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Continuity of homomorphism pairs into H^*-triple systems

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Abstract. We show that the separating subspaces for the component operators of a densely valued homomorphism pair into an H^*-triple system are contained in the annihilator ideal. Accordingly, the continuity of densely valued homomorphisms into H^*-algebras and H^*-triple systems with zero annihilator follows.

Keywords. Homomorphism pairs; triple system.

1. Introduction

From the celebrated theorem by Johnson [14] our knowledge of the continuity properties of homomorphisms onto Banach algebras is fairly complete and satisfactory, even in the non-associative context [19]. However, it is still not currently known if densely valued homomorphisms into semisimple associative Banach algebras are continuous, even when the domain algebra and the coming algebra are C^*-algebras. Rodríguez contributed in [20] to this old open problem stating the continuity of densely valued homomorphisms into (non-associative) H^*-algebras with zero annihilator (we note that, by [3; Lemma 34.9], an associative H^*-algebra has zero annihilator if and only if it is semisimple).

As it has been announced in the abstract, in the present paper our main goal is the study of the continuity properties of densely valued homomorphism pairs into an H^*-triple system.

Since the paper by Ambrose [1] introduced associative H^*- algebras and provided the corresponding structure theory, the frame of H^*-algebras has been extensively studied, even in the general non-associative context [11, 12] and successfully developed in several directions. Concerning results on automatic continuity in this context, the first author proved in [23] that the separating subspace for a derivation on an H^*-algebra is contained in the annihilator. The most important novelty of that paper was the introduction of techniques of central closeability in the treatment of automatic continuity problems. This line was successfully exploited in [20, 24, 25].

It should be pointed out the extensive treatment of some ternary H^*-structures [4–10, 16, 18, 25–27]. Specially outstanding are the Hilbert triple systems introduced by Kaup in [16], who used them to solve the classification problem for a certain type of manifolds. H^*-triple systems are a ternary version of the H^*-algebras introduced in [5] for the Jordan case and they are a generalization of the Hilbert triple systems of [16, 18].

We note that, in the papers cited above, a nice structure theory has been developed for H^*-algebras, Hilbert triple systems and H^*-triple systems. Actually two crucial results in the theory of H^*-triple systems will be used.

Theorem 1 [7]. *Every H^*-triple system W is an orthogonal sum*

$$W = \mathrm{Ann}(W) \oplus W_0$$

where Ann(W) *denotes the annihilator of W and W_0 is an H^*-triple system with zero annihilator.*

Theorem 2 [7]. *Every H^*-triple system with zero annihilator is the orthogonal sum of its minimal closed ideals which are topologically simple H^*-triple systems themselves.*

An extensive investigation about the continuity of derivation pairs on Hilbert triple systems was achieved in [25]. In that paper, the theorem on derivations on H^*-algebras was extended to Hilbert triple systems and central closeability was the fundamental tool. Now it is reasonable to extend the Rodríguez theorem on homomorphisms [20] to this ternary context.

2. Central closeability in triple systems

A linear space V over a field $\mathbb{K}$ endowed with a tri-linear triple product $[\cdots]: V \times V \times V \to V$ is said to be a *triple system*. A *normed triple system* is a real or complex triple system V whose underlying linear space is a normed space and the triple product is jointly continuous. We define *Banach triple systems* as those normed triple systems whose underlying normed space is a Banach space.

A real or complex triple system W is said to be an *H^*-triple system* when the underlying linear space is a Hilbert space endowed with an involutive mapping $*$, which is linear in the real case and conjugate-linear in the complex one, and satisfies $[xyz]^* = [x^*y^*z^*]$ and

$$([xyz]|w) = (x|[wz^*y^*]) = (y|[z^*wx^*]) = (z|[y^*x^*w])$$

for all $x, y, z, w \in W$. Here and subsequently $(\cdot|\cdot)$ denotes the inner product of the Hilbert space W. For a given continuous linear operator F on W we denote by $F^\bullet$ its adjoint operator.

It is easy to verify that, for a given H^*-triple system W, $\{x \in W: [xWW] = 0\} = \{x \in W: [WxW] = 0\} = \{x \in W: [WWx] = 0\}$. We write Ann($W$) to denote the above set and it shall be called the *annihilator* of W.

Let V be a triple system. A subspace U of V is said to be a *subtriple* when $[UUU] \subset U$. If $x, y, z \in V$, then we denote by $L(x, y)$, $M(x, z)$, and $R(y, z)$ the linear operators of left, middle, and right multiplication on V defined by $L(x, y)(z) = M(x, z)(y) = R(y, z)(x) = [xyz]$, respectively. We define the multiplication algebra of V, denoted by $M(V)$, as the subalgebra of $L(V)$ (the algebra of all linear operators on V) generated by all multiplication operators on V. For a given subtriple U of V let us denote by $M_U(V)$ the subalgebra of $M(V)$ generated by the multiplication operators $L(x, y)$, $M(x, z)$, and $R(y, z)$ where $x, y, z \in U$. Note that, for a normed triple system V, every element in $M(V)$ is obviously a continuous linear operator on V. Furthermore, if U is a dense subtriple of V, then the elements of $M(U)$ are nothing but the restriction to U of those of $M_U(V)$. This fact will be used in the sequel without previous comment. Given elements x, y, and z in an H^*-triple system W, we have $L(x, y)^\bullet = L(y^*, x^*)$, $M(x, z) = M(z^*, x^*)$, $R(y, z)^\bullet = R(z^*, y^*)$, which immediately implies the joint continuity of the triple product. Therefore every H^*-triple system is a Banach triple system.

