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Quasi-parabolic Siegel formula

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Abstract. The result of Siegel that the Tamagawa number of SL_r over a function field is 1 has an expression purely in terms of vector bundles on a curve, which is known as the Siegel formula. We prove an analogous formula for vector bundles with quasi-parabolic structures. This formula can be used to calculate the Betti numbers of the moduli of parabolic vector bundles using the Weil conjuctures.

Keywords. Siegel formula; quasi-parabolic divisors; Betti numbers; Weil conjucture.

1. Introduction

The Betti numbers of the moduli of stable vector bundles on a complex curve, in all the cases where the rank and degree are coprime, were first determined by Harder and Narasimhan [H-N] as an application of the Weil conjuctures. For this, they made use of the result of Siegel that the Tamagawa number of the special linear group over a function field is 1. In their refinement of the same Betti number calculation in [D-R], Desale and Ramanan expressed the result of Siegel in purely vector bundle terms. This result about the Tamagawa number, called the Siegel formula, was later given a simple proof in the language of vector bundles by Ghione and Letizia [G-L], by introducing a notion of effective divisors of higher rank on a curve, and counting the number of effective divisors which correspond to a given vector bundle. The purpose of this note is to introduce the notion of a quasi-parabolic divisor of higher rank on a curve (Definition 3.1 below), and to prove a quasi-parabolic analogue (Theorem 3.4 below) of the Siegel formula, which is done here by suitably generalizing the method of [G-L]. In a note to follow, this formula is used to calculate the zeta function and thereby the Betti numbers of the moduli of parabolic bundles in the case 'stable = semistable' (these Betti numbers have already been calculated by a guage theoretic method for genus $\geqslant 2$ in [N] and for genus 0 and 1 by Furuta and Steer in [F-S]).

2. Divisors supported on $X - S$

Let X be an absolutely irreducible, smooth projective curve over the finite field $k = \mathbf{F}_q$, and let S be any closed subset of X whose points are k-rational. Let K denote the function field of X, and let K_X denote the constant sheaf K on X. Let g denote the genus of X. Let r be a positive integer. Recall that (see [G-L]) a coherent subsheaf $D \subset K_X^r$ of generic rank r is called an r-divisor, and the r-divisor is called effective (or positive) if $\mathcal{O}_X^r \subset D$. The support of the divisor is by definition the support of the quotient $D/\mathcal{O}_X^r$, which is a torsion sheaf. The length n of $D/\mathcal{O}_X^r$ is called the degree of the divisor. Note that D is a locally free sheaf of rank r and degree n.

Remark 2.1. Let $Z_X(t)$ be the zeta function of X. Then as S consists of k-rational points, it can be seen that the zeta function Z_{X-S} of $X-S$ is given by the formula

$$Z_{X-S}(t)=(1-t)^s Z_X(t) \tag{1}$$

where s is the cardinality of S.

Note that an effective r-divisor on $X-S$ is the same as an effective r-divisor on X whose support is disjoint from S. Part (1) of the proposition 1 of [G-L] gives the following, with $X-S$ in place of X.

PROPOSITION 2.2

Let $b_n^{(r)}$ be the number of effective r-divisors of degree n on X whose support is disjoint from S. Let $Z_{X-S}^{(r)}(t)=\Sigma_{n\geqslant 0}b_n^{(r)}t^n$. Then we have

$$Z_{X-S}^{(r)}(t)=\prod_{1\leqslant j\leqslant r} Z_{X-S}(q^{j-1}t). \tag{2}$$

In order to have the analogue of part (2) of the proposition 1 of [G-L], we need the following lemmas.

Lemma 2.3. *Let V be a finite dimensional vector space over $k=\mathbf{F}_q$, and s a positive integer. For any $1\leqslant i\leqslant s$, let $\pi_i: k^s\to k$ be the linear projection. For any surjective linear map $\phi: V\to k^s$, let V_i be the kernel of $\pi_i\phi: V\to k$, which is a hyperplane in V as ϕ is surjective. Let $P=P(V)$, and $P_i=P(V_i)$ denote the corresponding projective spaces. Let $N(\phi)$ denote the number of k-rational points of $P-\cup_{1\leqslant i\leqslant s}P_i$. Then for any other surjective $\psi: V\to k^s$, we have $N(\phi)=N(\psi)$. In other words, given s, this number depends only on* $\dim(V)$.

Proof. Given any two surjective maps $\phi,\psi: V\to k^s$, there exists an $\eta\in GL(V)$ such that $\phi\eta=\psi$. From this, the results follows.

Lemma 2.4. *Let n be a positive integer, such that $n>2g-2+s$ where g is the genus of X and s is the cardinality of s. Let b_n be the total number of effective 1-divisors of degree n supported on $X-S$. Then for any line bundle L on X of degree n, the number of effective 1-divisors supported on $X-S$ which define L is $b_n/P_X(1)$, where $P_X(1)$ is the number of isomorphism classes of line bundles of any fixed degree on X.*

(Here, $P_X(t)$ is the polynomial $(1-t)(1-qt)Z_X(t)$.)

