$$= K_n^1(x) + K_n^2(x) + K_n^3(x) + K_n^4(x), \quad \text{say.} \tag{4.16}$$

By using Lemmas 2 and 3 we have

$$\|K_n^1\|_c = O(n^{-1}) \int_{b(N)}^{b(\delta)} t_1^{-1} t_2^{-1} |F_x(t) - F_x(t_1)| p_q^n(t_1) \mathrm{d}z$$

$$= O(n^{-1}) n \int_{1/n}^{\delta} t_1^{-1} |(F_x(t) - F_x(t_1))| p_q^n(t_1) \mathrm{d}z$$

$$= O(1) \left\{ \int_{1/n}^{\log n/(n)^{1/2}} t_1^{-1} |F_x(t) - F_x(t_1)| p_q^n(t_1) \mathrm{d}z + \int_{\log n/(n)^{1/2}}^{\delta} t_1^{-1} |F_x(t) - F_x(t_1)| p_q^n(t_1) \mathrm{d}z \right\}$$

$$= O(\|R_n^q\|) + O(1) \int_{\log n/(n)^{1/2}}^{\delta} t_1^{-1} |F_x(t) - F_x(t_1)| p_q^n(t_1) \mathrm{d}z$$

$$= O(\|R_n^q\|) + O(1) \int_{\log n/(n)^{1/2}}^{\delta} t_1^{\alpha - 1} \mathrm{e}^{-Ant_1^2} \mathrm{d}z$$

$$= O(\|R_n^q\|) + O(1) \int_{\log n/(n)^{1/2}}^{\delta} z^{\alpha - 1} \mathrm{e}^{-Anz^2} \mathrm{d}z$$

$$= O(\|R_n^q\|) + O(n^{-1}) \int_{\log n/(n)^{1/2}}^{\delta} z^{\alpha - 2} \frac{\mathrm{d}}{\mathrm{d}z}(-\mathrm{e}^{-Anz^2}) \mathrm{d}z$$

$$= O(\|R_n^q\|) + O(n^{-1})\left\{|[z^{\alpha-2}(-e^{-Anz^2})]_{\log n/(n)^{1/2}}^{\delta}| + \int_{\log n/(n)^{1/2}}^{\delta} z^{\alpha-3}e^{-Anz^2}dz\right\}$$

$$= O(\|R_n^q\|) + O(n^{-1})\frac{(\log n)^{\alpha-2}}{n^{\alpha/2-1}}\exp(-A(\log n)^2)$$

$$= O(\|R_n^q\|) + O(n^{-\alpha}). \tag{4.17}$$

Using Lemmas 2, 3, 4 and 6 we get

$$\|K_n^2\|_c = \left|\int_{b(N)}^{b(\delta)}\left(\frac{t_2-t_1}{tt_1}\right)\{p_q^n(t)g(z) - p_q^n(t_1)g(z+\pi/n)\}F_x(t)\sin nz\,dz\right|$$

$$= O(n^{-1})\int_{b(N)}^{b(\delta)}\frac{n^{-1}(z+\pi/n+n\sin t_1)}{tt_1}p_q^n(z)|F_x(t)||\sin nz|dz$$

$$= O(n^{-2})\int_{b(N)}^{b(\delta)}\frac{(nz)(nz)z^{\alpha}e^{-Anz^2}dz}{z^2} = O(1)\int_{1/n}^{\delta} z^{\alpha}e^{-Anz^2}dz$$

$$= O(n^{-1})\int_{1/n}^{\delta} z^{\alpha-1}\frac{d}{dz}(-e^{-Anz^2})dz$$

$$= O(n^{-1})\left\{|[z^{\alpha-1}(-e^{-Anz^2})]_{1/n}^{\delta}| + (1-\alpha)\int_{1/n}^{\delta} z^{\alpha-2}(e^{-Anz^2})dz\right\}$$

