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On unified fractional integral operators

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Abstract. The present paper is in continuation to our recent paper [6] in these proceedings. Therein, three composition formulae for a general class of fractional integral operators had been established. In this paper, we develop the Mellin transforms and their inversions, the Mellin convolutions, the associated Parseval–Goldstein theorem and the images of the multivariable H-function together with applications for these operators. In all, seven theorems and two corollaries (involving the Konhauser biorthogonal polynomials and the Jacobi polynomials) have been established in this paper. On account of the most general nature of the polynomials $S_n^m[x]$ and the multivariable H-function whose product form the kernels of our operators, a large number of (new and known) interesting results involving simpler polynomials and special functions (involving one or more variables) obtained by several authors and hitherto lying scattered in the literature follow as special cases of our findings. We give here exact references to the results (in essence) of seven research papers which follow as simple special cases of our theorems.

Keywords. Fractional integral operator; general class of polynomials; multivariable H-function; Mellin transform; Mellin convolution.

1. Introduction

We shall study in this paper the fractional integral operators defined by means of the following equations

$$R_x^{\eta,\alpha}[f(x)]$$

$$= R_{x;e;z_1,\ldots,z_r,a_j,\alpha'_j,\ldots,\alpha_j^{(r)},b_j,\beta'_j,\ldots,\beta_j^{(r)};c'_j,\gamma'_j,d'_j,\delta'_j,\ldots,c_j^{(r)},\gamma_j^{(r)};d_j^{(r)},\delta_j^{(r)}}^{\eta,\alpha;m,n,\mu,v;N,P,Q,M',N',P',Q',\ldots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},u_1,v_1,\ldots,u_r,v_r}[f(x)]$$

$$= x^{-\eta-\alpha-1}\int_0^x t^\eta (x-t)^\alpha S_n^m\left[e\left(\frac{t}{x}\right)^\mu\left(1-\frac{t}{x}\right)^v\right] H\left[z_1\left(\frac{t}{x}\right)^{u_1}\left(1-\frac{t}{x}\right)^{v_1},\ldots,\right.$$

$$\left. z_r\left(\frac{t}{x}\right)^{u_r}\left(1-\frac{t}{x}\right)^{v_r}\right] f(t)\mathrm{d}t \qquad (1)$$

$$W_x^{\eta,\alpha}[f(x)]$$

$$= W_{x;e;z_1,\ldots,z_r,a_j,\alpha'_j,\ldots,\alpha_j^{(r)},b_j,\beta'_j\ldots,\beta_j^{(r)};c'_j,\gamma'_j,d'_j,\delta'_j,\ldots,c_j^{(r)},\gamma_j^{(r)};d_j^{(r)},\delta_j^{(r)}}^{\eta,\alpha;m,n,\mu,v;N,P,Q,M',N',P',Q',\ldots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},u_1,v_1,\ldots,u_r,v_r}[f(x)]$$

$$= x^\eta\int_x^\infty t^{-\eta-\alpha-1}(t-x)^\alpha S_n^m\left[e\left(\frac{x}{t}\right)^\mu\left(1-\frac{x}{t}\right)^v\right] H\left[z_1\left(\frac{x}{t}\right)^{u_1}\left(1-\frac{x}{t}\right)^{v_1},\ldots,\right.$$

$$\left. z_r\left(\frac{x}{t}\right)^{u_r}\left(1-\frac{x}{t}\right)^{v_r}\right] f(t)\mathrm{d}t. \qquad (2)$$

Here $S_n^m[x]$ denotes the general class of polynomials introduced by Srivastava

[12, p. 1, eq. (1)]

$$S_n^m[x]=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}x^k,\quad n=0,1,2,\ldots,\tag{3}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}(n,k\geqslant 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,k}$, $S_n^m[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel polynomials, the Gould-Hopper polynomials, the Brafman polynomials and several others [16, pp. 158–161].

The H-function of r complex variables $z_1,\ldots,z_r$ [15] occurring in the paper will be represented in the following form [14, p. 251, eq. (C.1)]

$$H[z_1,\ldots,z_r]=H^{o,N:M',N';\ldots;M^{(r)},N^{(r)}}_{P,Q:P',Q';\ldots;P^{(r)},Q^{(r)}}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}(a_j;\alpha'_j,\ldots,\alpha_j^{(r)})_{1,P}:\\ (b_j;\beta'_j,\ldots,\beta_j^{(r)})_{1,Q}:\end{matrix}\right.$$
$$\left.\begin{matrix}(c'_j,\gamma'_j)_{1,P'};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,P^{(r)}}\\ (d'_j,\delta'_j)_{1,Q'};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,Q^{(r)}}\end{matrix}\right].\tag{4}$$

The defining integral and other details about this function can be found in the references given above. It is assumed throughout the present work that this function always satisfies the appropriate existence and convergence conditions of its defining integral [14, pp. 252–253, eqs (C.4–C.6)].

