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Degree of approximation of functions by their Fourier series in the generalized Hölder metric

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Abstract. The paper studies the degree of approximation of functions by matrix means of their Fourier series in the generalized Hölder metric, generalizing many known results in the literature.

Keywords. Banach space; Hölder metric; generalized Hölder metric; infinite matrix; deferred Cesàro mean.

1. Definition

Let f be a periodic function of period 2π and let $f \in L_p[0, 2\pi]$ for $p \geqslant 1$. Let the Fourier series of f at $t = x$ be given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{1.1}$$

In the case $0 < p < 1$, we can still regard (1.1) as the Fourier series of f by further assuming that $f(t)\cos nt$ and $f(t)\sin nt$ are integrable (see [21] p. 9).

The space $L_p[0, 2\pi]$ where $p = \infty$ includes the space $C_{2\pi}$ of all continuous functions defined over $[0, 2\pi]$. We write

$$\|f\|_c = \sup_{t \in [0, 2\pi]} |f(t)| \quad (p = \infty)$$

$$\|f\|_p = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(t)|^p \mathrm{d}t\right)^{1/p} & (p \geqslant 1) \\ \displaystyle\int_0^{2\pi} |f(t)|^p \mathrm{d}t & (0 < p < 1). \end{cases}$$

We write

$$w(\delta) = w(\delta, f) = \sup_{0 \leqslant h \leqslant \delta} \|f(x+h) - f(x)\|_c \tag{1.2}$$

$$w_p(\delta) = w_p(\delta, f) = \sup_{0 \leqslant h \leqslant \delta} \|f(x+h) - f(x)\|_p \tag{1.3}$$

$$w_p^{(2)}(\delta) = w_p^{(2)}(\delta, f) = \sup_{0 \leqslant h \leqslant \delta} \|f(x+h) + f(x-h) - 2f(x)\|_p \tag{1.4}$$

which are respectively called modulus of continuity, integral modulus of continuity and integral modulus of smoothness (see [21], p. 42).

In the case $0<\beta\leqslant 1$ and $w(\delta,f)=0(\delta^\beta)$ we write $f\in\mathrm{Lip}\,\beta$, and if $w_p(\delta,f)=0(\delta^\beta)$ we write $f\in\mathrm{Lip}(\beta,p)$. The case $\beta>1$ is of no interest as in this case f turns out to be constant. The class $\mathrm{Lip}(\beta,p)$ with $p=\infty$ will be taken as $\mathrm{Lip}\,\beta$.

Let

$$H_\alpha=\{f\in C_{2\pi}:|f(x)-f(y)|\leqslant K|x-y|^\alpha\}$$

where K is a positive constant, not necessarily the same at each occurrence. It is known [15] that H_α is a Banach space with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha=\|f\|_c+\sup_{x\neq y}\Delta^\alpha f(x,y) \tag{1.5}$$

where

$$\Delta^\alpha f(x,y)=\frac{|f(x)-f(y)|}{|x-y|^\alpha}\quad (x\neq y)$$

and

$$\Delta^0 f(x,y)=0.$$

The metric induced by the norm (1.5) on H_α is called the Hölder metric. Since

$$\|f\|_\beta\leqslant(2\pi)^{\alpha-\beta}\|f\|_\alpha,\quad 0\leqslant\beta<\alpha\leqslant 1$$

it follows that $H_\alpha\subseteq H_\beta\subseteq C_{2\pi}$; that is, $\{H_\alpha,\|\cdot\|_\alpha\}$ is a family of Banach space which decreases as α increases.

With a view to generalize Hölder metric, we proceed as follows. We define for $0<\alpha\leqslant 1$

$$H(\alpha,p)=\{f\in L_p,\quad 0<p\leqslant\infty: \|f(x+h)-f(x)\|_p\leqslant K|h|^\alpha\}$$

and introduce the following metric. For $\alpha>0$

$$\|f\|_{(\alpha,p)}=\|f\|_p+\sup\frac{\|f(x+h)-f(x)\|_p}{|h|^\alpha}$$

$$\|f\|_{(0,p)}=\|f\|_p. \tag{1.6}$$

It can be easily verified that (1.6) is a norm for $p\geqslant 1$ and p-norm in the case $0<p<1$. Further it can be verified that $H(\alpha,p)$ is a Banach space for $p\geqslant 1$ and a complete p-normed space for $0<p<1$. Note that $H(\alpha,\infty)$ is the familiar H_α space introduced earlier by Prösdorff [15].

Let $A=(a_{n,k})$ be an infinite matrix and let $S_n(x)$ be the nth partial sum of the series (1.1). We denote $T_n(f)$ the A-transform of the Fourier series of f by

$$T_n(f)=T_n(f;x)=\sum_{k=0}^{\infty}a_{n,k}S_k(x) \tag{1.7}$$

provided that the series converges for each $n=0,1,2,\ldots$.