$$= O(n^{-\alpha}). \tag{4.18}$$

Using Lemmas 2, 3 and 6 we get

$$\|K_n^3\|_c = \left\|\int_{b(N)}^{b(\delta)}\frac{(t_2-t_1)(t_2-t)}{tt_1t_2}p_q^n(t_1)g(z+\pi/n)F_x(t)\sin nz\,dz\right\|$$

$$= O(n^{-2})\int_{1/n}^{\delta} t^{-3}\|F_x(t)\|dz$$

$$= O(n^{-\alpha}). \tag{4.19}$$

Again by Lemmas 2, 3 and 6 we get

$$\|K_n^4\|_c = \left\|\int_{b(N)}^{b(\delta)}\left(\frac{2t_1-t-t_2}{tt_1}\right)p_q^n(t)F(t)g(z)\sin nz\,dz\right\|$$

$$= O(n^{-2})\int_{1/n}^{\delta}\frac{(z+\pi/n)\|F(t)\|dz}{tt_1}$$

$$= O(n^{-2})\int_{1/n}^{\delta} z^{\alpha-1}dz$$

$$= O(n^{-2}). \tag{4.20}$$

Using (4.18), (4.19), (4.20) and (4.21) in (4.17) we get Lemma 7.

5. Proof of the theorem

From (3.6) and (1.8) we have

$$E_n^q(x)=(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left[\frac{2}{\pi}\int_0^{\pi}\chi_x(u)D_k(u)\mathrm{d}u\right].$$

Hence

$$\begin{aligned}L_n(x)&=E_n^q(x)-\chi_x(N)\\&=E_n^q(x)-\frac{2}{\pi}\int_0^{\pi}\chi(N)D_k(u)\mathrm{d}u\\&=\frac{2}{\pi}\int_0^{\pi}\{\chi(u)-\chi(N)\}\left\{(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}D_k(u)\right\}\mathrm{d}u\\&=\frac{2}{\pi}\int_0^{\pi}F_x(u)\left\{(q+1)^{-n}\sum_{k=0}^{n}\binom{k}{n}q^{n-k}\frac{\sin(k+\frac{1}{2})u}{2\sin u/2}\right\}\mathrm{d}u\\&=\frac{2}{\pi}\int_0^{\pi}F_x(u)\left\{(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{\sin ku}{2\tan u/2}\right\}\mathrm{d}u\\&\quad+\frac{2}{\pi}\int_0^{\pi}F_x(u)\left\{(q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{\cos ku}{2}\right\}\mathrm{d}u\\&=\frac{2}{\pi}\int_0^{\pi}F_x(u)\frac{Q(n,u)}{2\tan u/2}\mathrm{d}u+\frac{1}{\pi}\int_0^{\pi}F_x(u)P(n,u)\mathrm{d}u.\end{aligned}\tag{5.1}$$

And hence

$$\begin{aligned}L_n(x)-L_n(y)&=\frac{2}{\pi}\int_0^{\pi}\{F_x(u)-F_y(u)\}\frac{Q(n,u)}{2\tan u/2}\mathrm{d}u\\&\quad+\frac{1}{\pi}\int_0^{\pi}\{F_x(u)-F_y(u)\}P(n,u)\mathrm{d}u\\&=\frac{2}{\pi}\int_0^{\pi}\frac{G(u)Q(n,u)\mathrm{d}u}{2\tan u/2}+\frac{1}{\pi}\int_0^{\pi}G(u)P(n,u)\mathrm{d}u\\&=I+J,\quad\text{say.}\end{aligned}\tag{5.2}$$

Now

$$\begin{aligned}J&=\frac{1}{\pi}\int_0^{\pi}G(u)P(n,u)\mathrm{d}u=\frac{1}{\pi}\left[\int_0^{N}+\int_N^{\log n/(n)^{1/2}}+\int_{\log n/(n)^{1/2}}^{\pi}\right]\\&=J_1+J_2+J_3,\ \text{say.}\end{aligned}\tag{5.3}$$