Observe that, for a subtriple U of the H^*-triple system W, $(M_U(W))^* = M_{U^*}(W)$. Accordingly, $M(W)$ is a self-adjoint subalgebra of the algebra of all continuous linear operator on the Hilbert space W.

A subspace I of a triple system V is said to be an *ideal* if $[IVV] + [VIV] + [VVI] \subset I$. The definition of the quotient triple system is left to the imagination of the reader.

We say that a triple system V is *prime* if $[IVJ] = 0$ implies either $I = 0$ or $J = 0$, when I and J are ideals of V. A normed triple system V is called *topologically simple* if $[VVV] \neq 0$ and V has no non-trivial closed ideals.

PROPOSITION 1

Let V be a prime (respectively, topologically simple) normed triple system. If U is a dense subtriple of V, then U is prime (respectively, topologically simple).

Proof. It is easy to check that the closure $\bar{I}$ in V of every ideal I of U is an ideal in V and the result follows. □

Central closeability seems to be a powerful tool in the treatment of the automatic continuity problems into H^*-structures, whose use was started in an earlier work [23] about the continuity of derivations on H^*-algebras. Successively this technique has been used in the investigation of the continuity of densely valued homomorphisms into H^*-algebras [20], random derivations on H^*-algebras [24] and derivation pairs on Hilbert triple systems [25]. So it is not surprising that in order to investigate the continuity of homomorphisms we also apply techniques of central closeability. In the present paper several essential and supporting results in this direction are stated.

A *centralizer* on a triple system V is a linear map $f: V \to V$ commuting with all the elements $P \in M(V)$. The set of all the centralizers on V is called the *centroid* of V and we denote it as $Z(V)$. Note that V may be viewed as a $Z(V)$-module. A *partially defined centralizer* on V is a linear operator f from a suitable non-zero ideal I of V into V satisfying:

$$f([xyz]) = [f(x)yz], \quad f([yxz]) = [yf(x)z], \quad \text{and} \quad f([zyx]) = [zyf(x)]$$

for all $x \in I$ and $y, z \in V$.

In the sequel the triple systems V satisfying the following conditions will play a prominent role:

C1: Every partially defined centralizer on V can be extended to an everywhere defined centralizer on V.

C2: The centroid $Z(V)$ determines a finite extension of the ground field.

Actually, if in addition $Z(V)$ is the ground field, then the triple system V is said to be *centrally closed*.

Theorem 3. *Let W be a topologically simple H^*-triple system. Then W satisfies the conditions* C1 *and* C2. *Actually, $Z(W) = \mathbb{C}$ in the complex case and $Z(W)$ is either $\mathbb{R}$ or $\mathbb{C}$ in the real case.*

Proof. On account of [10; Theorem 3], if W is a complex H^*-triple system, then $Z(W) = \mathbb{C}I_V$ and following the pattern established in [25; Lemmas 1–3 and Theorem 2], it may be concluded that W is centrally closed.

Now we assume that W is a real H^*-triple system. In this case, from [10; Theorem 3], $Z(W)$ is either the real or complex field and C2 follows. If W is the realification of a topologically simple complex H^*-triple system, then a straightforward verification, taking into account what has already been proved, shows that the condition C1 is satisfied and actually $Z(W)=\mathbb{C}I_W$. Otherwise, according to [10; Proposition 6], the complexified $W_{\mathbb{C}}$ (see [10; Proposition 5]) is a complex topologically simple H^*-triple system. Partially defined centralizers on W could be complexified to get partially defined centralizers on $W_{\mathbb{C}}$ which can be extended to everywhere defined centralizers on $W_{\mathbb{C}}$. Therefore C1 follows. □

PROPOSITION 2

Let U be a dense subtriple of a prime normed triple system V satisfying C1. *Then every partially defined centralizer on U can be extended to a centralizer on V. Accordingly, if in addition V satisfies* C2, *then the partially defined centralizers on U, $C(U)$, are a finite extension of the ground field with* $\dim_{\mathbb{K}} C(U) \leqslant \dim_{\mathbb{K}} Z(V)$.

Proof. Let f be a partially defined centralizer on U. Let I be the set of those x in V such that $x=\lim f(x_n)$ for some sequence $\{x_n\}$ in the domain of f, denoted by dom(f), converging to zero. It follows easily that I is a closed ideal of V.

Now let $x \in I$, then $x=\lim f(x_n)$ for some sequence $\{x_n\}$ in dom(f) converging to zero. For all $y \in U$ and $z \in \mathrm{dom}(f)$ we have:

$$[xyz]=\lim f(x_n)yz]=\lim[f(x_n)yz]=\lim[x_n y f(z)]=0.$$

Thus we have $[IU\,\mathrm{dom}(f)]=0$ and so $[IV\overline{\mathrm{dom}(f)}]=0$. By primeness $I=0$ and this means that f is closeable as a partially defined operator from V into V. In the next we prove that the closure of f, say f^-, is a partially defined centralizer on V. We have to verify first that the domain, say J, of f^- is an ideal of V. Given $x \in J$ and $y,z \in V$ there exist sequences $\{x_n\}$ in dom(f) and $\{y_n\}$, $\{z_n\}$ in U converging to x, y and z respectively, and $f^-(x)=\lim f(x_n)$. On the other hand $\lim[x_n y_n z_n]=[xyz]$ and $\lim f([x_n y_n z_n])=\lim[f(x_n)y_n z_n]=[f^-(x)yz]$, which shows that $[xyz]\in J$ and $f^-([xyz])=[f^-(x)yz]$. In a similar way we obtain that $[yxz]$, $[zyx]\in J$, $f^-([yxz])=[yf^-(x)z]$, and $f^-[zyx])=[zyf^-(x)]$. Therefore f^- is a partially defined centralizer with domain J into V. By C1, f^- can be extended to an everywhere defined centralizer on V. □

From Theorem 3, Proposition 1 and the preceding result we obtain the following.