To be specific, we shall assume throughout this paper that

$$f(x)=\begin{cases}O(|x|^{\gamma}), & |x|\to 0\\ O(|x|^{\delta}e^{-\lambda|x|}), & |x|\to\infty.\end{cases}$$

It is easy to verify that the operator defined by (1) exists if

(i) The quantities $\mu,v,u_1,v_1,\ldots,u_r,v_r$ are all positive (some of them may decrease to zero provided that the resulting operator has a meaning).

(ii) $\mathrm{Re}(\alpha)+\sum_{i=1}^{r}v_i\min_{1\leq j\leq M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0.$

(iii) $\mathrm{Re}(\eta+\gamma)+\sum_{i=1}^{r}u_i\min_{1\leq j\leq M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0.$

and the operator defined by (2) exists if $\mathrm{Re}(\lambda)>0$ or $\mathrm{Re}(\lambda)=0$ and $\mathrm{Re}(\eta-\delta)+\sum_{i=1}^{r}u_i\min_{1\leq j\leq M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]>0$, and the set of conditions (i) and (ii) specified for the existence of the operator (1) are satisfied.

2. The Mellin transforms and the inversion formulae

Theorem 1. *If* $M\{f(x);s\}$, $M\{R_x^{\eta,\alpha}[f(x)];s\}$ *exist,* $\mathrm{Re}(1+\alpha)>0$, $\mathrm{Re}(1+\eta-s)>0$ *and the conditions of the existence of the operator* $R_x^{\eta,\alpha}[f(x)]$ *are satisfied, then*

$$M\{R_x^{\eta,\alpha}[f(x)];s\}=\phi_1(s)M\{f(x);s\}\tag{5}$$

where

$$\phi_1(s)=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}e^kH^{o,N+2:*}_{P+2,Q+1:*}\left[\begin{array}{c}z_1\\ \vdots\\ z_r\end{array}\right|$$

$$\left.\begin{array}{l}(s-\eta-\mu k;u_1,\ldots,u_r),(-\alpha-\nu k;v_1,\ldots,v_r),(a_j;\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,P}:*\\ (-1+s-\eta-\alpha-(\mu+\nu)k;u_1+v_1,\ldots,u_r+v_r),(b_j;\beta'_j,\ldots,\beta^{(r)}_j)_{1,Q}:*\end{array}\right] \tag{6}$$

the asterisk (*) *in* (6) *indicates that the parameters at these places are the same as the parameters of the H-function of r complex variables occurring in* (4) *and* $M\{f(x);s\}$ *stands for the well known Mellin transform of the function* $f(x)$ *defined by the following equation*

$$F(s)=M\{f(x);s\}=\int_0^\infty x^{s-1}f(x)\mathrm{d}x. \tag{7}$$

Proof. From (7) and (1), $M\{R^{\eta,\alpha}_x[f(x)];s\}=\Delta$ (say) takes the following form

$$\Delta=\int_0^\infty x^{s-1}\left\{x^{-\eta-\alpha-1}\int_0^x t^\eta(x-t)^\alpha S^m_n\left[e\left(\frac{t}{x}\right)^\mu\left(1-\frac{t}{x}\right)^\nu\right]H\left[z_1\left(\frac{t}{x}\right)^{u_1}\left(1-\frac{t}{x}\right)^{v_1},\ldots,z_r\left(\frac{t}{x}\right)^{u_r}\left(1-\frac{t}{x}\right)^{v_r}\right]f(t)\mathrm{d}t\right\}\mathrm{d}x.$$

On changing the order of integration in the above equation (which is permissible under the conditions stated) we get

$$\Delta=\int_0^\infty t^\eta f(t)\left\{\int_t^\infty x^{s-\eta-\alpha-2}(x-t)^\alpha S^m_n\left[e\left(\frac{t}{x}\right)^\mu\left(1-\frac{t}{x}\right)^\nu\right]H\left[z_1\left(\frac{t}{x}\right)^{u_1}\left(1-\frac{t}{x}\right)^{v_1},\ldots,z_r\left(\frac{t}{x}\right)^{u_r}\left(1-\frac{t}{x}\right)^{v_r}\right]\mathrm{d}x\right\}\mathrm{d}t. \tag{8}$$