Using Lemmas 5 and 6, we obtain

$$J_1=O(1)n^{-\alpha}\int_0^{N}\mathrm{d}u=O(n^{-1-\alpha})\tag{5.4}$$

and

$$\begin{aligned}J_1&=O(1)|x-y|^{\alpha}\int_0^{N}\log\frac{1}{u}\mathrm{d}u\\&=O(1)|x-y|^{\alpha}\frac{\log n}{n}.\end{aligned}\tag{5.5}$$

Using Lemmas 2 and 6

$$\begin{aligned}
J_2 &= \frac{1}{\pi}\int_N^{\log n/(n)^{1/2}} G(u)P(n,u)\mathrm{d}u \\
&= O(1)\int_N^{\log n/(n)^{1/2}} u^{\alpha}\mathrm{e}^{-Anu^2}\mathrm{d}u \\
&= O(n^{-1})\int_N^{\log n/(n)^{1/2}} u^{\alpha-1}\frac{\mathrm{d}}{\mathrm{d}u}(-\mathrm{e}^{-Anu^2})\mathrm{d}u \\
&= O(n^{-1})\left\{|[u^{\alpha-1}(-\mathrm{e}^{-Anu^2})]_N^{\log n/(n)^{1/2}}| + (1-\alpha)\int_N^{\log n/(n)^{1/2}} u^{\alpha-2}\mathrm{e}^{-Anu^2}\mathrm{d}u\right\} \\
&= O(n^{-\alpha}).
\end{aligned} \tag{5.6}$$

Again by Lemmas 5 and 6

$$\begin{aligned}
J_2 &= \frac{1}{\pi}\int_N^{\log n/(n)^{1/2}} G(u)P(n,u)\mathrm{d}u \\
&= O(1)|x-y|^{\alpha}\log n\int_N^{\log n/(n)^{1/2}} \mathrm{d}u \\
&= O(1)|x-y|^{\alpha}\frac{(\log n)^2}{\sqrt{n}}.
\end{aligned} \tag{5.7}$$

By Lemmas 2 and 6

$$\begin{aligned}
J_3 &= \frac{1}{\pi}\int_{\log n/(n)^{1/2}}^{\pi} G(u)p_q^n(u)\cos n\theta\,\mathrm{d}u \\
&= O(1)\int_{\log n/(n)^{1/2}}^{\pi} u^{\alpha}p_q^n(u)\mathrm{d}u \\
&= O(1)\int_{\log n/(n)^{1/2}}^{\pi} u^{\alpha}\mathrm{e}^{-Anu^2}\mathrm{d}u \\
&= O(1)\mathrm{e}^{-A(\log n)^2}\int_{\log n/(n)^{1/2}}^{\pi} u^{\alpha}\mathrm{d}u \\
&= O\left(\frac{1}{n^{\Delta}}\right), \quad \Delta > 0 \text{ however large}
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
J_3 &= \frac{1}{\pi}\int_{\log n/(n)^{1/2}}^{\pi} G(u)p_q^n(u)\cos n\theta\,\mathrm{d}u \\
&= O(1)|x-y|^{\alpha}\log n\int_{\log n/(n)^{1/2}}^{\pi} \mathrm{e}^{-Anu^2}\mathrm{d}u \\
&= O(1)|x-y|^{\alpha}\log n\,\mathrm{e}^{-A(\log n)^2}\int_{\log n/(n)^{1/2}}^{\pi} \mathrm{d}u \\
&= O(1)|x-y|^{\alpha}\frac{\log n}{n^{\Delta}}, \quad \Delta > 0, \text{ however large.}
\end{aligned} \tag{5.9}$$

Collecting the estimates of $J_i (i = 1, 2, 3)$ from (5.4), (5.5), (5.6), (5.7), (5.8) and (5.9) and using them in (5.3) we get

$$J = O(1)\begin{cases} n^{-\alpha} \\ |x-y|^\alpha \end{cases} \frac{(\log n)^2}{\sqrt{n}}. \tag{5.10}$$