COROLLARY 1

Every dense subtriple U of a topologically simple H^*-triple system V with $Z(V)=\mathbb{K}$ is topologically simple and satisfies C1 *and* C2.

Following the pattern established in [23–25] our next goal consists of providing suitable sequences for the continuity problem. To this end we start obtaining an analogous result to [13; Theorem 3.1].

Lemma 1. *Let V be a triple system satisfying* C1. *Given $x_1,\ldots,x_n \in V$ $Z(V)$-linearly independent, there is $P\in M(V)$ such that $P(x_1)\neq 0$ and $P(x_k)=0$ for $1<k\leqslant n$.*

Proof. For $n=1$ the result is trivially true and we assume inductively that the result holds for n. Now we assume that for all $P \in M(V)$, if $P(x_k)=0$ for $k=2,\ldots,n+1$ then $P(x_1)=0$. Define $\mathscr{I}=\{P \in M(V): P(x_k)=0 \text{ for } k=2,\ldots,n\}$. Since obviously $\mathscr{I}$ is a left ideal of $M(V)$, $I=\{P(x_{n+1}): P \in \mathscr{I}\}$ is an ideal of V. Furthermore by induction assumption I is non-zero. The mapping $f: I \to V$ given by $P(x_{n+1}) \mapsto P(x_1)$ is well defined and is clearly a partially defined centralizer on V. So it can be extended to a centralizer on V which, for abbreviation, we continue to write it as f. Since $0=P(x_1)-f(P(x_{n+1}))=P(x_1-f(x_{n+1}))$ for each P in $\mathscr{I}$. By the induction assumption applied to $x_1-f(x_{n+1}), x_2, \ldots, x_n$ there exists $P \in M(V)$ such that $P(x_k)=0$, $k=2,\ldots,n$ (i.e. $P \in \mathscr{I}$) but $P(x_1-f(x_{n+1})) \neq 0$, which is a contradiction. □

Lemma 2. Let V be a topologically simple normed triple system for which $Z(V)$ is a field and let G be a non-empty open subset of V. If $P \in M(V)$ and $\dim_{Z(V)} P(V)=1$ then, there exists $Q \in M(V)$ such that $(QP)^2(V) \cap G \neq \varnothing$.

Proof. Let $y \in V$ such that $P(V)=Z(V)y$ and let $x \in V$ such that $P(x)=y$. Since $P \neq 0$, $G \not\subset \ker(P)$ and so $(V \backslash \ker(P)) \cap G$ is a nonempty open subset of V. From the topological simplicity of V it follows that there is $Q \in M(V)$ such that $Q(y) \in (V \backslash \ker(P)) \cap G$. Since $Q(y) \notin \ker(P)$ we have $Q(y)=f(x)+z$, for suitable $f \in Z(V)$ and $z \in \ker(P)$. Therefore

$$(QP)^2(f^{-1}x)=f^{-1}(QPQ)y=f^{-1}(QP)(fx+z)=f^{-1}Q(fy)=Q(y) \in G. \quad \square$$

Now we follow the traditional sliding hump procedure. To do this we construct as in [24] appropriate sequences having amazing properties, which allow us to put powerful automatic continuity principles into action.

Theorem 4. *Let V be a topologically simple normed triple system for which $Z(V)$ is a field and $\dim_{Z(V)} V=\infty$ and let G be a non-empty open subset of V which does not contain the zero. If V satisfies* C1, *then one of the following assertions holds:*

1. *There exist sequences $\{x_n\}$ in V and $\{P_n\}$ in $M(V)$ such that:*

$$P_{n+1} \cdots P_1 x_n = 0 \quad \text{and} \quad P_n \cdots P_1 x_n \in G \; \forall n \in \mathbb{N}.$$

2. *There exists a sequence $\{Q_n\}$ in $M(V)$ such that:*

$$\dim_{Z(V)} Q_n(V)=1 \quad \forall n \in \mathbb{N},$$

$$Q_n^2(V) \cap G \neq \varnothing \quad \forall n \in \mathbb{N},$$

$$Q_n Q_m = 0 \quad \text{if } m<n.$$

Proof. In the first step we prove that 1 is obtained if $\dim_{Z(V)} P(V) \geqslant 2$ for all $P \in M(V) \backslash \{0\}$. From the topological simplicity of V, we obtain $x_1 \in V$ and $P_1 \in M(V)$ such that $P_1 x_1 \in G$ and suppose that $x_1, \ldots, x_k$ and $P_1, \ldots, P_k$ have been choosen inductively such that $P_j \ldots P_1 x_j \in G$ and $P_j \ldots P_1 x_{j-1}=0$ for $j=2,\ldots,k$. Since $\dim_{Z(V)}(P_k \ldots P_1)(V) \geqslant 2$, there exists $x_{k+1} \in V$ such that $P_k \ldots P_1 x_k$ and $P_k \ldots P_1 x_{k+1}$ are $Z(V)$-linearly independent, and so by Lemma 1, there exists $P \in M(V)$ such that $PP_k \ldots P_1 x_k=0$ and $PP_k \ldots P_1 x_{k+1} \neq 0$. From the topological simplicity of V there is $Q \in M(V)$ such that $QPP_k \ldots P_1 x_{k+1} \in G$ and we define $P_{k+1}=QP$. The sequences $\{x_n\}$ and $\{P_n\}$ constructed in this way satisfy the requirements of the first assertion.