Now we express the general class of polynomials involved in the above equation in the series form and the multivariable H-function in terms of its well known Mellin-Barnes contour integral and interchange the order of summation and integration in the result thus obtained. The equation given by (8) now yields the following result after a little simplification

$$\Delta=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}e^k\int_0^\infty t^\eta f(t)\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\cdots\phi_r(\xi_r)\,\psi(\xi_1,\ldots,\xi_r)z_1^{\xi_1}\cdots z_r^{\xi_r}\mathrm{d}\xi_1\cdots\mathrm{d}\xi_r\left\{\int_t^\infty x^{s-\eta-\alpha-2-(\mu+\nu)k-(u_1+v_1)\xi_1-\cdots-(u_r+v_r)\xi_r}(x-t)^{\alpha+\nu k+v_1\xi_1+\cdots+v_r\xi_r}\mathrm{d}x\right\}t^{\mu k+u_1\xi_1+\cdots+u_r\xi_r}\mathrm{d}t. \tag{9}$$

Evaluating the x-integral occurring in (9) with the help of a known result [2, p. 201, eq. (6)] and reinterpreting the resulting multiple Mellin–Barnes contour integral so obtained in terms of the H-function of r variables, we easily arrive at the desired Theorem.

If in the left-hand side of (5) we take $n=0$ (the polynomial S^m_0 will reduce to $A_{0,0}$ which can be taken to be unity without loss of generality) and put $N=P=Q=0$, $M^{(i)}=Q^{(i)}=1$, $N^{(i)}=P^{(i)}=0$, $d^{(i)}_1=0$, $\delta^{(i)}_1=1$, $u_i=v_i=0$ and let $z_i\to 0$ $(i=2,\ldots,r)$, the

H-function of r variables occurring therein reduces to the H-function of Fox [3] and the above Theorem reduces to a theorem which is in essence same as that obtained by Saxena and Kumbhat [11, p. 3, eq. (3.3)].

Theorem 2. *If* $M\{f(x);s\}$, $M\{W_x^{\eta,\alpha}[f(x)];s\}$ *exist,* $\mathrm{Re}(s+\eta)>0$, $\mathrm{Re}(1+\alpha)>0$ *and the conditions of the existence of the operator* $W_x^{\eta,\alpha}[f(x)]$ *are satisfied, then*

$$M\{W_x^{\eta,\alpha}[f(x)];s\} = \phi_2(s)M\{f(x);s\} \tag{10}$$

where

$$\phi_2(s) = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} e^k H^{o,N+2:*}_{P+2,Q+1:*}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\right| \left.\begin{matrix} (1-s-\eta-\mu k; u_1,\ldots,u_r),(-\alpha-\nu k; v_1,\ldots,v_r),(a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P}:* \\ (-s-\eta-\alpha-(\mu+\nu)k; u_1+v_1,\ldots,u_r+v_r),(b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q}:* \end{matrix}\right] \tag{11}$$

the asterisk $(*)$ *in* (11) *indicates that the parameters at these places are the same as the parameters of the H-function of r complex variables occurring in* (4).

Proof. If we follow the lines of proof as given in Theorem 1 and make use of another well known result [2, p. 185, eq. (7)] we easily obtain the Theorem 2.

Again, if in the left-hand side of (10) we take $n=0$ and reduce the multivariable H-function occurring therein to the Fox's H-function in the manner explained earlier, the above Theorem reduces to the other similar theorem obtained by Saxena and Kumbhat [11, p. 4, eq. (3.6)].

It may be remarked that the Theorems 1 and 2 given above are also generalizations of the theorems obtained by Saxena [10, p. 289, eqs (6), (8)] and Kalla [7, pp. 270–271, eqs (34), (37)] for the operators studied by them.

On using the well known Mellin inversion theorem in (5) and (10) in succession, we arrive at the following interesting Theorems

Theorem 3.

$$\frac{1}{2}[f(t+0)+f(t-0)] = \frac{1}{(2\pi\omega)} \lim_{\tau\to\infty} \int_{c-\omega\tau}^{c+\omega\tau} \frac{t^{-s}}{\phi_1(s)} M\{R_x^{\eta,\alpha}[f(x)];s\}\,ds \tag{12}$$

where $f(t)$ *is of bounded variation at the point* $t=x$ $(x>0)$, *the conditions stated with Theorem* 1 *are satisfied and* $\phi_1(s)$ *is defined by* (6).

Theorem 4.

$$\frac{1}{2}[f(t+0)+f(t-0)] = \frac{1}{(2\pi\omega)} \lim_{\tau\to\infty} \int_{c-\omega\tau}^{c+\omega\tau} \frac{t^{-s}}{\phi_2(s)} M\{W_x^{\eta,\alpha}[f(x)];s\}\,ds \tag{13}$$

where $f(t)$ *is of bounded variation at the point* $t=x$ $(x>0)$, *the conditions stated with Theorem* 2 *are satisfied and* $\phi_2(s)$ *is as defined by* (11).