Now

$$\begin{aligned} I &= \frac{2}{\pi}\int_0^\pi \frac{G(u)Q(n,u)\mathrm{d}u}{2\tan u/2} \\ &= \frac{2}{\pi}\left\{\int_0^N + \int_N^\delta + \int_\delta^\pi\right\} \\ &= I_1 + I_2 + I_3, \text{ say.} \end{aligned} \tag{5.11}$$

By Lemmas 5 and 6

$$\begin{aligned} I_1 &= \frac{2}{\pi}\int_0^N \frac{G(u)Q(n,u)\mathrm{d}u}{2\tan u/2} \\ &= O(1)n^{-\alpha}\int_0^N \mathrm{d}u \\ &= O(n^{-\alpha}) \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} I_1 &= \frac{2}{\pi}\int_0^N \frac{G(u)Q(n,u)}{2\tan u/2}\mathrm{d}u \\ &= O(1)|x-y|^\alpha n\int_0^N \log\frac{1}{u}\mathrm{d}u \\ &= O(1)|x-y|^\alpha \log n. \end{aligned} \tag{5.13}$$

Combining (5.12) and (5.13) we get

$$I_1 = O(1)\begin{cases} n^{-\alpha} \\ |x-y|^\alpha \log n. \end{cases} \tag{5.14}$$

By Lemmas 2, 5 and 6, we get

$$\begin{aligned} I_3 &= \frac{2}{\pi}\int_\delta^\pi \frac{G(u)p_q^n(u)\sin n\theta\,\mathrm{d}u}{2\tan u/2} \\ &= O(1)\int_\delta^\pi u^{\alpha-1}\mathrm{e}^{-Anu^2}\mathrm{d}u \\ &= O(1)\mathrm{e}^{-An\delta^2}\int_\delta^\pi u^{\alpha-1}\mathrm{d}u \\ &= O(1)\mathrm{e}^{-An\delta^2} \\ &= O(n^{-\Delta}), \quad \Delta > 0 \text{ however large} \end{aligned} \tag{5.15}$$

and

$$I_3 = \frac{2}{\pi}\int_\delta^\pi \frac{G(u)p_q^n(u)\sin n\theta\,\mathrm{d}u}{2\tan u/2}$$

$$= O(1)|x-y|^\alpha \log n \int_\delta^\pi \frac{e^{-Anu^2}}{u}du$$

$$= O(1)|x-y|^\alpha e^{-An\delta^2}\log n \int_\delta^\pi u^{-1}du$$

$$= O(1)\frac{|x-y|^\alpha}{n^\Delta}, \quad \Delta > 0, \text{ however large.} \tag{5.16}$$

Combining (5.15) and (5.16) we get

$$I_3 = O(1)\begin{cases} n^{-\Delta}, & \Delta > 0 \text{ however large} \\ \dfrac{|x-y|^\alpha}{n^\Delta}. \end{cases} \tag{5.17}$$

By Lemma 5

$$I_2 = \frac{1}{\pi}\int_N^\delta G(u)\cot u/2\, p_q^n(u)\sin n\theta\, du$$

$$= \frac{2}{\pi}\int_N^\delta \left[\frac{G(u)}{u}p_q^n(u)\sin n\theta + G(u)\left\{\frac{1}{2\tan u/2} - \frac{1}{u}\right\}p_q^n(u)\sin n\theta\right]du$$

$$= L + M, \quad \text{say.} \tag{5.18}$$

Using Lemmas 2 and 6 and the fact that $\{(\tan u/2)^{-1} - u^{-1}\} = O(u)$ we get

$$M = O(1)\int_N^\delta u^{\alpha+1}e^{-Anu^2}du$$

$$= O(n^{-1})\int_N^\delta u^\alpha \frac{d}{du}(-e^{-Anu^2})du$$

$$= O(n^{-1})\left\{|[u^\alpha(-e^{-Anu^2})]_N^\delta| + \int_N^\delta u^\alpha e^{-Anu^2}du\right\}$$

$$= O(n^{-1}). \tag{5.19}$$

Now for the estimation of L, we use the transformation

$$u = t(z) = z + \sin^{-1}(q\sin z),$$

which is same as $z = \tan^{-1}(\sin u/(q+\cos u))$.