We assume that assertion 2 does not hold and we deduce the first one. Certainly we can assume that there is $P \in M(V)$ such that $\dim_{Z(V)} P(V) = 1$ and we apply Lemma 2 to obtain $R \in M(V)$ with $\dim_{Z(V)} R(V) = 1$ and $R^2(V) \cap G \neq \emptyset$. Let $\mathcal{Q}$ be the set of all finite sequences $\{Q_n\}_{n=1}^N$ in $M(V)$ satisfying:

$$\dim_{Z(V)} Q_n(V) = 1 \quad n = 1, 2, \ldots, N,$$

$$Q_n^2(V) \cap G \neq \emptyset \quad n = 1, 2, \ldots, N,$$

$$Q_n Q_m = 0 \text{ if } 1 \leqslant m < n \leqslant N.$$

Since the singleton $\{R\}$ lies in $\mathcal{Q}$, this set is non-empty. Moreover, we consider in $\mathcal{Q}$ the partial order defined by:

$$\{Q_n\}_{n=1}^N \leqslant \{Q'_m\}_{m=1}^M \quad \text{iff} \quad N \leqslant M \quad \text{and} \quad Q_n = Q'_n \quad \text{for} \quad n = 1, \ldots, N.$$

There is a maximal element $\{Q_n\}_{n=1}^N$ in $\mathcal{Q}$, because we assume that assertion 2 does not hold. For every $n = 1, \ldots, N$, let $y_n \in V$ such that $Q_n(V) = Z(V)y_n$ and $Q_n(y_n) \neq 0$. From the equality $Q_n Q_m = 0$ if $1 \leqslant m < n \leqslant N$, we deduce that $\{y_1, \ldots, y_N\}$ is a family of $Z(V)$-linearly independent elements in V. Now let $x_1 \in V$ such that $\{y_1, \ldots, y_N, x_1\}$ is a family of $Z(V)$-linearly independent elements and, by Lemma 1, let $P_1 \in M(V)$ such that $P_1(y_1) = \cdots = P_1(y_N) = 0$ and $P_1(x_1) \neq 0$. If $\dim_{Z(V)} P_1(V) = 1$ then we apply Lemma 2 to obtain $Q \in M(V)$ such that the element $Q_{N+1} = QP_1$ has the property $Q_{N+1}^2(V) \cap G \neq \emptyset$, trivially $\dim_{Z(V)} Q_{N+1}(V) = 1$, and also $Q_{N+1} Q_n = 0$ for all $n = 1, \ldots, N$. Thus, the sequence $\{Q_n\}_{n=1}^{N+1}$ lies in $\mathcal{Q}$ which contradicts the maximality of $\{Q_n\}_{n=1}^N$. Therefore $\dim_{Z(V)} P_1(V) \geqslant 2$, and thus there exists $x_2 \in V$ such that $P_1(x_1)$ and $P_1(x_2)$ are $Z(V)$-linearly independent. Now let $P_2 \in M(V)$ with $P_2 P_1(x_1) = 0$ and $P_2 P_1(x_2) \in G$. Reasoning as above, we deduce that $\dim_{Z(V)}(P_2 P_1(V) \geqslant 2$. We suppose that $x_1, \ldots, x_k$ and $P_1, \ldots, P_k$ have been choosen inductively such that $P_j, \ldots P_1 x_{j-1} = 0$ and $P_j \ldots P_1 x_j \in G$ with $\dim_{Z(V)}(P_j \ldots P_1)(V) \geqslant 2$ for $j = 2, \ldots, k$. We get $x_{k+1} \in V$ such that $P_k \ldots P_1 x_k$ and $P_k \ldots P_1 x_{k+1}$ are $Z(V)$-linearly independent. Therefore, by Lemma 1 and topological simplicity there exists an element $P_{k+1} \in M(V)$ such that $P_{k+1} \ldots P_1 x_k = 0$ and $P_{k+1} \ldots P_1 x_{k+1} \in G$. If $\dim_{Z(V)}(P_{k+1} \ldots P_1)(V) = 1$, we apply Lemma 2, to obtain an element $Q_{N+1} = QP_{k+1} \ldots P_1$ in $M(V)$ satisfying $Q_{N+1}^2(V) \cap G \neq \emptyset$, clearly $\dim_{Z(V)} Q_{N+1}(V) = 1$, and $Q_{N+1} Q_n = 0$ for $n = 1, \ldots, N$. Consequently $\dim_{Z(V)}(P_{k+1} \ldots P_1)(V) \geqslant 2$. Finally the sequences $\{x_n\}$ in V and $\{P_n\}$ in $M(V)$ constructed in this way satisfy assertion 1. □

According to Corollary 1, we may apply the preceding theorem to get the following.