Also when $f(t)$ is continuous at $t=x$ $(x>0)$ then the left-hand sides of (12) and (13) are equal to $f(t)$.

If in (1) and (2) we take $n=0$ and put $N=P=Q=0$, $M^{(i)}=Q^{(i)}=1$, $N^{(i)}=P^{(i)}=0$, $d_1^{(i)}=0, \delta_1^{(i)}=1$, $u_i=v_i=0$ and let $z_i\to 0 (i=2,\ldots,r)$, $u_1=0, v_1=1$ and further reduce the H-function of Fox so obtained in terms of the generalized hypergeometric function in the usual way the Theorems 3 and 4 yield the inversion formulae which are in essence same as given by Goyal and Jain [4, p. 257, eqs (3.10), (3.11)].

3. The Mellin convolutions

From a well known theorem by Titchmarsh [17, p. 60, Th. 44] we know that if $f\varepsilon L(0,\infty)$, $g\varepsilon L(0,\infty)$, then $(f*g)\varepsilon L(0,\infty)$, where

$$(f*g)(x)=\int_0^\infty t^{-1}f\left(\frac{x}{t}\right)g(t)\mathrm{d}t. \tag{14}$$

Following the lines adopted by Buschman [1], we shall define a function $R^{\eta,\alpha}(x)$ as follows

$$R^{\eta,\alpha}(x)=x^{-\eta-\alpha-1}(x-1)^{\alpha}U(x-1)S_n^m\left[e\left(\frac{1}{x}\right)^{\mu}\left(\frac{x-1}{x}\right)^{\nu}\right]H\left[z_1\left(\frac{1}{x}\right)^{u_1}\left(\frac{x-1}{x}\right)^{v_1},\ldots,z_r\left(\frac{1}{x}\right)^{u_r}\left(\frac{x-1}{x}\right)^{v_r}\right] \tag{15}$$

where U denotes the well known unit step function. It can be easily verified that $R^{\eta,\alpha}(x)\varepsilon L(0,\infty)$ if

$$\mathrm{Re}(\alpha)+\sum_{i=1}^{r}v_i\min_{1\leq j\leq M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0,$$

$$\mathrm{Re}(\eta)+\sum_{i=1}^{r}u_i\min_{1\leq j\leq M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]>0.$$

We can represent the operator (1) as a convolution of the form (14). Indeed, we have

$$R_x^{\eta,\alpha}f(x)=\int_0^\infty t^{-1}\left\{\left(\frac{x}{t}\right)^{-\eta-\alpha-1}\left(\frac{x}{t}-1\right)^{\alpha}U\left(\frac{x}{t}-1\right)S_n^m\left[e\frac{1}{\left(\frac{x}{t}\right)^{\mu}}\left(\frac{\frac{x}{t}-1}{\frac{x}{t}}\right)^{\nu}\right]\right.$$
$$\left.H\left[z_1\frac{1}{\left(\frac{x}{t}\right)^{u_1}}\left(\frac{\frac{x}{t}-1}{\frac{x}{t}}\right)^{v_1},\ldots,z_r\frac{1}{\left(\frac{x}{t}\right)^{u_r}}\left(\frac{\frac{x}{t}-1}{\frac{x}{t}}\right)^{v_r}\right]\right\}f(t)\mathrm{d}t$$
$$=(R^{\eta,\alpha}*f)(x)\quad\text{(with the help of (14))}. \tag{16}$$

Again, if we define

$$W^{\eta,\alpha}(x)=x^{\eta}(1-x)^{\alpha}U(1-x)S_n^m[e(x)^{\mu}(1-x)^{\nu}]H[z_1(x)^{u_1}(1-x)^{v_1},\ldots,z_r(x)^{u_r}(1-x)^{v_r}]. \tag{17}$$

On proceeding in a manner as indicated above, we have

$$W_x^{\eta,\alpha}f(x)=(W^{\eta,\alpha}*f)(x). \tag{18}$$

Also, $W^{\eta,\alpha}(x)\varepsilon L(0,\infty)$ for

$$\mathrm{Re}(\eta)+\sum_{i=1}^{r}u_i \min_{1\leq j\leq M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0,$$

$$\mathrm{Re}(\alpha)+\sum_{i=1}^{r}v_i \min_{1\leq j\leq M^{(i)}}[\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0.$$

Results given by (16) and (18) yield the corresponding results given by Buschman [1, pp. 99–101] if we reduce our operators to the simple operators studied by him.