By simple calculation, we have

$$du = g(z)dz, \quad \frac{\sin u}{q+\cos u} = \tan z$$

and

$$\sin n\theta = \sin n\left(\tan^{-1}\left(\frac{\sin u}{q+\cos u}\right)\right) = \sin nz$$

where $g(z)$ is defined in (3.11).

Throughout the present work, we write t for $t(z)$. Hence

$$L = \frac{2}{\pi}\int_N^\delta \frac{G(u)}{u}p_q^n(u)\sin n\theta\, du$$

$$=\frac{2}{\pi}\int_{b(N)}^{b(\delta)}\frac{G(u)}{t}p_q^n(t)\sin nz\,g(z)\mathrm{d}z$$

$$=\frac{1}{\pi}\left[\int_{b(N)}^{b(\delta)}+\int_{b(N)+\pi/n}^{b(\delta)+\pi/n}+\int_{b(N)}^{b(N)+\pi/n}-\int_{b(\delta)}^{b(\delta)+\pi/n}\right]. \tag{5.20}$$

Now

$$\frac{1}{\pi}\int_{b(N)+\pi/n}^{\delta+\pi/n}\frac{G(t)}{t}p_q^n(t)g(z)\sin nz\,\mathrm{d}z$$

$$=-\frac{1}{\pi}\int_{b(N)}^{b(\delta)}\frac{G(t_1)}{t_1}p_q^n(t_1)g(z+\pi/n)\sin nz\,\mathrm{d}z. \tag{5.21}$$

Using (5.21) in (5.20) we get

$$L=\frac{1}{\pi}\int_{b(N)}^{b(\delta)}\{t^{-1}G(t)p_q^n(t)g(z)-t_1^{-1}G(t_1)p_q^n(t_1)g(z+\pi/n)\}\sin nz\,\mathrm{d}z$$

$$+\frac{1}{\pi}\left(\int_{b(N)}^{b(N)+\pi/n}-\int_{b(\delta)}^{b(\delta)+\pi/n}\right)\frac{G(t)}{t}p_q^n(t)g(z)\sin nz\,\mathrm{d}z$$

$$=\frac{1}{\pi}T+\frac{1}{\pi}\left(\int_{b(N)}^{b(N)+\pi/n}-\int_{b(\delta)}^{b(\delta)+\pi/n}\right)\frac{G(t)}{t}p_q^n(t)g(z)\sin nz\,\mathrm{d}z. \tag{5.22}$$

Using Lemma 6 and the fact that $g(z)=O(1)$, we get

$$\int_{b(N)}^{b(\delta)+\pi/n}\frac{G(t)}{t}p_q^n(t)g(z)\sin nz\,\mathrm{d}z$$

$$=O(1)\int_{b(N)}^{b(N)+\pi/n}t^{\alpha-1}\mathrm{d}z=O(1)\int_{b(N)}^{b(N)+\pi/n}z^{\alpha-1}\mathrm{d}z$$

$$=O(n^{-\alpha}). \tag{5.23}$$

Again by Lemmas 2 and 6

$$\int_{b(\delta)}^{b(\delta)+\pi/n}\frac{G(t)}{t}p_q^n(t)g(z)\sin nz\,\mathrm{d}z$$

$$=O(1)\int_{b(\delta)}^{b(\delta)+\pi/n}t^{\alpha-1}\mathrm{e}^{-Ant^2}\mathrm{d}z$$

$$=O(1)\mathrm{e}^{-An\delta^2}=O(n^{-\Delta}),\quad \Delta>0 \text{ however large.} \tag{5.24}$$