Throughout the present paper we assume that elements of the matrix $A=(a_{n,k})$ satisfy the conditions

$$\|A\|=\sup_n\sum_{k=0}^{\infty}|a_{n,k}|<\infty \tag{1.8}$$

and

$$\sum_{k=0}^{\infty}a_{n,k}=1\quad\text{for each}\quad n=0,1,2,\ldots. \tag{1.9}$$

We write $A \in \tau$ if conditions (1.8) and (1.9) hold. We use the following notations throughout:

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x) \tag{1.10}$$

$$l_n(x) = T_n(f;x) - f(x) \tag{1.11}$$

$$K_n(t) = \sum_{k=0}^{\infty} a_{n,k} \frac{\sin(k+\frac{1}{2})t}{2\sin(t/2)} \tag{1.12}$$

$$\psi(n) = \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}| \tag{1.13}$$

$$\bar{a}_n(k) = \sum_{r=0}^{k} a_{n,r} \tag{1.14}$$

$$a'_{n,k} = \sum_{r=k}^{n} a_{n,r}. \tag{1.15}$$

2. Introduction

Prösdorff [15] studied the degree of approximation in the Hölder metric and proved the following theorems:

Theorem A [15]. *Let $f \in H_\alpha (0 < \alpha \leqslant 1)$ and $0 \leqslant \beta < \alpha \leqslant 1$. Then*

$$\|\sigma_n(f) - f\|_\beta = O(1) \begin{cases} n^{\beta-\alpha} & (0<\alpha<1) \\ n^{\beta-1}(1+\log n)^{1-\beta} & (\alpha = 1) \end{cases}$$

where $\sigma_n(f)$ is the Fejér means of the Fourier series of f.

The case $\beta = 0$ of Theorem A is due to Alexits [2].

Chandra [6] obtained a generalization of Theorem A in the Nörlund or (N,p) and $(\bar{N},p)$ transform set up. Later Mohapatra and Chandra [13] considered the problem by matrix means and obtained the above results as corollaries. Also see Singh ([18], [19]), who claims to have improved the results of Mohapatra and Chandra [13].

With regard to the approximation of functions in L_p norm, the following theorem is due to Quade [16].

Theorem B [16]. *Let $f \in \mathrm{Lip}(\alpha, p)$, $0 < \alpha \leqslant 1$. Then*

$$\|\sigma_n(f) - f\|_p = O(1) \begin{cases} n^{-\alpha} & (p>1) \\ n^{-\alpha} & (p=1, 0<\alpha<1) \\ (\log n)/n & (p=1, \alpha=1). \end{cases}$$

With a view to generalize the above results in Nörlund transformation set up, attempts were made by Sahney and Rao [17], Chandra [5], [6]. Mohapatra and Russel [14] considered this in generalized Nörlund mean set up.

The object of the present paper is to make a comprehensive study of the above problems and to bring as many corollaries possible within the fold and offer suitable generalization in the following directions:

(a) Hölder metric is to be replaced by generalized Hölder metric,
(b) Transformation is by means of infinite matrix so as to include deferred means introduced by Agnew [1].
(c) Modulus of continuity is to be replaced by more general integral modulus of continuity for $0 < p \leqslant \infty$.

The degree of approximation problem for a class of continuous functions of bounded variation, integral modulus of continuity of order 1 have been extensively studied by Mazhar [10], [11], Mazhar and Totik [12] and Bojanic and Mazhar [3], [4]. However, these studies do not come within the fold of our theorem of the present paper.

3. Main results

We prove the following theorems:

Theorem 1. *Suppose that* $A \in \tau$ *and let there exist a positive non-decreasing sequence* (μ_n) *such that*

$$\sum_{k=\mu_n}^{\infty} (k+1)|a_{n,k}| = O(\mu_n). \tag{3.1}$$

Then for $p \geqslant 1$ *and* $f \in H(\alpha, p)$, $0 < \alpha \leqslant 1$, $0 \leqslant \beta < \alpha$

$$\|l_n(x)\|_{(\beta,p)} = O(1)\begin{cases} (1+\log(\mu_n/\lambda_n))^{\beta/\alpha} + \psi(n)\lambda_n^{1-\alpha+\beta}, & (0<\alpha<1) \\ \dfrac{(1+\log(\mu_n/\lambda_n))^{\beta}}{\lambda_n^{1-\beta}} + \psi(n)\lambda_n^{\beta}(\log\lambda_n)^{1-\beta} & (\alpha = 1) \end{cases}$$

where $l_n(x)$ *and* $\psi(n)$ *are respectively defined in* (1.11) *and* (1.13) *and* λ_n *is any positive non-decreasing sequence such that* $\lambda_n \leqslant \mu_n$.

Theorem 2. *Let* $0 < p < 1$ *and let* $A = (a_{n,k})$ *satisfy the conditions of Theorem* 1. *Let* $f \in H(\alpha, p)$, $0 \leqslant \beta < \alpha$, $0 < \alpha \leqslant 1$. *Then*

$$\|l_n(x)\|_{(\beta,p)} = O(1)(\lambda_n^{p-1})^{\beta/\alpha}\lambda_n^{\beta-\alpha} + (\psi(n))^p \begin{cases} O(1) & (\alpha > 0, 2p < 1) \\ \lambda_n^{(2p-1)\beta/\alpha} & (\alpha > 2p-1, 2p > 1) \\ (\log\lambda_n)^{\beta/\alpha} & (\alpha > 2p-1, 2p = 1) \\ \lambda_n^{2p+\beta-\alpha-1} & (0 < \alpha < 2p-1, 2p > 1) \\ (\log\lambda_n)^{1-\beta/\alpha}\lambda_n^{(2p-1)\beta/\alpha} & \\ & (\alpha = 2p-1, 2p > 1). \end{cases}$$

