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Hardy's theorem for zeta-functions of quadratic forms*

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Abstract. Let $Q(u_1,\ldots,u_l)=\Sigma d_{ij}u_iu_j$ $(i,j=1$ to $l)$ be a positive definite quadratic form in $l(\geqslant 3)$ variables with integer coefficients $d_{ij}(=d_{ji})$. Put $s=\sigma+it$ and for $\sigma>(l/2)$ write

$$Z_Q(s)=\sum{}'(Q(u_1,\ldots,u_l))^{-s},$$

where the accent indicates that the sum is over all l-tuples of integers $(u_1,\ldots,u_l)$ with the exception of $(0,\ldots,0)$. It is well-known that this series converges for $\sigma>(l/2)$ and that $(s-(l/2))Z_Q(s)$ can be continued to an entire function of s. Let δ be any constant with $0<\delta<\frac{1}{100}$. Then it is proved that $Z_Q(s)$ has $\gg_\delta T\log T$ zeros in the rectangle $(|\sigma-\frac{1}{2}|\leqslant\delta, T\leqslant t\leqslant 2T)$.

Keywords. Quadratic forms; zeta-function; zeros near the line sigma equal to half.

1. Introduction

Let $Q(u_1,u_2,\ldots,u_l)$ be a positive definite quadratic form $\Sigma\, d_{ij}u_iu_j$, $(i,j=1$ to $l)$ in $l(\geqslant 3)$ variables and with integer coefficients d_{ij} $(=d_{ji}$ for $i,j)$. Put (with $s=\sigma+it$).

$$Z_Q(s)=\sum{}'(Q(u_1,u_2,\ldots,u_l))^{-s},$$

where the accent indicates that the summation is over all integer l-tuples $(u_1,u_2,\ldots,u_l)$ with the exception of $(0,0,\ldots,0)$. (It is known that $Z_Q(s)(s-(l/2))$ is an entire function.) Let $N(\alpha,T)$ denote the number of zeros of $Z_Q(s)$ in $\sigma\geqslant\alpha, T\leqslant t\leqslant 2T$. We prove the following theorem.

Main Theorem. *We have*

$$N(\alpha,T)\gg T\log T$$

if $\alpha=(l-1)/2-\delta$, *($\delta>0$ any constant) provided* $l\geqslant 3$. *Also we have*

$$N(\beta,T)\ll T$$

if $\beta=(l-1)/2+\delta$.

For a neat consequence of this see Remark 2 below.

Remark 1. The proof of this theorem depends on the following two important results.

*Dedicated to Professor R P Bambah on his seventy-first birthday

First the lower bound

$$\frac{1}{T}\int_T^{2T} |Z_Q(\sigma+it)|\,\mathrm{d}t \gg T^{\delta}, \quad \left(\sigma=\frac{l-1}{2}-\delta\right),$$

where $\delta>0$ is a constant if $l\geqslant 3$. Next for $\frac{1}{2}+\varepsilon\leqslant(l-1)/2-\delta\leqslant(l-1)/2-\varepsilon$ (ε is a small positive constant), we have

$$\frac{1}{T}\int_T^{2T}\left|Z_Q\left(\frac{l-1}{2}-\delta+it\right)\right|^2\mathrm{d}t \ll T^{2\delta}$$

for $l\geqslant 3$. Both the results follow from the ideas of R Balasubramanian and K Ramachandra (see $[\mathrm{RB},\mathrm{KR}]_1$, $[\mathrm{RB},\mathrm{KR}]_2$, $[\mathrm{KR}]_1$, $[\mathrm{KR}]_2$ and $[\mathrm{KR}]_3$). Also one has to use Theorem 3 of $[\mathrm{RB},\mathrm{KR}]_1$.

Remark 2. Using the functional equation of $Z_{\bar{Q}}(s)$ (with some associated quadratic form $\bar{Q}$) and applying the theorem we have the following corollary: $Z_Q(s)$ has $\gg T\log T$ zeros in ($|\sigma-\frac{1}{2}|\leqslant\delta$, $T\leqslant t\leqslant 2T$). In a rough way we may say that the critical line (for $Z_Q(s)$) gets blown up into an inner critical strip $\frac{1}{2}\leqslant\sigma\leqslant(l-1)/2$ and that in the neighbourhood of the vertical borders there are plenty of zeros of $Z_Q(s)$. This is the justification for the title of the present paper.