Using (5.23) and (5.24) in (5.22) we get

$$L=\frac{1}{\pi}T+O(n^{-\alpha}). \tag{5.25}$$

Using (5.19) and (5.25) in (5.18) we get

$$I_2=\frac{1}{\pi}T+O(n^{-\alpha}). \tag{5.26}$$

Now

$$T=\int_{b(N)}^{b(\delta)}\left\{\frac{1}{t}G(t)-\frac{1}{t_1}G(t_1)\right\}p_q^n(t)g(z)\sin nz\,\mathrm{d}z$$

$$+\int_{b(N)}^{b(\delta)}\{p_q^n(t)g(z)-p_q^n(t_1)g(z+\pi/n)\}\frac{G(t_1)}{t_1}\sin nz\,\mathrm{d}z$$

$$=T_1+T_2,\quad \text{say.} \tag{5.27}$$

By Lemmas 2 and 4

$$\begin{aligned}T_2&=O(1)\int_{b(N)}^{b(\delta)}n^{-1}\{z+\pi/n+n\sin t_1\}p_q^n(z)t_1^{\alpha-1}\mathrm{d}z\\&=O(1)\int_{1/n}^{\delta}p_q^n(z)t_1^{\alpha}\mathrm{d}z\\&=O\left(\frac{1}{n}\right)\int_{1/n}^{\delta}z^{\alpha-1}\frac{\mathrm{d}}{\mathrm{d}z}(-\mathrm{e}^{-Anz^2})\mathrm{d}z\\&=O(n^{-\alpha}),\end{aligned} \tag{5.28}$$

integrating by parts. Now

$$\begin{aligned}T_1&=\int_{b(N)}^{b(\delta)}\frac{\{G(t)-G(t_1)\}}{t_1}p_q^n(t)\mathrm{d}z\\&\quad+\left(\int_{b(N)}^{b(N)+(\pi/n)}+\int_{b(N)+\pi/n}^{b(\delta)}\right)\left(\frac{1}{t}-\frac{1}{t_1}\right)G(t)p_q^n(t)g(z)\sin nz\,\mathrm{d}z\\&=T_{1,1}+T_{1,2}+T_{1,3},\quad\text{say.}\end{aligned} \tag{5.29}$$

As

$$G(t)-G(t_1)=(F_x(t)-F_x(t_1))-(F_y(t)-F_y(t_1)),$$

we have

$$\begin{aligned}T_{1,1}&=O(1)\int_{1/n}^{\delta}\frac{|(G(t)-G(t_1)|}{t_1}p_q^n(t)\mathrm{d}z\\&=O(1)\left[\int_{1/n}^{\log n/(n)^{1/2}}\frac{|G(t)-G(t_1)|}{t_1}p_q^n(t)\mathrm{d}z\right.\\&\quad\left.+\int_{\log n/(n)^{1/2}}^{\delta}\frac{|G(t)-G(t_1)|}{t_1}p_q^n(t)\mathrm{d}z\right]\\&=O(1)\left[\int_{1/n}^{\log n/(n)^{1/2}}\frac{|F_x(t)-F_x(t_1)|}{t_1}p_q^n(t)\mathrm{d}z\right.\\&\quad\left.+\int_{1/n}^{\log n/(n)^{1/2}}\frac{|F_y(t)-F_y(t_1)|}{t_1}p_q^n(t)\mathrm{d}z+\int_{\log n/(n)^{1/2}}^{\delta}\frac{|G(t)-G(t_1)|}{t_1}p_q^n(t)\mathrm{d}z\right]\\&=O(1)\left[R_n^q(x)+R_n^q(y)+\int_{\log n/(n)^{1/2}}^{\delta}\frac{|G(t)-G(t_1)|}{t_1}p_q^n(t)\mathrm{d}z\right].\end{aligned} \tag{5.30}$$