Theorem 3. *Let* $0 < \alpha \leqslant 1$, $0 \leqslant \beta < \alpha$, $\alpha p > 1$. *Let* $A = (a_{n,k})$ *satisfy the same conditions as in Theorem* 1. *Let* $f \in H(\alpha, p)$. *Then*

$$\|l_n(x)\|_{(\beta-(\beta/\alpha p),p)} = O(1)\left(1+\log\left(\frac{\mu_n}{\lambda_n}\right)\right)^{\beta/\alpha}\lambda_n^{((1/p)-\alpha)(1-(\beta/\alpha))}$$

$$+ O(1)\psi(n) \begin{cases} \lambda_n^{1+(1/p)+\beta-\alpha-(\beta/\alpha p)} & (\alpha < 1+(1/p)) \\ \lambda_n^{\beta/\alpha}(\log\lambda_n)^{(1-\beta/\alpha)} & (\alpha = 1+(1/p)) \end{cases}$$

Remarks. (i) In case

$$\theta = \liminf\left(\frac{n}{\mu_n}\right) > 0, \tag{3.2}$$

for sufficiently large n

$$\frac{\theta}{2}\sum_{k=\mu_n}^{\infty}|a_{n,k}| \leqslant \frac{n}{\mu_n}\sum_{k=\mu_n}^{\infty}|a_{n,k}| \leqslant \frac{1}{\mu_n}\sum_{k=\mu_n}^{\infty}(k+1)|a_{n,k}|(n\geqslant k)$$

so that hypothesis (3.1) includes (1.8).
(ii) The hypothesis (3.1) seems to be unusual and interesting. It was first introduced by Mohapatra and Chandra [13] in $\mu_n = n+1$. The importance of (3.1) seems to lie in the fact that it moderates the effect of $a_{n,k}$ for large k; in fact (3.1) annihilates it for lower triangular or even deferred matrices. For example if $a_{n,k}=0$ for $k>\mu_n$, then the hypothesis (3.1) is automatically satisfied. See § 4 for a beautiful application of this in the case of deferred Cesàro mean (Corollaries 4 and 5).

We require the following lemmas for the proof of the theorems.

Lemma 1. *Let* $0<p\leqslant\infty$. *Then*

(i) $w_p^{(2)}(\delta,f)\leqslant 2w_p(\delta,f)$
(ii) $\|\phi_x-\phi_{x+h}\|_p\leqslant 4K\|f(x)-f(x+h)\|_p$,

where K is some positive constant.

Proof. For $p\geqslant 1$ and by Minkowski's inequality, we have

$$\left(\int_0^{2\pi}|\phi_x(t)|^p\mathrm{d}x\right)\leqslant\left(\int_0^{2\pi}|f(x+t)-f(x)|^p\mathrm{d}x\right)^{1/p} + \left(\int_0^{2\pi}|f(x-t)-f(x)|^p\mathrm{d}x\right)^{1/p} \tag{3.3}$$

and for $0<p<1$, we have by the modified Minkowski type inequality

$$\int_0^{2\pi}|\phi_x(t)|^p\mathrm{d}x\leqslant\int_0^{2\pi}|f(x+t)-f(x)|^p\mathrm{d}x+\int_0^{2\pi}|f(x-t)-f(x)|^p\mathrm{d}x. \tag{3.4}$$

Now Lemma 1(i) follows from (3.3) and (3.4). For proving (ii) we first note that

$$\phi_x(t)-\phi_{x+h}(t)=\{f(x+t)-f(x+t+h)\}+\{f(x-t)-f(x+h-t)\} - 2\{f(x)-f(x+h)\}$$

and then apply Minkowski's inequality separately for $p\geqslant 1$ and for $0<p<1$.

Lemma 2 [8]. *Suppose that* $f\in\mathrm{Lip}(\alpha,p)$, $p\geqslant 1, 0<\alpha\leqslant 1$.

(i) *If* $\alpha p\leqslant 1$, $p<q<\dfrac{p}{1-\alpha p}$, *then* $f\in\mathrm{Lip}\left(\alpha-\dfrac{1}{p}+\dfrac{1}{q},q\right)$

(ii) *If* $\alpha p>1$, *then* $f\in\mathrm{Lip}\left(\alpha-\dfrac{1}{p}+\dfrac{1}{q},q\right)$ *for* $q>p$ *and* f *is equivalent to a function of* $\mathrm{Lip}\left(\alpha-\dfrac{1}{p}\right)$, *i.e., two functions are same almost everywhere.*