2. Notation and preliminaries

1. $C_1, C_2,\ldots, A_1, A_2,\ldots$ denote effective positive constants, sometimes absolute.
2. $\varepsilon_1,\varepsilon_2,\ldots,\delta_1,\delta_2,\ldots$ denote small positive constants.
3. $f(x)\ll g(x)$ and $f(x)=O(g(x))$ will mean that $|f(x)|\leqslant C_1 g(x)$.
4. We write $s=\sigma+it$, $w=u+iv$.
5. $f(x)=o(g(x))$ means that $f(x)/g(x)$ as $x\to\infty$.

In any fixed strip $\alpha\leqslant\sigma\leqslant\beta$, as $t\to\infty$ we have

$$\Gamma(\sigma+it)=t^{\sigma+it-(1/2)}\mathrm{e}^{-(\pi/2)t-it+(i\pi/2)(\sigma-(1/2))}.\sqrt{2\pi}\left(1+O\left(\frac{1}{t}\right)\right). \tag{2.1}$$

$Z_Q(s)$ satisfies the functional equation (see [EH] or [HMS])

$$\left(\frac{\Delta^{1/l}}{2\pi}\right)^s\Gamma(s)Z_Q(s)=\left(\frac{\Delta^{1-(1/l)}}{2\pi}\right)^{(l/2)-s}\Gamma\left(\frac{l}{2}-s\right)Z_{\bar{Q}}\left(\frac{l}{2}-s\right), \tag{2.2}$$

where $\Delta=|\det((d_{ij}))|$. If we write

$$Z_Q(s)=\psi_Q(s)Z_{\bar{Q}}\left(\frac{l}{2}-s\right), \tag{2.3}$$

then, from (2.1) and (2.2), we obtain,

$$\begin{aligned}\psi_Q(s)&=\left(\frac{\Delta^{1-(1/l)}}{2\pi}\right)^{l/2}\left(\frac{\Delta}{(2\pi)^2}\right)^{-\sigma}\mathrm{e}^{-i\pi(\sigma-(l/4))}t^{2((l/4)-\sigma)}\left(\frac{t\sqrt{\Delta}}{2\pi\mathrm{e}}\right)^{-2it}\\&\times\left(1+O\left(\frac{1}{t}\right)\right)=Ct^{2((l/4)-\sigma)}\left(\frac{t\sqrt{\Delta}}{2\pi\mathrm{e}}\right)^{-2it}\left(1+O\left(\frac{1}{t}\right)\right).\end{aligned} \tag{2.4}$$

Hereafter, we write

$$Z_Q(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \left(\text{in Re } s > \frac{l}{2} \right)$$

and its analytic continuations. The analytic continuation of $Z_Q(s)$ shows that in $|t| \geqslant 10$ we have in $-1 \leqslant \sigma \leqslant l$ the bound $|Z_Q(s)| < t^A$ for some constant A depending on the quadratic form Q.

3. Some Lemmas

Lemma 3.1. *We have*

$$\sum_{n \leqslant x} a_n = C_2 x^{(l/2)} + O(x^{(l/2 - 1/2)})$$

where C_2 depends on Δ and l.

Proof. See for example [EL] Hilfssatz 16.

Lemma 3.2. *Let Q be a primitive positive definite quadratic form in l-variables with integer coefficients. For $l \geqslant 3$, we have*

$$\sum_{n \leqslant x} a_n^2 = C_3 x^{l-1} + O(x^{(l-1)(4l-5)/(4l-3)}),$$

where C_3 is a positive real constant which depends on Q.

Proof. See Theorem 6.1 of [WM].

Lemma 3.3. *Let $\{b_n\}$ and $\{b'_n\}$, $n = 1, 2, \ldots, M$ be any set of complex numbers. Then*

$$\int_0^T \left(\sum_{n=1}^{M} b_n n^{-it} \right) \left(\sum_{n=1}^{M} b'_n n^{it} \right) dt = T \sum_{n=1}^{M} b_n b'_n + O\left(\left(\sum_{n=1}^{M} n|b_n|^2 \right)^{1/2} \times \left(\sum_{n=1}^{M} n|b'_n|^2 \right)^{1/2} \right).$$

Proof. See [HLM, RCV] or $[\text{KR}]_4$.