COROLLARY 2

Let U be an infinite-dimensional dense subtriple of a topologically simple H^*-triple system V with $Z(V) = \mathbb{K}$. Then one of the following conditions holds:

S1: *There exist sequences $\{x_n\}$ in U and $\{P_n\}$ in $M(U)$ such that:*

$$P_{n+1} \ldots P_1 x_n = 0 \quad \textit{and} \quad P_n \ldots P_1 x_n \neq 0 \quad \forall n \in \mathbb{N}.$$

S2: *There exists a sequence $\{Q_n\}$ in $M(U)$ such that:*

$$\dim_{\mathbb{K}} Q_n(U) = 1 \qquad \forall n \in \mathbb{N},$$

$$Q_n^2 \neq 0 \qquad \forall n \in \mathbb{N},$$

$$Q_n Q_m = 0 \quad \textit{if} \quad m < n.$$

3. Continuity of homomorphisms and homomorphism pairs

Let U and V be triple systems. A linear map $\Phi: U \to V$ is called an *homomorphism* if

$$\Phi([xyz]) = [\Phi(x)\Phi(y)\Phi(z)]$$

for all $x, y, z \in U$.

For a given homomorphism $\Phi: U \to V$, it is easy to check that the image $\Phi(U)$ is a subtriple of V and the kernel $\ker \Phi$ is an ideal in U.

Given $x, y, z \in U$, it follows easily that $\Phi L(x,y) = L(\Phi(x), \Phi(y))\Phi$, $\Phi M(x,z) = M(\Phi(x), \Phi(z))\Phi$, and $\Phi R(y,z) = R(\Phi(y), \Phi(z))\Phi$. So the subalgebra $\mathcal{M}$ of those $P \in M(U)$ for which there is $Q \in M_{\Phi(U)}(V)$ such that $\Phi P = Q\Phi$, contains all left, middle, and right multiplication operators on U and therefore equals $M(U)$. If U and V are normed triple systems and Φ is a densely valued homomorphism from U into V we obtain for every P in $M(U)$ a unique element $\phi(P)$ in $M_{\Phi(U)}(V)$ such that $\Phi P = \phi(P)\Phi$. So we can define an algebra homomorphism ϕ from $M(U)$ onto $M_{\Phi(U)}(V)$.

Recall that we can measure the continuity of an operator F acting between Banach spaces X and Y by considering its so-called *separating subspace*

$$\mathscr{S}(F) = \{y \in Y : \lim F(x_n) = y \text{ and } \lim x_n = 0\}.$$

By the closed graph theorem it follows that F is continuous if, and only if, $\mathscr{S}(F) = 0$.

It is easy to check that the separating subspace $\mathscr{S}(\Phi)$ for a densely valued homomorphism Φ between Banach triple systems U and V is a closed ideal in V.

For studying the continuity properties of densely valued homomorphisms into H^*-triple systems we first require two fundamental principles from automatic continuity. The first one was stated by M P Thomas in [22; Proposition 1.3], but it should be pointed out that the underlying principle was used earlier by many authors [15, 17, 23, 24].

PROPOSITION 3

Let X be a Banach space, $\{S_n\}$ a sequence of continuous linear operator from X into itself and $\{R_n\}$ be a sequence of continuous linear operators whose domain is X but which may map into others Banach spaces Y_n. If F is a possibly discontinuous linear operator from X into itself, such that $R_n F S_1 \ldots S_m$ is continuous for $m > n$, then $R_n F S_1 \ldots S_n$ is continuous for sufficiently large n.

The second principle seems to be unstated, but as before the underlying principle was used by many authors [2, 24].

PROPOSITION 4

Let X and Y be Banach spaces and F a possibly discontinuous linear operator from X into Y. If $\{S_n\}$ and $\{R_n\}$ are sequences of continuous linear operators on X and Y respectively such that $FS_n - R_n F$ is continuous for all $n \in \mathbb{N}$ and $S_n S_m = 0$ if $n < m$ then FS_n^2, and so $R_n^2 F$, is continuous for sufficiently large n.

Proof. Suppose that the result fails. It is then possible to choose a strictly increasing sequence $\{n_k\}$ of natural numbers and a sequence $\{x_k\}$ in X satisfying:

(i) $\|x_k\| < \dfrac{1}{2^k \|S_{n_k}\|} \forall k \in \mathbb{N}$,

(note that $S_{n_k} \neq 0$).

(ii) $\|FS_{n_k}^2(x_k)\| > k\|R_{n_k}\| + \left\|F\left(\sum_{j=1}^{k-1} S_{n_k}S_{n_j}(x_j)\right)\right\| + \|R_{n_k}F - S_{n_k}F\| \forall k \in \mathbb{N}$.