Proof of Theorem 1. We know that

$$S_n(x)-f(x)=\frac{1}{\pi}\int_0^\pi \phi_x(t)\frac{\sin(n+\frac{1}{2})t}{2\sin t/2}\mathrm{d}t$$

and therefore

$$\begin{aligned} l_n(x) &= \sum_{k=0}^{\infty} a_{n,k}S_k(x)-f(x)\sum_{k=0}^{\infty} a_{n,k} \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty} a_{n,k}\int_0^\pi \phi_x(t)\frac{\sin(n+\frac{1}{2})t}{2\sin t/2}\mathrm{d}t \\ &= \frac{1}{\pi}\int_0^\pi \phi_x(t)K_n(t)\mathrm{d}t \end{aligned} \tag{3.5}$$

where $K_n(t)$ is defined in (1.12). Note that the change of order of summation and integration is justified provided either side is convergent. We observe that by (1.8) the series for $K_n(t)$ is convergent (even absolutely) and $K_n(t)=O(t^{-1})$ for all $0<t\leqslant\pi$ and hence the integral given in (3.5) exists by (3.6) and by the fact that $f\in H(\alpha,p)$ in which case

$$\|f(x+t)-f(x)\|_p=O(|t|^\alpha)$$

and by Lemma 1(i)

$$w_p^{(2)}(t,f)=O(|t|^\alpha).$$

Now by generalized Minkowski's inequality for $p\geqslant 1$, we have

$$\|l_n(x)-l_n(x+y)\|_p\leqslant\frac{1}{\pi}\int_0^\pi \|\phi_x(t)-\phi_{x+y}(t)\|_p|K_n(t)|\mathrm{d}t. \tag{3.6}$$

We split the integral in (3.6) as I_1 and I_2 with limits of integration from 0 to $1/\lambda_n$ and from $1/\lambda_n$ to π respectively.

By making use of the fact that

$$\left|\sum_{k=0}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right|\leqslant\|A\|<\infty$$

and by Lemma 1(i), we obtain

$$\begin{aligned} I_1 &= \frac{1}{\pi}\int_0^{1/\lambda_n} \|\phi_x-\phi_{x+y}\|_p|K_n(t)|\mathrm{d}t \\ &= O(1)\int_0^{1/\lambda_n} t^\alpha|K_n(t)|\mathrm{d}t \\ &= O(1)\int_0^{1/\lambda_n} t^{\alpha-1}\mathrm{d}t \\ &= O(1)\left(\frac{1}{\lambda_n^\alpha}\right). \end{aligned} \tag{3.7}$$

We make use of the fact that, by Abel's transformation

$$\sum_{k=0}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)t = O(t^{-1})\sum_{k=0}^{\infty}|a_{n,k}-a_{n,k+1}| \tag{3.8}$$

and Lemma 1(i), we obtain

$$\begin{aligned} I_2 &= \frac{1}{\pi}\int_{1/\lambda_n}^{\pi} \|\phi_x(t)-\phi_{x+y}\|_p |K_n(t)|\,\mathrm{d}t \\ &= O(1)\int_{1/\lambda_n}^{\pi} t^{\alpha}|K_n(t)|\,\mathrm{d}t \\ &= O(1)\begin{cases}\psi(n)\lambda_n^{1-\alpha} & (0<\alpha<1)\\ \psi(n)\log\lambda_n & (\alpha=1).\end{cases} \end{aligned} \tag{3.9}$$

To obtain alternative set of order estimates for I_1 and I_2 we make use of Lemma 1(ii). Now we split I_1 into I_{11} and I_{12} with limits of integration from 0 to $1/\mu_n$ and from $1/\mu_n$ to $1/\lambda_n$ respectively.

We first note that

$$\begin{aligned} K_n(t) &= \frac{1}{2\sin(\frac{1}{2})t}\left(\sum_{k=0}^{\mu_n} + \sum_{k=\mu_n+1}^{\infty}\right) a_{n,k}\sin\left(k+\frac{1}{2}\right)t \\ &= O(t^{-1})\left[\sum_{k=0}^{\mu_n}|a_{n,k}|(k+1)t + \sum_{k=\mu_n+1}^{\infty}(k+1)|a_{n,k}|t\right] \\ &= O(\mu_n)+O(\mu_n)=O(\mu_n) \end{aligned} \tag{3.10}$$

by (1.8) and (3.1).

Hence by Lemma 1(ii) and (3.10)

$$\begin{aligned} I_{11} &= \left(|y|^{\alpha}\int_0^{1/\mu_n}|K_n(t)|\,\mathrm{d}t\right) \\ &= O\left(|y|^{\alpha}\mu_n\int_0^{1/\mu_n}\mathrm{d}t\right) = O(|y|^{\alpha}) \end{aligned} \tag{3.11}$$

$$\begin{aligned} I_{12} &= O(|y|^{\alpha})\int_{1/\mu_n}^{1/\lambda_n}\frac{\mathrm{d}t}{t}\left|\sum_{k=0}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right| \\ &= O(|y|^{\alpha})\|A\|\int_{1/\mu_n}^{1/\lambda_n}\frac{\mathrm{d}t}{t} \\ &= O(|y|^{\alpha})\log\frac{\mu_n}{\lambda_n}. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) we obtain

$$I_1 = O(|y|^{\alpha})\left(1+\log\frac{\mu_n}{\lambda_n}\right). \tag{3.13}$$

By Lemma 1(ii) and (3.8) we have

$$\begin{aligned} I_2 &= O(|y|^\alpha)\int_{1/\lambda_n}^{\pi} |K_n(t)|\mathrm{d}t \\ &= O(|y|^\alpha)\psi(n)\int_{1/\lambda_n}^{\pi} \frac{\mathrm{d}t}{t^2} \\ &= O(|y|^\alpha)\psi(n)\lambda_n. \end{aligned} \tag{3.14}$$