Lemma 3.4. *For $T \geqslant 100$, we have*

$$\frac{1}{T} \int_T^{2T} |\zeta(\tfrac{1}{2} + \delta_1 + it)|^4 \, dt \ll_{\delta_1} 1,$$

where δ_1 is a fixed positive constant.

Proof. See for example [ECT].

Lemma 3.5. (*see* [KR, AS]). *Let I be any unit interval in $[T, 2T]$ and define*

$$m(I) = \max_{\substack{t \in I \\ (1/2) + \delta_1 \leqslant \sigma \leqslant 2}} |\zeta(\sigma + it)|.$$

Then, we have

$$\sum_{I}(m(I))^4 \ll_{\delta_2} T.$$

COROLLARY

If $M(I)=\max_{t\in I}|\zeta(\frac{1}{4}+it)|$ *where I is any unit interval contained in* $[T,2T]$, *then*

$$\sum_{I}(M(I))^4 \ll T^2.$$

Proof. Let $m(I)=|\zeta(\sigma_j+it_j)|$ and let $D(s_0)=D_{(\delta_2/2)}(s_0)$ denote the disc of radius $(\delta_2/2)$ with centre s_0. By Cauchy's theorem, we have $(s_j=\sigma_j+it_j)$,

$$|\zeta(s_j)|^4 \leqslant \frac{1}{A}\int\int_{D(s_j)}|\zeta(s)|^4\,\mathrm{d}\sigma\,\mathrm{d}t,$$

where $A=\pi(\delta_2/2)^2$ is the area of $D(s_j)$. For any fixed j, $D(s_j)$ intersects $D(s_{j'})$ only for $O(1)$ values of j'. Now, summing over j, we obtain

$$\begin{aligned}\sum_{I}(m(I))^4 &= \sum_{j}|\zeta(s_j)|^4\\ &\ll \delta_2^{-2}\int_{(1+\delta_2)/2}^{100}\left(\int_{T-1}^{2T+1}|\zeta(s)|^4\,\mathrm{d}t\right)\mathrm{d}\sigma\\ &\ll_{\delta_2} T.\end{aligned}$$

Now, the corollary follows on using the functional equation for $\zeta(s)$.

4. First power mean-lower bound

Theorem 4.1. *Let* $\delta>0$ *be any fixed constant such that* $\frac{1}{2}+\varepsilon\leqslant(l-1)/2-\delta\leqslant(l-1)/2-\varepsilon$. *We make only the following hypothesis (which is satisfied by* a_n *in* $Z_Q(s)$ *from Lemmas* 3.1 *and* 3.2):

Hypothesis $(*)^+$. For each fixed l, we assume that for the corresponding a_n, the inequalities

$$\sum_{x\leqslant n\leqslant 2x}\frac{a_n}{n^{(l/2)-1}} \gg x$$

and

$$\sum_{x\leqslant n\leqslant 2x}\left(\frac{a_n}{n^{(l/2)-1}}\right)^2 \ll x$$

hold.

$^+$ *Postscript.* Instead of Hypothesis $(*)$ of Theorem 4.1 we can manage with the following hypothesis

$$\mathrm{Re}\sum_{x\leqslant n\leqslant 2x}a_n \gg x^{l/2} \quad\text{and}\quad \sum_{x\leqslant n\leqslant 2x}|a_n|^2 \ll x^{l-1}.$$

Then for $T \geqslant 10$, we have

$$\frac{1}{T}\int_{T}^{2T}\left|Z_Q\left(\frac{l-1}{2}-\delta+it\right)\right|\mathrm{d}t \gg T^{\delta}.$$

Note. We use the notation in the proof,

$$A \equiv A\left(\frac{l-1}{2}-\delta+it\right) \quad \text{and} \quad \zeta^* \equiv \zeta^*(\tfrac{1}{4}+it).$$