Defining x in X as $x = \sum_{j=1}^{\infty} S_{n_j}(x_j)$ we have, for all $k \in \mathbb{N}$:

$$FS_{n_k}(x) = F\left(\sum_{j=1}^{\infty} S_{n_k}S_{n_j}(x_j)\right) = FS_{n_k}^2(x) + F\left(\sum_{j=1}^{k-1} S_{n_k}S_{n_j}(x_j)\right),$$

and so

$$\begin{aligned}\|R_{n_k}F(x)\| &\geqslant \|FS_{n_k}x\| - \|(R_{n_k}F - FS_{n_k})x\| \\ &\geqslant \|FS_{n_k}^2 x\| - \left\|F\left(\sum_{j=1}^{k-1} S_{n_k}S_{n_j}(x_j)\right)\right\| - \|R_{n_k}F - FS_{n_k}\| \\ &> k\|R_{n_k}\|.\end{aligned}$$

Hence $\|F(x)\| > k$ for every $k \in \mathbb{N}$ which is impossible. □

Lemma 3. *Let Φ be a densely valued homomorphism from a Banach triple system V into an infinite-dimensional topologically simple H^*-triple system W. If $(\Phi(V))^*$ satisfies* S1, *then Φ is continuous.*

Proof. There exist sequences $\{x_n\}$ in $(\Phi(V))^*$ and $\{P_n\}$ in $M_{(\Phi(V))^*}(W) = (M_{\Phi(V)}(W))^*$ such that: $P_{n+1} \ldots P_1 x_n = 0$ and $P_n \ldots P_1 x_n \neq 0$ $\forall n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we choose $S_n \in M(V)$ such that $\phi(S_n) = P_n^*$. Let R_n be the continuous linear operator defined on W by $R_n(x) = (x|x_n)$. For all $x \in V$ and $m, n \in \mathbb{N}$, we have:

$$\begin{aligned}R_n\Phi S_1 \ldots S_m x &= (\Phi S_1 \ldots S_m x | x_n) = (\phi(S_1 \ldots S_m)\Phi x | x_n) \\ &= (\phi(S_1) \ldots \phi(S_m)\Phi x | x_n) = (P_1^* \cdots P_m^* \Phi x | x_n) \\ &= (\Phi x | P_m \ldots P_1 x_n)\end{aligned}$$

which equals zero if $m > n$. From Proposition 3 it follows the continuity of the operator $R_n\Phi S_1 \ldots S_n$, and so the continuity of the functional $x \mapsto (\Phi x | P_n \ldots P_1 x_n)$ on V, for sufficiently large n. The linear subspace of W defined by $I = \{y \in W : x \mapsto (\Phi(x)|y)$ is continuous on $V\}$ is closed by the Banach-Steinhauss theorem and it is easy to check that

$$[I(\Phi(V))^*(\Phi(V))^*] + [(\Phi(V))^* I(\Phi(V))^*] + [(\Phi(V))^*(\Phi(V))^* I] \subset I.$$

Since $(\Phi(V))^*$ is dense in W, I is a closed ideal of W containing the non-zero elements $P_n \ldots P_1 x_n$, for sufficiently large n. Hence I equals W by topological simplicity. The continuity of every functional $x \mapsto (\Phi(x)|y)$ $(y \in W)$ shows that $\mathscr{S}(\Phi) = 0$ and consequently the continuity of Φ. □

Lemma 4. *Let Φ be a densely valued homomorphism from a Banach triple system V into a topologically simple H^*-triple system W with $Z(V) = \mathbb{K}$. Then* $\ker\Phi$ *is closed in V.*

Proof. It is obvious that $\ker\Phi$ is an ideal in V and $\overline{\Phi(\overline{\ker\Phi})}$ is a closed ideal in W. If $\overline{\Phi(\overline{\ker\Phi})} = 0$, then $\overline{\ker\Phi} = \ker\Phi$ and the claim follows. Otherwise, by topological simplicity, $\overline{\Phi(\overline{\ker\Phi})} = W$ and the restriction $\Psi = \Phi_{|\overline{\ker\Phi}}$ is a densely valued homomorphism from $\overline{\ker\Phi}$ into W. Let us denote by ψ the algebra homomorphism from $M(\overline{\ker\Phi})$ onto $M_{\Psi(\overline{\ker\Phi})}(W)$ associated to Ψ.

Assume that W has infinite dimension. $(\Psi(\overline{\ker\Phi}))^*$ is infinite-dimensional and, by Corollary 2, satisfies either S1 or S2.

If $(\Psi(\overline{\ker\Phi}))^*$ satisfies S1, then the preceding lemma shows that Ψ is continuous and so $\ker\Psi$ is closed in $\overline{\ker\Phi}$. Since $\ker\Psi = \ker\Phi$, this is a contradiction.

On the other hand, if $(\Psi(\overline{\ker\Phi}))^*$ satisfies S2, then there is Q in $M_{(\Psi(\overline{\ker\Phi}))^*}(W)$ with $\dim_{\mathbb{K}} Q(W) = 1$ and $Q^2 \neq 0$. Since $\dim_{\mathbb{K}} Q^{\bullet}(W) = 1$ there exist $x_1, x_2 \in W$ such that $Q^{\bullet}(x) = (x|x_1)x_2 \forall x \in W$. Furthermore $(x_2|x_1) \neq 0$, since $(Q^{\bullet})^2 \neq 0$. Set $S = (x_2|x_1)^{-1} Q^{\bullet} \in M_{\Psi(\overline{\ker\Phi})}(W)$ and note that $S \neq 0$ and $S^2 = S$. Let $\overline{\ker\psi}$ denote the closure in $M(\overline{\ker\Phi})$ of $\ker\psi$, which is obviously a subalgebra of $M(\overline{\ker\Phi})$ containing all multiplication operators on $\overline{\ker\Phi}$. Therefore $\overline{\ker\psi} = M(\overline{\ker\Phi})$. Since $S \in M_{\Psi(\overline{\ker\Phi})}(W)$ there is P in $\overline{\ker\psi}$ such that $\psi(P) = S$ and we choose $R \in \ker\psi$ such that $\|R - P\| < 1$. The completeness of the domain, $\overline{\ker\Phi}$, implies that $(I_{\overline{\ker\Phi}} - P + R)F = I_{\overline{\ker\Phi}}$ for a suitable invertible continuous operator F from $\overline{\ker\Phi}$ onto itself. Hence