Combining the results (3.7) and (3.13) we obtain

$$\begin{aligned} I_1 &= I_1^{\beta/\alpha} I_1^{1-\beta/\alpha} \\ &= O(|y|^\beta)\left(1+\log\frac{\mu_n}{\lambda_n}\right)^{\beta/\alpha} \lambda_n^{\beta-\alpha} \end{aligned} \tag{3.15}$$

and combining (3.9) and (3.14)

$$\begin{aligned} I_2 &= I_2^{\beta/\alpha} I_2^{1-\beta/\alpha} \\ &= O(|y|^\beta)(\psi(n)\lambda_n)^{\beta/\alpha}\begin{cases}(\psi(n)\lambda_n^{1-\alpha})^{1-\beta/\alpha} & (0<\alpha<1)\\ (\psi(n)\log\lambda_n)^{1-\beta} & (\alpha=1)\end{cases} \\ &= O(|y|^\beta)\psi(n)\begin{cases}\lambda_n^{1+\beta-\alpha} & (0<\alpha<1)\\ \lambda_n^{\beta}(\log\lambda_n)^{1-\beta} & (\alpha=1).\end{cases} \end{aligned} \tag{3.16}$$

Hence

$$\begin{aligned} &\sup_{y\neq 0}\frac{\|l_n(x+y)-l_n(x)\|_p}{|y|^\beta} \\ &= O(1)\left(1+\log\frac{\mu_n}{\lambda_n}\right)^{\beta/\alpha}\lambda_n^{\beta-\alpha} + O(1)\psi(n)\begin{cases}\lambda_n^{1-\alpha+\beta} & (0<\alpha<1)\\ \lambda_n^{\beta}(\log\lambda_n)^{1-\beta} & (\alpha=1).\end{cases} \end{aligned} \tag{3.17}$$

It follows from the analysis of the proofs of (3.7) and (3.9) that

$$\|l_n(x)\|_p = O\left(\frac{1}{\lambda_n^\alpha}\right) + O(1)\psi(n)\begin{cases}\lambda_n^{1-\alpha} & (0<\alpha<1)\\ \log\lambda_n & (\alpha=1).\end{cases} \tag{3.18}$$

Now we combine (3.17) and (3.18) to obtain the degree of approximation for $\|l_n(x)\|_{(\beta,p)}$ as

$$\begin{aligned} \|l_n(x)\|_{(\beta,p)} = O(1)\Bigg[&\frac{1}{\lambda_n^\alpha} + \psi(n)\begin{cases}\lambda_n^{1-\alpha} & (0<\alpha<1)\\ \log\lambda_n & (\alpha=1)\end{cases} \\ &+\left(1+\log\frac{\mu_n}{\lambda_n}\right)^{\beta/\alpha}\lambda_n^{\beta-\alpha} + \psi(n)\begin{cases}\lambda_n^{1-\alpha+\beta} & (0<\alpha<1)\\ \lambda_n^{\beta}(\log\lambda_n)^{1-\beta} & (\alpha=1)\end{cases}\Bigg] \end{aligned} \tag{3.19}$$

whence the result follows. This completes the proof of Theorem 1.

We omit the proof of Theorem 2 as it follows the lines of proof of Theorem 1. By Lemma 2(ii) we can make use of the fact that

$$f\in \mathrm{Lip}\left(\alpha-\frac{1}{p}\right) \quad (\alpha p>1) \tag{3.20}$$

for the proof of Theorem 3. Using (3.20) and adopting the argument similar to those used in proving Theorem 1, we can prove Theorem 3.

4. Corollaries

We specialize the matrix A to obtain the following corollaries from Theorem 1.

COROLLARY 1

Let $p \geqslant 1, 0 < \alpha \leqslant 1, 0 \leqslant \beta < \alpha$. Let A be a lower triangular matrix such that

$$a_{n,k} \geqslant 0, \quad a_{n,k} \leqslant a_{n,k+1} (k = 0, 1, 2, \ldots, n-1), \sum_{k=0}^{n} a_{n,k} = 1. \tag{4.1}$$

Then for $f \in H(\alpha, p)$

$$\|l_n(x)\|_{(\beta,p)} = O(1) \begin{cases} \dfrac{1}{n^{\alpha-\beta}} + a_{n,n} n^{1-\alpha+\beta} & (0 < \alpha < 1) \\ \dfrac{1}{n^{1-\beta}} + a_{n,n} n^{\beta} (\log n)^{1-\beta} & (\alpha = 1). \end{cases}$$

Proof. In this case

$$\psi(n) = \sum_{k=0}^{n} |a_{n,k} - a_{n,k+1}| = \sum_{k=0}^{n-1} (a_{n,k+1} - a_{n,k}) + a_{n,n}$$
$$= 2a_{n,n} - a_{n,0} \leqslant 2a_{n,n}.$$

Now, we take $\lambda_n = \mu_n = n + 1$ so that condition (3.1) is automatically satisfied.

COROLLARY 2

Let the conditions of Corollary 1 hold. Then

$$\|l_n(x)\|_{(\beta,p)} = O(1) \begin{cases} a_{n,n}^{\alpha-\beta} (1 + \log(n+1) a_{n,n})^{\beta/\alpha} & (0 < \alpha < 1) \\ a_{n,n}^{1-\beta} \left[(1 + \log(n+1) a_{n,n})^{\beta} + \left(\log \dfrac{1}{a_{n,n}} \right)^{1-\beta} \right] & (\alpha = 1). \end{cases}$$

Proof. Since $1 = \Sigma_{k=0}^{n} a_{n,k} \leqslant (n+1) a_{n,n}$, we choose $\mu_n = n + 1$, $\lambda_n = 1/a_{n,n}$ so as to satisfy condition (3.1). Also we have $\psi(n) \leqslant 2a_{n,n}$.