Proof. Let L be any line bundle on X of degree n, where $n>2g-2+s$. Then $H^1(X,L(-S))=0$, so the natural map $\phi: H^0(X,L)\to H^0(X,L|S)$ is surjective. Let $V=H^0(X,L)$. Then $\dim(V)=n+1-g$. Choose a basis for each fiber L_P for $P\in S$. This given an identification of $H^0(X,L|S)$ with k^s. Now it follows that the number $N(\phi)$ defined in the preceeding lemma depends only on n, and is independent of the choice of L as long as it has degree n. But $N(\phi)$ is precisely the number of effective 1-divisors supported on $X-S$, which define the line bundle L on X.

Using the above lemma, the following proposition follows, by an argument similar to the proof of part (2) of proposition 1 in [G-L]. The proof in [G-L] expresses the number of r-divisors in terms of the number of 1-divisors, and the above lemma tells us the number of 1-divisors with support in $X-S$ corresponding to a given line bundle on X.

PROPOSITION 2.5

For L a line bundle of degree n, let $b_n^{(r,L)}$ be the number of effective r-divisors on X supported on $X - S$, having determinant isomorphic to L. Then provided that $n > 2g - 2 + s$, we have

$$b_n^{(r,L)} = b_n^{(r)}/P_X(1). \tag{3}$$

PROPOSITION 2.6

$$\lim_{n\to\infty} \frac{b_n^{(r)}}{q^{rn}} = P_X(1)\frac{(q-1)^{s-1}}{q^{g-1+s}} Z_{X-S}(q^{-2})\dots Z_{X-S}(q^{-r}). \tag{4}$$

Proof. The above statement is the analogue of proposition 2 of [G-L], with the following changes. Instead of all r-divisors on X in [G-L], we consider only those which are supported over $X - S$, and instead of $Z_X(t)$, we use $Z_{X-S}(t)$. As $Z_{X-S}(t) = (1-t)^s Z_X(t)$, the property of $Z_X(t)$ that it has a simple pole at $t = q^{-1}$ and is regular at $1/q^j$ for $j \geqslant 2$ is shared by $Z_{X-S}(t)$. Hence the proof in [G-L] also works in our case, proving the proposition.

Remark 2.7. There is a minor misprint in the equation labeled (1) in [G-L] (p. 149); the factor q^{g-1} should be read as q^{1-g}.

Let L be any given line bundle on X. Choose any closed point $P \in X - S$, and let l denote its degree. For any $\mathcal{O}_X$ module E, set $E(m) = E \otimes \mathcal{O}_X(mP)$. If a vector bundle E of rank r, degree n has determinant L, then $E(m)$ has determinant $L(rm)$, degree $n + rml$ and Euler characteristic $\chi(m) = n + rml + r(1-g)$.

The equations (3) and (4) above imply the following

$$\lim_{m\to\infty} \frac{b_{n+rml}^{(r,L(rm))}}{q^{r\chi(m)}} = (q-1)^{s-1} q^{(r^2-1)(g-1)-s} Z_{X-S}(q^{-2})\dots Z_{X-S}(q^{-r}). \tag{5}$$

3. Quasi-parabolic divisors

For basic facts about parabolic bundles, see [S] and [M-S]. We now introduce the notion of a quasi-parabolic effective divisor of rank r. Let $S \subset X$ be a finite subset consisting of k-rational points. For each $P_i \in S$, let there be given positive integers p_i and $r_{i,1}, \dots, r_{i,p_i}$, with $r_{i,1} + \dots + r_{i,p_i} = r$. This will be called, as usual, the quasi-parabolic data. Recall that a quasi-parabolic structure on a vector bundle E of rank r on X by definition consists of flags $E_{P_i} = F_{i,1} \supset \dots \supset F_{i,p_i} \supset F_{i,p_i+1} = 0$ of vector subspaces in the fibers over the points of S such that $\dim(F_{i,j}/F_{i,j+1}) = r_{i,j}$ for each j from 1 to p_i.

DEFINITION 3.1

Let X, S and the numerical data $(r_{i,j})$ be as above. A positive quasi-parabolic divisor (F, D) on X consists of (i) a quasi-parabolic structure F on the trivial bundle $\mathcal{O}_X^r$, consisting of flags F_i in k^r at points $P_i \in S$ of the given numerical type $(r_{i,j})$, together with (ii) an effective r-divisor D on X, supported on $X - S$.