Proof. Let $M(I)=\max_{t\in I}|\zeta(\frac{1}{4}+it)|$ where t runs over all points in the unit interval I contained in $[T, 2T]$. From the corollary of Lemma 3.5, we have

$$\#\{I/M(I)\geqslant C_4T^{1/4}\} \ll \frac{T}{C_4^4}, \tag{4.1.1}$$

where C_4 is a large positive constant. We define

$$A(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}(\mathrm{e}^{-(n/C_5T)}-\mathrm{e}^{-(n/C_6T)}), \tag{4.1.2}$$

where C_5 and C_6 satisfy $0<C_6<C_5<1$ (will be chosen suitably) and

$$\zeta^* \equiv \zeta^*\left(\frac{1}{4}+it\right)=\sum_{n\leqslant T}n^{-1/4-it}. \tag{4.1.3}$$

We divide the interval $[T, 2T]$ into disjoint unit intervals I. Now, consider

$$\int_{T}^{2T}|A|\mathrm{d}t \geqslant \sum_{I}\int_{I}\frac{|A\bar{\zeta}^*|\mathrm{d}t}{M(I)} \gg \frac{1}{C_4T^{1/4}}\sum_{I}{}'\int_{I}|A\bar{\zeta}^*|\mathrm{d}t, \tag{4.1.4}$$

where accent in the above sum indicates that the sum is over those I for which $M(I)\leqslant C_4T^{1/4}$. Hence from (4.1.4), we obtain

$$\begin{aligned}\int_{T}^{2T}|A|\mathrm{d}t &\gg \frac{1}{C_4T^{1/4}}\left\{\int_{T}^{2T-1}|A\bar{\zeta}^*|\mathrm{d}t-\int_{T}^{2T}\psi(t)|A\zeta^*|\mathrm{d}t\right\}\\ &\gg \frac{1}{C_4T^{1/4}}\left\{\left|\int_{T}^{2T-1}A\bar{\zeta}^*\mathrm{d}t\right|-\int_{T}^{2T}\psi(t)|A\zeta^*|\mathrm{d}t\right\},\end{aligned} \tag{4.1.5}$$

where $\psi(t)$ is the characteristic function of those I for which $M(I)\geqslant C_4T^{1/4}$. We note that from (4.1.1),

$$\int_{T}^{2T}\psi(t)\mathrm{d}t \ll \frac{T}{C_4^4}. \tag{4.1.6}$$

Now, from Lemma 3.3, we have

$$\begin{aligned}\int_{T}^{2T-1}A\bar{\zeta}^*\mathrm{d}t &= T\sum_{n\leqslant T}\frac{a_n(\mathrm{e}^{-(n/C_5T)}-\mathrm{e}^{-(n/C_6T)})}{n^{(l-1)/2-\delta+1/4}}\\ &\quad +O\left(\left(\sqrt{\sum_{n=1}^{\infty}\frac{a_n^2(\mathrm{e}^{-(n/C_5T)}-\mathrm{e}^{-(n/C_6T)})^2}{n^{l-1-2\delta}}\cdot n}\right)\left(\sqrt{\sum_{n\leqslant T}\frac{n}{n^{1/2}}}\right)\right)\\ &= J_1+O(J_2).\ \text{(say)}\end{aligned} \tag{4.1.7}$$

Now, for $\lambda < 1$, we have

$$\begin{aligned} J_1 &\geqslant T \sum_{n \leqslant T\lambda C_6} \frac{a_n}{n^{(l-1)/2-\delta+1/4}} \left\{ 1 - \frac{n}{C_5 T} - \left(1 - \frac{n}{C_6 T} \right) + O\left(\frac{n^2}{C_6^2 T^2} \right) \right\} \\ &\geqslant T \sum_{n \leqslant T\lambda C_6} \frac{a_n}{n^{(l-1)/2-\delta+1/4}} \left\{ \frac{n}{2C_6 T} + O\left(\frac{n^2}{C_6^2 T^2} \right) \right\} \\ &\geqslant C_7 T \sum_{n \leqslant T\lambda C_6} \frac{1}{n^{(3/4-\delta)}} \left\{ \frac{n}{2C_6 T} + O\left(\frac{n^2}{C_6^2 T^2} \right) \right\} \\ &\geqslant T \left\{ C_8 \frac{(T\lambda C_6)^{(5/4+\delta)}}{C_6 T} - C_9 \frac{(T\lambda C_6)^{(9/4+\delta)}}{C_6^2 T^2} \right\}, \end{aligned} \tag{4.1.8}$$