Remark. The case $p = \infty$ of Corollaries 1 and 2 are respectively Theorems 1 and 2 of Mohapatra and Chandra [13].

COROLLARY 3

Let $p \geqslant 1, 0 < \alpha \leqslant 1, 0 \leqslant \beta < \alpha$. Let A be a lower triangular matrix such that

$$a_{n,k} \geqslant 0, \sum_{k=0}^{n} a_{n,k} = 1, a_{n,k} \geqslant a_{n,k+1} (k = 0, 1, 2, \ldots, n-1). \tag{4.2}$$

Then for $f \in H(\alpha, p)$

$$\|l_n(x)\|_{(\beta,p)} = O(1) \begin{cases} (1 + \log(n+1) a_{n,0})^{\beta/\alpha} a_{n,0}^{\alpha-\beta}, & (0 < \alpha < 1) \\ (1 + \log(n+1) a_{n,0})^{\beta} a_{n,0}^{1-\beta} + a_{n,0}^{1-\beta} \left(\log \dfrac{1}{a_{n,0}} \right)^{1-\beta} & (\alpha = 1). \end{cases} \tag{4.3}$$

Also

$$\|l_n(x)\|_{(\beta,p)} = O(1)\begin{cases} n^{\beta-\alpha} + a_{n,0}n^{1-\alpha+\beta} & (0<\alpha<1) \\ n^{\beta-1} + a_{n,0}n^{\beta}(\log n)^{1-\beta} & (\alpha=1). \end{cases} \tag{4.4}$$

Proof. In this case

$$\psi(n) = \sum_{k=0}^{n} |a_{n,k} - a_{n,k+1}| = \sum_{k=0}^{n} (a_{n,k} - a_{n,k+1}) = a_{n,0}.$$

Since

$$1 = \sum_{k=0}^{n} a_{n,k} \leqslant (n+1)a_{n,0}$$

we choose

$$\mu_n = n+1, \quad \lambda_n = \frac{1}{a_{n,0}}$$

to obtain (4.3) or we may choose

$$\mu_n = \lambda_n = n+1$$

to obtain (4.4).

Remark. In the case $a_{n,k} = (A^{\delta-1}_{n-k}/A^{\delta}_n)$ (Cesàro matrix), $a_{n,0} = (A^{\delta-1}_n/A^{\delta}_n) \sim (1/n)$. Cesàro matrix satisfies the condition (4.2) in the case $\delta \geqslant 1$ and it satisfies (4.1) in the case $0 < \delta \leqslant 1$. Thus Corollaries 1 (or 2) and 3 together cover the degree of approximation of Cesàro matrix for all $\delta > 0$.

Before we give a few more corollaries, we first describe below a method of summation called 'deferred Cesàro transformation' introduced by Agnew [1].

Deferred Cesàro mean. Let p_n and q_n be sequences of non-negative integers satisfying

$$p_n < q_n \tag{4.5}$$

$$\lim_{n\to\infty} q_n = \infty. \tag{4.6}$$

The deferred Cesàro mean $D(p_n, q_n)$ is defined by ([1], p. 414),

$$D_n(S_n) = \frac{S_{p_n+1} + S_{p_n+2} + \cdots + S_{q_n}}{q_n - p_n}. \tag{4.7}$$

In the notation of matrix transformation

$$D_n(S_n) = \sum_{k=0}^{\infty} a_{n,k} S_k \tag{4.8}$$

where

$$a_{n,k} = \begin{cases} \dfrac{1}{q_n - p_n} & p_n < k \leqslant q_n \\ 0 & \text{(elsewhere)}. \end{cases} \tag{4.9}$$

It is known [1] that $D(p_n, q_n)$ is regular under conditions (4.5) and (4.6). Note that $D_n(n-1, n)$ is the identity transformation and $D(0, n)$ is the $(C, 1)$ transformation.

$D(n, n+k)$ is called the delayed first arithmetic mean (Zygmund [21], p. 80) and it is known [21] that

$$(C,1) \subset D(n, n+k) \quad \text{if } \frac{n}{k} = O(1) \text{ as } k \to \infty \text{ with } n.$$

In fact, more generally, it is known [1] that

$$(C,1) \subset D(p_n, q_n) \text{ if and only if } \frac{p_n}{q_n - p_n} = O(1).$$

It is also known [1] that

$$D(p_n, n) \subset (C,1)$$

and

$$D(p_n, n) \sim (C,1) \text{ if and only if } \frac{p_n}{q_n - p_n} = O(1).$$

Let $\lambda = \{\lambda_n\}$ be a monotone non-decreasing sequence of positive integers such that $\lambda_1 = 1$ and $\lambda_{n+1} - \lambda_n \leqslant 1$. Then $D(n - \lambda_n, n)$ is same as the nth generalized de la vallee Poussin mean $V_n(\lambda)$ [9] generated by the sequence $\{\lambda_n\}$.