4. An analogue of the Parseval–Goldstein theorem for the operators defined by (1) and (2)

Theorem 5. *If*

$$\phi_1(x)=R_x^{\eta,\alpha}[f_1(x)] \tag{19}$$

and

$$\phi_2(x)=W_x^{\eta,\alpha}[f_2(x)] \tag{20}$$

then

$$\int_0^\infty f_1(x)\phi_2(x)\mathrm{d}x=\int_0^\infty f_2(x)\phi_1(x)\mathrm{d}x \tag{21}$$

provided that the various integrals involved converge absolutely.

Proof. We have from (2), $\int_0^\infty f_1(x)\phi_2(x)\mathrm{d}x$

$$=\int_0^\infty f_1(x)\left\{x^\eta\int_0^\infty t^{-\eta-\alpha-1}(t-x)^\alpha S_n^m\left[e\left(\frac{x}{t}\right)^\mu\left(1-\frac{x}{t}\right)^\nu\right]\right.$$
$$\left.H\left[z_1\left(\frac{x}{t}\right)^{u_1}\left(1-\frac{x}{t}\right)^{v_1},\ldots,z_r\left(\frac{x}{t}\right)^{u_r}\left(1-\frac{x}{t}\right)^{v_r}\right]f_2(t)\mathrm{d}t\right\}\mathrm{d}x.$$

Changing the order of integration in the right-hand side of the above equation (which is permissible under the conditions stated), it takes the following form after a little simplification

$$=\int_0^\infty f_2(t)\left\{t^{-\eta-\alpha-1}\int_0^t x^\eta(t-x)^\alpha S_n^m\left[e\left(\frac{x}{t}\right)^\mu\left(1-\frac{x}{t}\right)^\nu\right]H\left[z_1\left(\frac{x}{t}\right)^{u_1}\left(1-\frac{x}{t}\right)^{v_1},\right.\right.$$
$$\left.\left.\ldots,z_r\left(\frac{x}{t}\right)^{u_r}\left(1-\frac{x}{t}\right)^{v_r}\right]f_1(x)\mathrm{d}x\right\}\mathrm{d}t.$$

On reinterpreting the x-integral given in the above expression with the help of (1) we arrive at the required Theorem.

The above Theorem is a generalization of a theorem obtained by Kalla [7, p. 271, eq. (38)] for his operators and provides an analogue of the Parseval–Goldstein theorem for the operators studied in this paper.

5. Images of the multivariable *H*-function in the operators of our study

(i) $$R_x^{\eta,\alpha}\{x^l H[z_{r+1}x^{u_{r+1}},\ldots,z_{r+s}x^{u_{r+s}}]\}$$

$$= x^l \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} e^k H^{o,N+N_1+2\,:M',N',\ldots;M^{(r+s)},N^{(r+s)}}_{P+P_1+2,Q+Q_1+1\,:P',Q';\ldots;P^{(r+s)},Q^{(r+s)}} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \\ z_{r+1}x^{u_{r+1}} \\ \vdots \\ z_{r+s}x^{u_{r+s}} \end{matrix}\right|$$

$$(-l-\eta-\mu k; u_1,\ldots,u_{r+s}),\ \left(-\alpha-\nu k; v_1,\ldots,v_r,\frac{o,\ldots,o}{s}\right),$$

$$(-1-l-\eta-\alpha-(\mu+\nu)k\,;\, u_1+v_1,\ldots,u_r+v_r,u_{r+1},\ldots,u_{r+s}),$$

$$\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,N},\ \left(a'_j;\frac{o,\ldots,o}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(r+s)}\right)_{1,P_1},$$

$$\left(b_j;\beta'_j,\ldots,\beta_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,Q},\ \left(b'_j;\frac{o,\ldots,o}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(r+s)}\right)_{1,Q_1},$$

$$\left.\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{N+1,P} \begin{matrix} :(c'_j,\gamma'_j)_{1,P'};\ldots;(c_j^{(r+s)},\gamma_j^{(r+s)})_{1,P^{(r+s)}} \\ :(d'_j,\delta'_j)_{1,Q'};\ldots;(d_j^{(r+s)},\delta_j^{(r+s)})_{1,Q^{(r+s)}} \end{matrix}\right] \tag{22}$$

where the function occurring on the right-hand side of (22) is the H-function of $r+s$ variables and the following conditions are satisfied.