provided $1 > C_5 \geqslant 20 C_6 > 0$. This implies that for sufficiently small λ, there exists an absolute constant C_{10} such that

$$J_1 \geqslant C_{10} T^{(5/4+\delta)} C_6^{(1/4+\delta)}. \tag{4.1.9}$$

Now,

$$\begin{aligned} J_2 &\ll T^{3/4} \sqrt{\sum_{n=1}^{\infty} \frac{a_n^2 \cdot n}{n^{l-1-2\delta}} e^{-(2n/C_5 T)}} \\ &\ll \sqrt{\sum_{n=1}^{\infty} n^{2\delta} e^{-(n/2C_5 T)}} \\ &\ll T^{(5/4+\delta)} C_5^{(1/2+\delta)} \end{aligned} \tag{4.1.10}$$

since

$$\sum_{n=1}^{\infty} n^{2\delta} e^{-(n/2C_5 T)} = \frac{1}{2\pi i} \int_{\operatorname{Re} w = 2} \zeta(-2\delta + w) \Gamma(w) (2C_5 T)^w \mathrm{d}w \tag{4.1.11}$$

and move the line of integration in (4.1.11) to $\operatorname{Re} w = 1 + 2\delta$ so that the residue of the pole at $w = 1 + 2\delta$ is $(2C_5 T)^{1+2\delta}\, \Gamma(1+2\delta)$. Note that, we have used the hypothesis (*) in estimating J_1 and J_2. Therefore from (4.1.7), (4.1.9) and (4.1.10) we obtain

$$\begin{aligned} \int_T^{2T-1} A\bar{\zeta}^* \mathrm{d}t &> C_{10} T^{(5/4+\delta)} C_6^{(1/4+\delta)} - C_{11} T^{(5/4+\delta)} C_5^{(1/2+\delta)} \\ &= T^{(5/4+\delta)} C_{10} \cdot C_6^{(1/4+\delta)} \left(1 - \frac{C_{11}}{C_{10}} \cdot \frac{C_{12}^{(1/2+\delta)}}{C_6^{(1/4+\delta)}} \right). \end{aligned} \tag{4.1.12}$$

We choose C_6 small and then C_5 such that

$$C_5^{(1/2+\delta)} = \frac{C_{10}}{2C_{11}} C_6^{(1/4+\delta)}$$

i.e.

$$C_5 = \left(\frac{C_{10}}{2C_{11}} \right)^{1/(1/2+\delta)} \cdot C_6^{(1/4+\delta)/(1/2+\delta)} \geqslant 20 C_6.$$

This is satisfied since C_6 is small and $(1/4+\delta)/(1/2+\delta) < 1$. Hence we have

$$\int_T^{2T-1} A\zeta^* \mathrm{d}t > C_{12} T^{(5/4+\delta)}, \tag{4.1.13}$$

where C_{12} depends only on δ. Now, from Hölder's inequality, it follows that

$$\int_T^{2T} \psi(t)|A\zeta^*|\mathrm{d}t \leqslant \left(\int_T^{2T} |A|^2 \mathrm{d}t\right)^{1/2} \left(\int_T^{2T} \psi^4(t)\mathrm{d}t\right)^{1/4} \left(\int_T^{2T} |\zeta^*|^4 \mathrm{d}t\right)^{1/4} \tag{4.1.14}$$

From (4.1.1) and from Lemma 3.5, using the functional equation for $\zeta(s)$, we notice that

$$\left(\int_T^{2T} \psi^4(t)\mathrm{d}t\right)^{1/4} \left(\int_T^{2T} |\zeta^*|^4 \mathrm{d}t\right)^{1/4} \ll \left(\frac{T}{C_4^4}\right)^{1/4} (T^2)^{1/4} \ll C_4^{-1} T^{3/4}. \tag{4.1.15}$$