By Lemmas 4 and 6 we have

$$\int_{\log n/(n)^{1/2}}^{\delta}\frac{|G(t)-G(t_1)|}{t_1}p_q^n(t)\mathrm{d}z$$

$$= O(1)\int_{\log n/(n)^{1/2}}^{\delta} t_1^{\alpha-1} e^{-Anz^2} dz$$

$$= O\left(\frac{1}{n^{\Delta}}\right), \quad \Delta > 0 \text{ however large.} \tag{5.31}$$

Using (5.31) in (5.30), we get

$$T_{1,1} = O(1)(R_n^q(x) + R_n^q(y)) + O(n^{-\Delta}). \tag{5.32}$$

By Lemmas 3 and 6, we get

$$\begin{aligned} T_{1,2} &= O(1)\int_{b(N)}^{b(N)+\pi/n} \frac{n^{-1} t^{\alpha} nz\,dz}{t_1 t} \\ &= O(1)\int_{b(N)}^{b(N)+\pi/n} z^{\alpha-1} dz \\ &= O(n^{-\alpha}) \end{aligned} \tag{5.33}$$

$$\begin{aligned} 2T_{1,3} &= 2\int_{b(N)+\pi/n}^{\delta} \left(\frac{1}{t} - \frac{1}{t_1}\right) G(t) p_q^n(t) g(z) \sin nz\,dz \\ &= \left(\int_{b(N)}^{b(\delta)} + \int_{b(N)+\pi/n}^{b(\delta)+\pi/n}\right) + \int_{b(N)}^{b(N)+\pi/n} - \int_{b(\delta)}^{b(\delta)+\pi/n} \\ &= \int_{b(N)}^{b(\delta)} + \int_{b(N)+\pi/n}^{b(\delta)+\pi/n} + T_{1,2} + O(1)\int_{b(\delta)}^{b(\delta)+\pi/n} \left(\frac{1}{t} - \frac{1}{t_1}\right) |G(t)|\,dz. \end{aligned} \tag{5.34}$$

Since $t_1(z) = t(z + \pi/n)$ and $t_2(z) = t_1(z + \pi/n)$, replacing z by $z + \pi/n$, we get

$$\begin{aligned} &\int_{b(N)+\pi/n}^{b(\delta)+\pi/n} \left(\frac{1}{t} - \frac{1}{t_1}\right) G(t) p_q^n(t) g(z) \sin nz\,dz \\ &= -\int_{b(N)}^{b(\delta)} \left(\frac{1}{t_1} - \frac{1}{t_2}\right) G(t_1) p_q^n(t_1) g(z + \pi/n) \sin nz\,dz. \end{aligned} \tag{5.35}$$

By Lemmas 6

$$\begin{aligned} &\int_{b(\delta)}^{b(\delta)+\pi/n} \left(\frac{1}{t} - \frac{1}{t_1}\right) |G(t)|\,dz \\ &= O(n^{-1})\int_{b(\delta)}^{b(\delta)+\pi/n} \frac{t^{\alpha-1}}{t_1} dz \\ &= O(n^{-1}). \end{aligned} \tag{5.36}$$

Using (5.33), (5.35) and (5.36) in (5.34), we have

$$\begin{aligned} 2T_{1,3} &= \int_{b(N)}^{b(\delta)} \left\{ \left(\frac{1}{t} - \frac{1}{t_1}\right) G(t) p_q^n(t) g(z) \right. \\ &\quad \left. - \left(\frac{1}{t_1} - \frac{1}{t_2}\right) G(t_1) p_q^n(t_1) g(z + \pi/n) \right\} \sin nz\,dz + O(n^{-\alpha}) \\ &= K_n(F_x) - K_n(F_y) + O(n^{-\alpha}). \end{aligned}$$