The quantities $\mu, \nu, u_1, v_1, \ldots, u_r, v_r, u_{r+1}, \ldots, u_{r+s}$ are all positive (some of them may however decrease to zero provided that the resulting image has a meaning),

$$\mathrm{Re}(\eta+l) + \sum_{i=1}^{r+s} u_i \min_{1\le j\le M^{(i)}} [\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})] + 1 > 0 \quad \text{and}$$

$$\mathrm{Re}(\alpha) + \sum_{i=1}^{r} v_i \min_{1\le j\le M^{(i)}} [\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})] + 1 > 0.$$

Also the number occurring below the line at any place on the right-hand side of (22) and throughout the paper indicates the total number of zeros covered by it. Thus $(o,\ldots,o)/r$ would mean r zeros, and so on.

Proof. We have from (1), $R_x^{\eta,\alpha}\{x^l H[z_{r+1}x^{u_{r+1}},\ldots,z_{r+s}x^{u_{r+s}}]\}$

$$= x^{-\eta-\alpha-1}\int_0^x t^\eta (x-t)^\alpha S_n^m\left[e\left(\frac{t}{x}\right)^\mu\left(1-\frac{t}{x}\right)^\nu\right] H\left[z_1\left(\frac{t}{x}\right)^{u_1}\left(1-\frac{t}{x}\right)^{v_1},\ldots,\right.$$

$$\left. z_r\left(\frac{t}{x}\right)^{u_r}\left(1-\frac{t}{x}\right)^{v_r}\right]\{t^l H[z_{r+1}t^{u_{r+1}},\ldots,z_{r+s}t^{u_{r+s}}]\}\,dt. \tag{23}$$

Now expressing the general class of polynomials involved in the right-hand side of (23) in the series form given by (3) and both the multivariable H-functions in terms of their well known Mellin–Barnes contour integrals and interchanging the order of summation and integration in the result thus obtained (which is permissible under the conditions stated) it takes the following form after a little simplification

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} e^k \frac{1}{(2\pi\omega)^{r+s}} \int_{L_1} \cdots \int_{L_{r+s}} \phi_1(\xi_1)\cdots\phi_{r+s}(\xi_{r+s})$$

$$\psi(\xi_1,\ldots,\xi_r)\psi'(\xi_{r+1},\ldots,\xi_{r+s})z_1^{\xi_1}\cdots z_{r+s}^{\xi_{r+s}}\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{r+s}$$

$$\left\{x^{-\eta-\alpha-1}\int_0^x t^{l+\eta+\mu k+u_1\xi_1+\cdots+u_{r+s}\xi_{r+s}}(x-t)^{\alpha+vk+v_1\xi_1+\cdots+v_r\xi_r}\mathrm{d}t\right\}$$

$$x^{-(\mu+v)k-(u_1+v_1)\xi_1-\cdots-(u_r+v_r)\xi_r}. \tag{24}$$

Evaluating the t-integral occurring in (24) with the help of a known result [2, p. 185, eq. (7)] and reinterpreting the resulting multiple Mellin–Barnes contour integrals so obtained in terms of the H-function of $r+s$ variables, we easily arrive at the desired image (22).

(ii) $W_x^{\eta,\alpha}\{x^l H[z_{r+s}x^{-u_{r+1}},\ldots,z_{r+s}x^{-u_{r+s}}]\}$

$$= x^l \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} e^k H^{o,N+N_1+2}_{P+P_1+2,Q+Q_1+1} \begin{matrix} :** \\ :** \end{matrix} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \\ z_{r+1}x^{-u_{r+1}} \\ \vdots \\ z_{r+s}x^{-u_{r+s}} \end{matrix}\right|$$

$$(1+l-\eta-\mu k;u_1,\ldots,u_{r+s}),\left(-\alpha-vk;v_1,\ldots,v_r,\frac{o,\ldots,o}{s}\right),$$

$$(l-\eta-\alpha-(\mu+v)k\,;u_1+v_1,\ldots,u_r+v_r,u_{r+1},\ldots,u_{r+s}),$$

$$\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,N},\left(a'_j;\frac{o,\ldots,o}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(r+s)}\right)_{1,P_1},$$

$$\left(b_j;\beta'_j,\ldots,\beta_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,Q},\left(b'_j;\frac{o,\ldots,o}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(r+s)}\right)_{1,Q_1},$$

$$\left.\left(a_j;\alpha',\ldots,\alpha^{(r)},\frac{o,\ldots,o}{s}\right)_{N+1,P}\begin{matrix} :** \\ :** \end{matrix}\right] \tag{25}$$

where the asterisks (∗∗) in (25) indicate that the parameters at these places are the same as the parameters of the H-function of $r+s$ complex variables occurring in (22) and the quantities $\mu, v, u_1, v_1, \ldots, u_r, v_r, u_{r+1}, \ldots, u_{r+s}$ are all positive (some of them may however decrease to zero provided that the resulting image has a meaning),

$$\mathrm{Re}(\alpha)+\sum_{i=1}^{r} v_i \min_{1\leq j\leq M^{(i)}} [\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]+1>0 \quad \text{and}$$

$$\mathrm{Re}(\eta-l)+\sum_{i=1}^{r+s} u_i \min_{1\leq j\leq M^{(i)}} [\mathrm{Re}(d_j^{(i)}/\delta_j^{(i)})]>0.$$