$$
\begin{aligned}
\Psi &= \Psi(I_{\overline{\ker\Phi}} - P + R)F \\
&= (\Psi - \Psi P + \Psi R)F = (\Psi - \psi(P)\Psi + \psi(R)\Psi)F \\
&= (\Psi - S\Psi)F = (I_W - S)\Psi F
\end{aligned}
$$

and so $I_W - S$ is a densely valued continuous linear operator from W into itself. Since $S(I_W - S) = 0$, we conclude that $S = 0$, a contradiction.

Assume that W has finite dimension. Let $\{y_1, \ldots, y_m\}$ a basis in W. By Lemma 1, there is $Q \in M(W)$ such that $Q(y_1) \neq 0$ and $Q(y_k) = 0$ $k = 2, \ldots, m$. It is clear that $\dim_{\mathbb{K}} Q(W) = 1$ and we apply Lemma 2 to get $R \in M(W)$ with $\dim_{\mathbb{K}} R(W) = 1$ and $R^2 \neq 0$. Since $\Psi(\overline{\ker\Phi}) = W$, $R^{\bullet} \in M_{\Psi(\overline{\ker\Phi})}(W)$ and we take $S = \psi^{-1}(R^{\bullet}) \in M(\overline{\ker\Phi})$. Now we argue as in the preceding step to get a contradiction. □

PROPOSITION 5

Let Φ be a densely valued homomorphism from a Banach triple system V into a topologically simple H^*-triple system W with $Z(W) = \mathbb{K}$. Then Φ is continuous.

Proof. By the preceding result we can drop the homomorphism Φ into a densely valued homomorphism Ψ from the Banach triple system $U = V/\ker\Phi$ into W. Let us denote by ψ the associated algebra isomorphism from $M(U)$ onto $M_{\Psi(U)}(W)$. It suffices to prove that Ψ is continuous. To do this we need only consider an infinite-dimensional W. In such a case $(\Psi(U))^*$ satisfies either S1 or S2.

If $(\Psi(U))^*$ satisfies S1, then according to Lemma 3, we have Ψ continuous.

Otherwise, $(\Psi(U))^*$ satisfies S2. For each $n \in \mathbb{N}$, we define $S_n = \psi^{-1}(Q_n^{\bullet})$. Also we define $R_n = Q_n^{\bullet}$. The operators $\Psi S_n - R_n \Psi$ equals zero, and so they are continuous. Moreover, it is easy to check that $S_n S_m = 0$ if $n < m$. Proposition 4 shows that the operator $R_n^2 \Psi$ is continuous, and thus $\mathscr{S}(R_n^2 \Psi) = 0$, for sufficiently large n. By [21; Lemma 1.3] $\mathscr{S}(\Psi) \subset \ker(R_n^2)$ for sufficiently large n. Since $\mathscr{S}(\Psi)$ is a closed ideal in W and $R_n^2 \neq 0$, we have $\mathscr{S}(\Psi) = 0$. Therefore Ψ is continuous as required. □

Theorem 5. *Let Φ be a densely valued homomorphism from a Banach triple system V into an H^*-triple system W. Then the separating subspace $\mathcal{S}(\Phi)$ is contained in the annihilator of W. Accordingly, Φ is continuous if W has zero annihilator.*

Proof. We have divided the proof into many steps.

I. Assume that the ground field is $\mathbb{C}$. If W is an H^*-triple system with zero annihilator, then we apply Theorem 2. Let I be a minimal closed ideal of W and π_I the orthogonal projection from W onto I. We consider the densely valued homomorphism $\pi_I\Phi$ from the Banach triple system V into the topologically simple H^*-triple system I for which $Z(I)=\mathbb{C}$ ([10; Theorem 3]). It follows from Proposition 5 that $\pi_I\Phi$ is continuous. By [21; Lemma 1.3], $\mathcal{S}(\Phi)\subset\ker(\pi_I)$ and so $(\mathcal{S}(\Phi)|I)=0$. The equality $(\mathcal{S}(\Phi)|I)=0$ holds for every minimal closed ideal I of W and applying Theorem 2 we get $(\mathcal{S}(\Phi)|W)=0$ and hence $\mathcal{S}(\Phi)=0$.

The general case will be reduced to that proved above by means of Theorem 1. W is an orthogonal sum $W=\mathrm{Ann}(W)\oplus W_0$ where W_0 is a complex H^*-triple system with zero annihilator. Let π denote the orthogonal projection from W onto W_0. It can be easily verified that $\pi\Phi$ is a densely valued homomorphism from V into W_0. $\pi\Phi$ is continuous and so [20; Lemma 1.3] $\mathcal{S}(\Phi)\subset\ker\pi=\mathrm{Ann}(W)$.