We now obtain the degree of approximation of deferred Cesàro mean.

It may be noted that the condition (3.1) is automatically satisfied by choosing a suitable μ_n for which $a_{n,k}$ vanishes if $k > \mu_n$. For example

$$\mu_n = q_n + 1, \quad \lambda_n = q_n - p_n.$$

Now

$$\begin{aligned}\psi(n) &= \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}| \\ &= \sum_{k=p_n}^{q_n} |a_{n,k} - a_{n,k+1}| \\ &= |a_{n,p_n} - a_{n,p_n+1}| + |a_{n,q_n} - a_{n,q_n+1}| \\ &= \left| -\frac{1}{q_n - p_n} \right| + \frac{1}{q_n - p_n} = \frac{2}{q_n - p_n}. \end{aligned} \tag{4.10}$$

Thus from Theorem 1, we obtain Corollary 4.

COROLLARY 4

Let $p \geqslant 1$, $0 < \alpha \leqslant 1$, $0 \leqslant \beta < \alpha$ and $f \in H(\alpha, p)$. Then

$$\| l_n(x) \|_{(\beta, p)} =$$

$$O(1) \begin{cases} \left(1 + \log \dfrac{q_n + 1}{q_n + p_n}\right)^{\beta/\alpha} \dfrac{1}{(q_n - p_n)^{\alpha - \beta}} & (0 < \alpha < 1) \\ \dfrac{1}{(q_n - p_n)^{1-\beta}} \left[\left(1 + \log \dfrac{q_n + 1}{q_n - p_n}\right)^{\beta} + (\log(q_n - p_n))^{1-\beta} \right] & (\alpha = 1). \end{cases}$$

Remark. In the case

$$\delta = \limsup_{n \to \infty} \frac{p_n}{q_n} < 1 \tag{4.11}$$

we have

$$\limsup_{n\to\infty} \frac{q_n+1}{q_n-p_n} = \limsup_{n\to\infty} \left(\frac{1+(1/q_n)}{1-(p_n/q_n)}\right) \leqslant \frac{1}{1-\delta}$$

so that from Corollary 4, we obtain Corollary 5.

COROLLARY 5

Let $p \geqslant 1, 0<\alpha \leqslant 1, 0 \leqslant \beta < \alpha$. Suppose that (4.11) *holds. Then for $f \in H(\alpha, p)$*

$$\|l_n(x)\|_{(\beta,p)} = O(1) \begin{cases} \dfrac{1}{(q_n-p_n)^{\alpha-\beta}} & (0<\alpha<1) \\ \dfrac{1}{(q_n-p_n)^{1-\beta}}[1+(\log(q_n+p_n))^{1-\beta}] & (\alpha=1). \end{cases}$$

At this stage we remark that the case $p=\infty$ of Corollary 4 for $D(n-\lambda_n, n)$ is due to Stypinski [20] where one log factor seems to have been overlooked. Corollary 4 too covers the case of Chandra [7] in relevant cases.

5. Additional theorems

We establish the following theorems:

Theorem 4. *Let $A=(a_{n,k})$ satisfy the same condition as in Theorem* 1. *Further let*

$$a_{n,k} \geqslant a_{n,k+1} \geqslant 0. \tag{5.1}$$

Then for $f \in L_p, p \geqslant 1$

$$\|l_n(x)\|_p = O(1) w_p^{(2)}\left(\frac{\pi}{\mu_n}\right) + O(1) \sum_{k=1}^{\mu_n} \frac{1}{k} w_p^{(2)}\left(\frac{\pi}{k}\right) \bar{a}_n(k+1).$$

Theorem 5. *Let $A=(a_{n,k})$ satisfy the same conditions as in Theorem* 1. *Then for $p \geqslant 1$ and $f \in H(\alpha,p)$, $0<\alpha \leqslant 1, 0 \leqslant \beta < \alpha$*

$$\|l_n(x)\|_{(\beta,p)} = O(1)\left[\frac{1}{\lambda_n^{\alpha-\beta}}\left(1+\log\frac{\mu_n}{\lambda_n}\right)^{\beta/\alpha} + \left(\sum_{k=1}^{\lambda_n} \frac{\theta(n,k)+\psi(n,k)}{k}\right)^{\beta/\alpha}\left(\sum_{k=1}^{\lambda_n} \frac{\theta(n,k)+\psi(n,k)}{k^{1+\alpha}}\right)^{1-(\beta/\alpha)}\right]$$

where

$$\theta(n,k) = \sum_{m=0}^{k} |a_{n,m}|, \quad \psi(n,k) = (k+1) \sum_{m=k+1}^{\infty} |a_{n,m} - a_{n,m+1}| \tag{5.2}$$

amd μ_n and λ_n are defined in Theorem 1.

Theorem 6. *Let $A=(a_{n,k})$ satisfy condition of Theorem* 4. *For $p \geqslant 1$, suppose that $f \in L_p$ and $w_p^{(2)}(t)/t^\theta$ is monotonic non-increasing as t increases for some θ with $0<\theta \leqslant 1$. Then*

$$\|l_n(x)\|_p = O(1) w_p^{(2)}\left(\frac{\pi}{\mu}\right)\left[1+\mu_n^\theta \sum_{k=1}^{\mu_n} \frac{\bar{a}_n(k+1)}{k^{1+\theta}}\right].$$

We need the following lemma.