Note that if (F, D) is a quasi-parabolic r-divisor, then the rank r vector bundle D has a quasi-parabolic structure given by F. We denote by $P_E^{(r)}$ the set of all effective quasi-parabolic r-divisors whose associated quasi-parabolic bundle is isomorphic to a given quasi-parabolic bundle E. For any vector bundle E of rank r, let $\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E)$ denote the set of all injective sheaf homomorphisms $\mathcal{O}^r_X \to E$ which are injective when restricted to S. For any quasi-parabolic bundle E, the group of all quasi-parabolic automorphisms of E will be denoted by $\mathrm{ParAut}(E)$. Then $\mathrm{ParAut}(E)$ acts on $\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E)$ by composition. This action is free, and $P^{(r)}_E$ has a canonical bijection with the quotient set $\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E)/\mathrm{ParAut}(E)$. Hence the cardinality of $P^{(r)}_E$ is given by

$$|P^{(r)}_E| = \frac{|\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E)|}{|\mathrm{ParAut}(E)|}. \tag{6}$$

For $1 \leqslant i \leqslant s$, let Flag_i be the variety of flags in k^r of the numerical type $(r_{i,1}, \ldots, r_{i,p_i})$. Let $\mathrm{Flag}_S = \Pi_{1 \leqslant i \leqslant s} \mathrm{Flag}_i$. Let $f(q, r_{i,j})$ denote the number of k-rational points of Flag_S. If $a^{(r,L)}_n$ denotes the number of quasi-parabolic divisors of flag data $(r_{i,j})$ with degree n, rank r and determinant L, then we have

$$a^{(r,L)}_n = f(q, r_{i,j}) b^{(r,L)}_n. \tag{7}$$

Now let $J(r, L)$ denote the set of all isomorphism classes of quasi-parabolic vector bundles of rank r, degree n, determinant L having the given quasi-parabolic data $(r_{i,j})$ over S. Hence (6) implies the following

$$a^{(r,L)}_n = \sum_{E \in J(r,L)} \frac{|\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E)|}{|\mathrm{ParAut}(E)|}. \tag{8}$$

For any integer m, the map from $J(r, L) \to J(r, L(rm))$ which sends E to $E(m) = E \otimes \mathcal{O}_X(mP)$ is a bijection which preserves $|\mathrm{ParAut}|$. Hence for each m, we have

$$a^{(r,L(rm))}_{n+rml} = \sum_{E \in J(r,L)} \frac{|\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E(m))|}{|\mathrm{ParAut}(E)|}. \tag{9}$$

Lemma 3.2. *With the above notations,*

$$\lim_{m \to \infty} \frac{|\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E(m))|}{q^{r\chi(E(m))}} = 1. \tag{10}$$

Proof. Let m be large enough, so that $E(m)$ is generated by global sections. Then the subset $\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E(m)) \subset \mathrm{Hom}_{\mathrm{inj}}(\mathcal{O}^r_X, E(m))$ is the intersection of the subset $\mathrm{Hom}_{\mathrm{inj}}(\mathcal{O}^r_X, E(m)) \subset \mathrm{Hom}(\mathcal{O}^r_X, E(m))$ with the complement of a union of sr number of r-codimensional linear subspaces of $\mathrm{Hom}(\mathcal{O}^r_X, E(m))$. (Here, the values sr and r are not important: all that matters is that the number sr of these linear subspaces is a constant independent of m, and each is a proper subspace.) Hence the above limit equals $\lim_{m \to \infty}(|\mathrm{Hom}_{\mathrm{inj}}(\mathcal{O}^r_X, E(m))|/q^{r\chi(E(m))})$, which has the value 1 by lemma 3 of [G-L].

Lemma 3.3. *The following sum and limit can be interchanged to give*

$$\sum_{E \in J(r,L)} \lim_{m \to \infty} \frac{|\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E(m))|}{q^{r\chi(E(m))}|\mathrm{ParAut}(E)|} = \lim_{m \to \infty} \sum_{E \in J(r,L)} \frac{|\mathrm{Hom}^S_{\mathrm{inj}}(\mathcal{O}^r_X, E(m))|}{q^{r\chi(E(m))}|\mathrm{ParAut}(E)|}.$$

This lemma has a proof entirely analogous to the corresponding statement in [G-L], so we omit the details.

By (10), the left hand side in the above lemma equals

$$\sum_{E\in J(r,L)} \frac{1}{|\mathrm{ParAut}(E)|}.$$

On the other hand, by (9), the right hand side is $\lim_{m\to\infty} a_{n+rml}^{(r,L(rm))}/q^{r\chi(m)}$. By (5) and (7), this limit has the following value

$$f(q, r_{i,j})(q-1)^{s-1} q^{((r^2-1)(g-1)-s} Z_{X-S}(q^{-2}) \dots Z_{X-S}(q^{-r}).$$

Hence we get theorem 3.4.

Theorem 3.4. (*Quasi-parabolic Siegel formula*)

$$\sum_{E\in J(r,L)} \frac{1}{|\mathrm{ParAut}(E)|} = f(q, r_{i,j})(q-1)^{s-1} q^{(r^2-1)(g-1)-s} Z_{X-S}(q^{-2}) \dots Z_{X-S}(q^{-r})$$

Remark 3.5. Using the expression $Z_{X-S}(t) = (1-t)^s Z_X(t)$, the above equation becomes

$$\sum_{E\in J(r,L)} \frac{1}{|\mathrm{ParAut}(E)|} = f(q, r_{i,j})(q-1)^{s-1} q^{(r^2-1)(g-1)-s} \prod_{2\leqslant j\leqslant r} (1-q^{-j}) Z_X(q^{-j}).$$

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