Proof. If we follow the same method as given for obtaining the result given by (22) and use another well known result [2, p. 201, eq. (6)], we obtain the image (25).

Images given by (22) and (25) are generalizations of the results obtained by Gupta (R.) [5, p. 73, eqs (2.12), (2.13)].

6. Applications

Now we shall make use of the Theorem 5 and the images obtained earlier in establishing two further theorems.

Theorem 6. *If*

$$\phi(x) = R_x^{\eta,\alpha}[f(x)] \tag{26}$$

then

$$\int_0^\infty x^l H[z_{r+1}x^{-u_{r+1}},\ldots,z_{r+s}x^{-u_{r+s}}]\phi(x)\mathrm{d}x$$

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} e^k \int_0^\infty x^l H^{o,N+N_1+2}_{P+P_1+2,Q+Q_1+1} \begin{matrix} :** \\ :** \end{matrix} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \\ z_{r+1}x^{-u_{r+1}} \\ \vdots \\ z_{r+s}x^{-u_{r+s}} \end{matrix} \right|$$

$$(1+l-\eta-\mu k; u_1,\ldots,u_{r+s}), \left(-\alpha - vk; v_1,\ldots,v_r,\frac{o,\ldots,o}{s}\right),$$

$$(l-\eta-\alpha-(\mu+v)k\,;\,u_1+v_1,\ldots,u_r+v_r,u_{r+1},\ldots,u_{r+s}),$$

$$\left(a_j;\alpha_j',\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,N}, \left(a_j';\frac{o,\ldots,o}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(r+s)}\right)_{1,P_1},$$

$$\left(b_j;\beta_j',\ldots,\beta_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,Q}, \left(b_j';\frac{o,\ldots,o}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(r+s)}\right)_{1,Q_1},$$

$$\left.\left(a_j;\alpha_j',\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{N+1,P} \begin{matrix} :** \\ :** \\ :** \end{matrix}\right] f(x)\mathrm{d}x \tag{27}$$

where the asterisks $(**)$ *in* (27) *indicate that the parameters at these places are the same as the parameters of the H-function of* $r+s$ *complex variables occurring in* (22) *and the conditions of the existence of the operator* $R_x^{\eta,\alpha}[f(x)]$ *mentioned earlier are satisfied and the integrals occurring in* (27) *are absolutely convergent.*

Proof. On substituting the results given by (25) and (26) in the analogue of the Parseval–Goldstein theorem given by (21), we easily arrive at the required Theorem after a little simplification.

Theorem 7. *If*

$$\phi(x) = W_x^{\eta,\alpha}[f(x)] \tag{28}$$

then

$$\int_0^\infty x^l H[z_{r+1}x^{u_{r+1}},\ldots,z_{r+s}x^{u_{r+s}}]\phi(x)\mathrm{d}x$$

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} e^k \int_0^\infty x^l H^{o,N+N_1+2}_{P+P_1+2,Q+Q_1+1} \begin{matrix} :** \\ :** \end{matrix} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \\ z_{r+1}x^{u_{r+1}} \\ \vdots \\ z_{r+s}x^{u_{r+s}} \end{matrix} \right|$$

$$(-l-\eta-\mu k;u_1,\ldots,u_{r+s}),\ \left(-\alpha-vk;v_1,\ldots,v_r,\frac{o,\ldots,o}{s}\right),$$

$$(-1-l-\eta-\alpha-(\mu+v)k;u_1+v_1,\ldots,u_r+v_r,u_{r+1},\ldots,u_{r+s}),$$

$$\left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,N},\ \left(a'_j;\frac{o,\ldots,o}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(r+s)}\right)_{1,P_1},$$

$$\left(b_j;\beta'_j,\ldots,\beta_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,Q},\ \left(b'_j;\frac{o,\ldots,o}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(r+s)}\right)_{1,Q_1}$$

$$\left.\left(a_j;\alpha',\ldots,\alpha^{(r)},\frac{o,\ldots,o}{s}\right)_{N+1,P}\ \begin{matrix}:**\\:**\end{matrix}\right]f(x)\mathrm{d}x \tag{29}$$

where the asterisks (**) in (29) indicate that the parameters at these places are the same as the parameters of the H-function of $r+s$ complex variables occurring in (22) and the conditions of the existence of the operator $W_x^{\eta,\alpha}[f(x)]$ are satisfied and the integrals occurring in (29) are absolutely convergent.