Lemma 3. *Let* (5.1) *hold. Then*

$$\sum_{k=0}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)t = O(\bar{a}_n(T))$$

where $T=[\pi/t]$.

Proof. We have

$$\left|\sum_{k=0}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right| \leqslant \left|\sum_{k=0}^{T-1} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right| + \left|\sum_{k=T}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right|$$
$$\leqslant \sum_{k=0}^{T-1} a_{n,k} + a_{n,T}O(t^{-1}),$$

by Abel's lemma. Since $(T+1)a_{n,T}\leqslant \Sigma_{k=0}^{T} a_{n,k}$, Lemma 3 follows at once from the above inequality.

Proof of Theorem 4. We know that (see the proof of Theorem 1)

$$l_n(x) = \frac{1}{\pi}\int_0^{\pi} \phi_x(t)K_n(t)\mathrm{d}t.$$

By generalized Minkowski's inequality

$$\|l_n(x)\|_p \leqslant \frac{1}{\pi}\int_0^{\pi} \|\phi_x(t)\|_p|K_n(t)|\mathrm{d}t$$
$$= \frac{1}{\pi}\left(\int_0^{\pi/\mu_n} + \int_{\pi/\mu_n}^{\pi}\right)\|\phi_x(t)\|_p|K_n(t)|\mathrm{d}t$$
$$= I_1 + I_2. \tag{5.3}$$

Now

$$I_1 = O(1)\int_0^{\pi/\mu_n} \frac{w_p^{(2)}(t)}{t}\left|\sum_{k=0}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right|\mathrm{d}t$$
$$= O(1)\mu_n\int_0^{\pi/\mu_n} w_p^{(2)}(t)\mathrm{d}t \quad \text{(as in (3.10))}$$
$$= O(1)\mu_n w_p^{(2)}(\pi/\mu_n)\int_0^{\pi/\mu_n}\mathrm{d}t = O(1)w_p^{(2)}(\pi/\mu_n). \tag{5.4}$$

By Lemma 3

$$I_2 = O(1)\int_{\pi/\mu_n}^{\pi} \frac{w_p^{(2)}(t)}{t}\left|\sum_{k=0}^{\infty} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right|\mathrm{d}t$$
$$= O(1)\int_{\pi/\mu_n}^{\pi} \frac{w_p^{(2)}(t)}{t}\bar{a}_n(T)\mathrm{d}t$$
$$= O(1)\sum_{k=1}^{\mu_n-1}\int_{\pi/(k+1)}^{\pi/k} \frac{w_p^{(2)}(t)}{t}\bar{a}_n([\pi/t])\mathrm{d}t$$

$$= O(1)\sum_{k=1}^{\mu_n} \bar{a}_n(k+1) w_p^{(2)}(\pi/k) \int_{\pi/(k+1)}^{\pi/k} \frac{dt}{t}$$

$$= O(1)\sum_{k=1}^{\mu_n} \frac{\bar{a}_n(k+1)}{k} w_p^{(2)}(\pi/k). \tag{5.5}$$

Now Theorem 4 follows at once from (5.3), (5.4) and (5.5).

We omit the proof of Theorems 5 and 6 as these are easy and follow the lines of proof of Theorem 4.

6.

In this section, we specialize the matrix $A = (a_{n,k})$ to obtain the following corollary to Theorem 5.

COROLLARY 6

Let $A = (a_{n,k})$ be an infinite matrix satisfying the following.

$$a_{n,k} \geqslant 0, \quad \sum_{k=0}^{\infty} a_{n,k} = 1, \quad a_{n,k} \geqslant a_{n,k+1} \tag{6.1}$$

$$\sum_{k=n+1}^{\infty} (k+1) a_{n,k} = O(n). \tag{6.2}$$

Then for $0 \leqslant \beta < \alpha \leqslant 1$ and $f \in H(\alpha, p)$

$$\|l_n(x)\|_{(\beta,p)} = O(1)\left[\frac{1}{n^{\alpha-\beta}} + \left(\sum_{k=1}^{n} \frac{\bar{a}_n(k+1)}{k}\right)^{\beta/\alpha} \left(\sum_{k=1}^{n} \frac{\bar{a}_n(k+1)}{k^{1+\alpha}}\right)^{1-(\beta/\alpha)}\right].$$

Proof. In this case

$$\begin{aligned}\theta(n,k) + \psi(n,k) &= \sum_{m=0}^{k} |a_{n,m}| + (k+1) \sum_{m=k+1}^{\infty} |a_{n,m} - a_{n,m+1}| \\ &= \sum_{m=0}^{k} a_{n,m} + (k+1) \sum_{m=k+1}^{\infty} (a_{n,m} - a_{n,m+1}) \\ &= \sum_{m=0}^{k} a_{n,m} + (k+1) a_{n,k+1} \\ &\leqslant \sum_{m=0}^{k} a_{n,m} + \frac{k+1}{k+1} \sum_{m=0}^{k} a_{n,m} \\ &= O(1) \bar{a}_n(k).\end{aligned}$$

Now taking $\mu_n = \lambda_n = n$ in Theorem 5, we obtain Corollary 6.

Remark. The case $p = \infty$ of Corollary 6 is Theorem 3 of Mohapatra and Chandra [14].

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