Hence by Lemma 7 we get

$$2T_{1,3}=O(\|R_n^q\|)+O(n^{-\alpha}). \tag{5.37}$$

Using (5.32), (5.33) and (5.37) in (5.29), we get

$$T_1=R_n^q(x)+R_n^q(y)+O(n^{-\alpha})+O(\|R_n^q\|). \tag{5.38}$$

Using (5.28) and (5.38) in (5.27) we have

$$\begin{aligned}T&=R_n^q(x)+R_n^q(y)+O(\|R_n^q\|)+O(n^{-\alpha})\\&=O(\|R_n^q\|)+O(n^{-\alpha}).\end{aligned} \tag{5.39}$$

Using (5.39) in (5.26), we can write

$$I_2=O(\|R_n^q\|)+O(n^{-\alpha}). \tag{5.40}$$

By Lemmas 5 and 6

$$\begin{aligned}I_2&=\frac{2}{\pi}\int_N^\delta\frac{G(u)Q(n,u)\mathrm{d}u}{2\tan u/2}\\&=O(1)|x-y|^\alpha\log n\int_N^\delta\frac{\mathrm{d}u}{u}\\&=O(1)|x-y|^\alpha(\log n)^2.\end{aligned} \tag{5.41}$$

Since $f\in H_\alpha$, we can write

$$\begin{aligned}2\{F_x(t)-F_x(t_1)\}&=2\int_t^{t_1}\frac{\Phi_x(u)\mathrm{d}u}{2\tan u/2}\\&=O(1)\int_t^{t_1}u^{\alpha-1}\mathrm{d}u=O(n^{-\alpha}).\end{aligned} \tag{5.42}$$

Hence

$$\begin{aligned}\|R_n^q\|&=\int_{1/n}^{\log n/(n)^{1/2}}t^{-1}\,\|F(t)-F(t_1)\|\,p_q^n(t)\mathrm{d}z\\&=O(n^{-\alpha})\int_{1/n}^{\log n/(n)^{1/2}}\frac{\mathrm{d}z}{z}\\&=O(1)n^{-\alpha}\log n.\end{aligned} \tag{5.43}$$

Using (5.43) in (5.40) and combining with (5.41) we get

$$I_2=O(1)\begin{cases}n^{-\alpha}\log n\\|x-y|^\alpha(\log n)^2.\end{cases} \tag{5.44}$$

Using (5.10), (5.11), (5.14), (5.17), (5.44) in (5.2), we get

$$|L_n(x)-L_n(y)|=O(1)\begin{cases}n^{-\alpha}\log n\\|x-y|^\alpha\,(\log n)^2.\end{cases} \tag{5.45}$$

Using (5.45), we get

$$|L_n(x)-L_n(y)|=|L_n(x)-L_n(y)|^{\beta/\alpha}\,|L_n(x)-L_n(y)|^{1-\beta/\alpha}$$

$$= O(1)(|x-y|^{\alpha}(\log n)^2)^{\beta/\alpha}(n^{-\alpha}\log n)^{1-\beta/\alpha}$$
$$= O(1)|x-y|^{\beta}n^{\beta-\alpha}(\log n)^{1+\beta/\alpha}$$

which further ensures that

$$\sup_{\substack{x,y\\ x\neq y}}|\Delta^{\beta}L_n(x,y)| = \sup_{\substack{x,y\\ x\neq y}}\frac{|L_n(x)-L_n(y)|}{|x-y|^{\beta}}$$
$$= O(1)n^{\beta-\alpha}(\log n)^{1+\beta/\alpha}. \tag{5.46}$$

Again $f\in H_{\alpha}\Rightarrow\Phi_x(\upsilon)=O(|\upsilon|^{\alpha})$ and so proceeding as above, we obtain

$$\|L_n(\cdot)\|_c = \sup_{-\pi\leqslant x<\pi}|L_n(x)| = O(1)n^{-\alpha}\log n. \tag{5.47}$$

Now the theorem is completely proved by combining (5.46) and (5.47).

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