Also from Lemma 3.3, it follows that

$$\begin{aligned}\int_T^{2T} |A|^2 \mathrm{d}t &\ll \sum_{n=1}^{\infty} \frac{(T+n)|a_n|^2(\mathrm{e}^{-(n/C_5T)} - \mathrm{e}^{-(n/C_6T)})^2}{n^{l-1-2\delta}} \\ &\ll \sum_{n=1}^{\infty} \frac{(T+n)|a_n|^2 \mathrm{e}^{-(2n/C_5T)}}{n^{l-1-2\delta}} \\ &\ll T\sum_{n=1}^{\infty} \frac{\mathrm{e}^{-(2n/C_5T)}}{n^{1-2\delta}} + \sum_{n=1}^{\infty} n^{2\delta}\mathrm{e}^{-(2n/C_5T)} \\ &\ll T^{1+2\delta}\end{aligned} \tag{4.1.16}$$

on using the hypothesis and noticing the fact similar to (4.1.11). Therefore from (4.1.14), (4.1.15) and (4.1.16), we obtain

$$\int_T^{2T} \psi(t)|A\zeta^*|\mathrm{d}t \ll C_4^{-1} T^{(5/4+\delta)}. \tag{4.1.17}$$

Therefore from (4.1.5), (4.1.13) and (4.1.17), we get

$$\begin{aligned}\int_T^{2T} |A|\mathrm{d}t &> \frac{1}{C_4 T^{1/4}}\{C_{12} T^{(5/4+\delta)} - C_4^{-1} C_{13} T^{(5/4+\delta)}\} \\ &\gg T^{1+\delta},\end{aligned} \tag{4.1.18}$$

since C_4 is large enough. Here C_{12} and C_{13} depend only on δ. Now let $\mathrm{Re}\, s = ((l-1)/2) - \delta$. By Mellin's transform, we have

$$\begin{aligned}A(s) &= \frac{1}{2\pi i}\int_{\mathrm{Re}\, w=100} Z_Q(s+w)((C_5T)^w - (C_6T)^w)\Gamma(w)\mathrm{d}w \\ &= \frac{1}{2\pi i}\int_{\substack{\mathrm{Re}\, w=100 \\ |v|\leqslant(\log T)^2}} Z_Q(s+w)((C_5T)^w - (C_6T)^w)\Gamma(w)\mathrm{d}w + O(T^{-1}).\end{aligned} \tag{4.1.19}$$

We move the line of integration in (4.1.19) to $\mathrm{Re}\, w = 0$ and we obtain

$$|A(s)| \ll \int_{|v|\leqslant(\log T)^2} \left|Z_Q(s+iv)\right| \left|\frac{C_5^{iv} - C_6^{iv}}{v}\right| \left|\Gamma(1+iv)\right| \mathrm{d}v + O(T^{-1})$$

and hence

$$\int_T^{2T} |A(s)|$$

$$\ll \int_{|v|\leqslant(\log T)^2}\int_T^{2T}\left|Z_Q\left(\frac{l-1}{2}-\delta+it+iv\right)\right|\left|\frac{C_5^{iv}-C_6^{iv}}{v}\right|\left|\Gamma(1+iv)\right|\mathrm{d}v\mathrm{d}t$$

$$\ll \int_{|v|\leqslant(\log T)^2}\int_{T-(\log T)^2}^{2T+(\log T)^2}\left|Z_Q\left(\frac{l-1}{2}-\delta+i\tau\right)\right|\left|\frac{C_5^{iv}-C_6^{iv}}{v}\right|\left|\Gamma(1+iv)\right|\mathrm{d}v\mathrm{d}\tau$$

$$\ll \int_{T-(\log T)^2}^{2T+(\log T)^2}\left|Z_Q\left(\frac{l-1}{2}-\delta+i\tau\right)\right|\mathrm{d}\tau. \tag{4.1.20}$$

From (4.1.18) and (4.1.20), the theorem follows, since we can define the integrand to be zero outside the interval $[T, 2T]$.

5. Mean-square upper bound

Theorem 5.1. *Let δ satisfy the condition as in Theorem 4.1. We make only the following hypothesis (which is satisfied by a_n in $Z_Q(s)$, from Lemma 3.2).*

Hypothesis (*, *) For each l for the corresponding a_n, the inequality

$$\sum_{n\leqslant x}\left(\frac{a_n}{n^{(l/2-1)}}\right)^2 \ll x$$

hold.

Then for $T \geqslant 100$, we have

$$\frac{1}{T}\int_T^{2T}\left|Z_Q\left(\frac{l-1}{2}-\delta+it\right)\right|^2 \mathrm{d}t \ll T^{2\delta}.$$

Proof. It follows from the papers $[\mathrm{KR}]_2$ and $[\mathrm{KR}]_3$.