II. Assume that the ground field is $\mathbb{R}$. We consider the algebraic complexified $V_\mathbb{C}$ and $W_\mathbb{C}$ of V and W, respectively, and the only complex homomorphism $\Phi_\mathbb{C}$ from $V_\mathbb{C}$ into $W_\mathbb{C}$ extending Φ. $W_\mathbb{C}$ is endowed with a natural structure of complex H^*-triple system [10; Proposition 5] and $V_\mathbb{C}$ may be endowed with a norm for which $V_\mathbb{C}$ becomes a complex Banach triple system (this can be made as in the algebra case [3; Proposition 13.3]). From what has already been proved, it may be concluded that $\mathcal{S}(\Phi_\mathbb{C})\subset\mathrm{Ann}(W_\mathbb{C})$. Since $\mathcal{S}(\Phi_\mathbb{C})=\mathcal{S}(\Phi)\oplus i\mathcal{S}(\Phi)$ and $\mathrm{Ann}(W_\mathbb{C})=\mathrm{Ann}(W)\oplus i\mathrm{Ann}(W)$, our claim follows. □

A couple (V_+,V_-) of linear spaces over a field $\mathbb{K}$ together with trilinear triple products $[\cdots]_+: V_+\times V_-\times V_+\to V_+$ and $[\cdots]_-: V_-\times V_+\times V_-\to V_-$ is said to be a *pair*. *Normed pairs* are defined as those real or complex pairs whose underlying vector spaces are normed spaces and the triple products are jointly continuous. We define *Banach pairs* as those normed pairs (V_+, V_-) whose underlying normed spaces V_+ and V_- are Banach spaces.

Examples. 1. Let X and Y be normed spaces and let us denote by $BL(X, Y)$, the normed space of all continuous linear operators from X into Y. Then $(BL(X, Y), BL(Y, X))$ is the classical example of normed pair where the triple products are both defined by $[abc]=abc$.

2. If V is a triple system, then (V, V) becomes a pair for the products $[xyz]_+=[xyz]_-=[xyz]$.

Given two pairs (U_+, U_-) and (V_+, V_-), a couple (Φ_+, Φ_-) of linear mappings Φ_σ: $U_\sigma\to V_\sigma$ is said to be an *homomorphism pair* if

$$\Phi_\sigma([xyz]_\sigma)=[\Phi_\sigma(x)\Phi_{-\sigma}(y)\Phi_\sigma(z)]_\sigma$$

for all $x,z\in V_\sigma$, $y\in V_{-\sigma}$, and $\sigma\in\{+,-\}$.

Theorem 6. *Let (Φ_+,Φ_-) be a homomorphism pair from a Banach pair (V_+, V_-) into an H^*-triple system W. If the operators Φ_+ and Φ_- are densely valued into W, then the separating subspaces $\mathcal{S}(\Phi_+)$ and $\mathcal{S}(\Phi_-)$ are contained in the annihilator of W. Accordingly, Φ_+ and Φ_- are continuous if W has zero annihilator.*

Proof. We consider the Banach space $V = V_+ \oplus V_-$ and define a jointly continuous triple product on V by

$$[(x_+, x_-)(y_+, y_-)(z_+, z_-)] = ([x_+ y_- z_+]_+, [x_- y_+ z_-]_-).$$

The same construction applied to W gives the so-called H^*-triple system polarized of W (see [8]), and we denote it as W_p. It is a simple matter to see that the linear map Φ from V into W_p given by $\Phi(x_+, x_-) = (\Phi_+ x_+, \Phi_- x_-)$ is a densely valued homomorphism. From Theorem 5 it follows that $\mathscr{S}(\Phi) \subset \mathrm{Ann}(W_p) = \mathrm{Ann}(W) \oplus \mathrm{Ann}(W)$ and this completes the proof, since $\mathscr{S}(\Phi) = \mathscr{S}(\Phi_+) \oplus \mathscr{S}(\Phi_-)$. □

From Theorem 5 we deduce the Rodríguez's theorem on the continuity of densely valued homomorphisms into H^*-algebras. Recall that an *H^*-algebra* is a real or complex (possibly non-associative) algebra B with an algebra involution $*$, which is linear in the real case and conjugate-linear in the complex one, and whose underlying linear space is a Hilbert space in which the equalities $(ab|c) = (a|cb^*) = (b|a^*c)$ hold for all $a, b, c \in B$.

COROLLARY 3

Let Φ be a densely valued homomorphism from a complete normed algebra A into an H^*-algebra B. Then the separating subspace $\mathscr{S}(\Phi)$ for Φ is contained in the annihilator of B. Accordingly, Φ is continuous if B has zero annihilator.

Proof. We endow the Hilbert space $W = B \oplus B$ with the jointly continuous triple product given by $[(a_1, a_2)(b_1, b_2)(c_1, c_2)] = ((a_1 b_2)c_1, c_2(b_1 a_2))$ and the involution given by $(a_1, a_2)^* = (a_2^*, a_1^*)$. W becomes an H^*-triple system. In the same way the Banach space $V = A \oplus A$ becomes a Banach triple system. The map Ψ from V into W given by $\Psi(a_1, a_2) = (\Phi a_1, \Phi a_2)$ is a densely valued homomorphism and therefore $\mathscr{S}(\Phi) \oplus \mathscr{S}(\Phi) = \mathscr{S}(\Psi) \subset \mathrm{Ann}(W) = \mathrm{Ann}(B) \oplus \mathrm{Ann}(B)$, which ends the proof. □

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