6. Balasubramanian–Ramachandra principle

Theorem 6.1. *For $T \geqslant T_0$, if*

$$\frac{1}{T}\int_T^{2T} |G(\sigma_1+it)|\mathrm{d}t > A_1\psi \tag{6.1.1}$$

and

$$\frac{1}{T}\int_T^{2T} |G(\sigma_1+it)|^2\mathrm{d}t < A_2\psi^2 \tag{6.1.2}$$

hold for a Dirichlet series $G(s)$ on a certain line σ_1 with positive constants A_1 and A_2, then there exists at least $\geqslant [(A_1^2/2A_2)(T/H)] - 1$ intervals I of length H such that in each of the intervals I, the inequality

$$\frac{1}{|I|}\int_I |G(\sigma_1+it)|\mathrm{d}t > \frac{A_1}{10}\psi \tag{6.1.3}$$

holds where $H \leqslant T^{1-\varepsilon_1}$, and $\psi = \psi(T)$ tends to ∞.

Remark. This principle has been used in several occasions (for example see [RB, KR]$_1$, [RB, KR]$_2$, ...). For the sake of completeness, we sketch the proof.

Proof. We divide the interval $[T, 2T]$ into smaller disjoint (but abutting) intervals of length H (but with length $\leqslant H$ for an end interval). By defining G to be zero if $t \leqslant T$ or $t \geqslant 2T$, we get

$$A_1 \psi T < \int_T^{2T} |G(\sigma_1 + it)| \mathrm{d}t \leqslant \sum_I \int_I |G(\sigma_1 + it)| \mathrm{d}t. \tag{6.1.4}$$

Now, we omit these intervals I appearing in the sum of (6.1.4) for which

$$\int_I |G(\sigma_1 + it)| \mathrm{d}t \leqslant \frac{A_1}{2} H \psi. \tag{6.1.5}$$

Let N_1 be the number of those intervals I for each of which the inequality

$$\int_I |G(\sigma_1 + it)| \mathrm{d}t \geqslant \frac{A_1}{2} H \psi \tag{6.1.6}$$

holds. Therefore applying Hölder's inequality, we find that from (6.1.4), (6.1.5) and (6.1.6),

$$\begin{aligned}
\frac{A_1}{2} \psi T &\leqslant \sum_I{}' \int_I |G(\sigma_1 + it)| \mathrm{d}t \\
&\leqslant \sqrt{N_1} \left\{ \sum_I{}' \left(\int_I |G(\sigma_1 + it)| \mathrm{d}t \right)^2 \right\}^{1/2} \\
&\leqslant \sqrt{N_1} \left\{ \sum_I \left(\int_I |G(\sigma_1 + it)| \mathrm{d}t \right)^2 \right\}^{1/2} \\
&\leqslant \sqrt{N_1} \left\{ \sum_I H \int_I |G(\sigma_1 + it)|^2 \mathrm{d}t \right\}^{1/2} \\
&\leqslant \sqrt{N_1 H} \left(\int_T^{2T} |G(\sigma_1 + it)|^2 \mathrm{d}t \right)^{1/2} \\
&\leqslant \sqrt{N_1 H} \psi T^{1/2} A_2^{1/2},
\end{aligned} \tag{6.1.7}$$

i.e. $N_1 \geqslant A_1^2 / A_2 \cdot T/H$, and the accent in the first two steps of the inequality (6.1.7) indicates that the sum runs over those intervals I for each of which the inequality (6.1.6) holds. This proves the theorem.

7. Proof of the main theorem

Taking $H = 1$, from Theorem 6.1, there are $\gg T$ well-spaced points t_r at which $|Z_Q(l-1)/2 - \delta + it_r)|$ is large. Now from Theorem 3 of [RB, KR]$_1$, each such point gives rise to $\gg \log T$ zeros of $Z_Q(s)$ in $\sigma \geqslant (l-1)/2 - \delta$. This completes the proof of the first part. Second part of the main theorem follows from the fact that

$$\frac{1}{T} \int_T^{2T} \left| Z_Q\left(\frac{l-1}{2} + it \right) \right|^2 \mathrm{d}t \ll T^\varepsilon \quad \forall \varepsilon > 0.$$

(For explanation see [ECT]).

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