Proof. On substituting the results given by (22) and (28) in (21), we get the required Theorem after a little simplification.

If we take $n=0$, $N=P=Q=0$, $M^{(i)}=Q^{(i)}=1$, $N^{(i)}=P^{(i)}=0$, $d_1^{(i)}=0$, $\delta_1^{(i)}=1$, $u_i=v_i=0$ and let $z_i\to 0$ $(i=1,\ldots,r)$ in Theorems 6 and 7, we easily obtain the results which are in essence same as those obtained by Mathur [9, p. 108, Th. 1; p. 112, Th. 2].

If we take $e=\mu=1$, $v=0$, $m=1$, $A_{n,k}=\Gamma(1+\zeta+\beta n)/n!\Gamma(1+\zeta+\beta k)$ in (26), the polynomial $S_n^1[t/x]$ reduces to the Konhauser biorthogonal polynomials $Z_n^\zeta[(t/x)^{1/\beta};\beta]$ [13, p. 225, eq. (3.23); 8, p. 304, eq. (5)] and if we further let $v_i=0$ $(i=1,\ldots,r)$ therein, Theorem 6 takes the following interesting form.

COROLLARY 1

If

$$\phi(x)=x^{-\eta-\alpha-1}\int_0^x t^\eta(x-t)^\alpha Z_n^\zeta\left[\left(\frac{t}{x}\right)^{1/\beta};\beta\right]H\left[z_1\left(\frac{t}{x}\right)^{u_1},\ldots,z_r\left(\frac{t}{x}\right)^{u_r}\right]f(t)\mathrm{d}t \tag{30}$$

then

$$\int_0^\infty x^l H[z_{r+1}x^{-u_{r+1}},\ldots,z_{r+s}x^{-u_{r+s}}]\phi(x)\mathrm{d}x$$

$$=\Gamma(1+\alpha)\sum_{k=0}^n\frac{(-n)_k\Gamma(1+\zeta+\beta n)}{k!\,n!\,\Gamma(1+\zeta+\beta k)}\int_0^\infty x^l H^{o,N+N_1+1\,:**}_{P+P_1+1,Q+Q_1+1\,:**}$$

$$\left[\begin{matrix}z_1\\ \vdots\\ z_r\\ z_{r+1}x^{-u_{r+1}}\\ \vdots\\ z_{r+s}x^{-u_{r+s}}\end{matrix}\ \middle|\ \begin{matrix}(1+l-\eta-k;u_1,\ldots,u_{r+s}),\ \left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,N},\\ \\ (l-\eta-\alpha-k;u_1,\ldots,u_{r+s}),\ \left(b_j;\beta'_j,\ldots,\beta_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,Q},\end{matrix}\right.$$

$$\left.\begin{matrix} \left(a_j';\frac{o,\ldots,o}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(r+s)}\right)_{1,P_1}, & \left(a_j;\alpha_j',\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{N+1,P} & :** \\ \left(b_j';\frac{o,\ldots,o}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(r+s)}\right)_{1,Q_1} & & :** \end{matrix}\right] f(x)\mathrm{d}x \quad (31)$$

where the asterisks (**) in (31) indicate that the parameters at these places are the same as the parameters of the H-function of $r+s$ complex variables occurring in (22) and the conditions easily obtainable from Theorem 6 are satisfied.

If we take $\beta = 1$, in (30), we get the corresponding result involving the Laguerre polynomials $L_n^{(\zeta)}[t/x]$.

If we take $e=\mu=1$, $v=0$, $m=1$, $A_{n,k}=\binom{n+\zeta}{n}(\zeta+\beta+n+1)_k/(\zeta+1)_k$ in (26), the polynomial $S_n^1[t/x]$ reduces to the Jacobi polynomials [16, p. 159, eq. (1.6)] and if we further let $v_i = 0$ $(i=1,\ldots,r)$ therein, we arrive at the following result.

COROLLARY 2

If

$$\phi(x) = x^{-\eta-\alpha-1}\int_0^x t^\eta (x-t)^\alpha P_n^{(\zeta,\beta)}\left[1-\frac{2t}{x}\right] H\left[z_1\left(\frac{t}{x}\right)^{u_1},\ldots,z_r\left(\frac{t}{x}\right)^{u_r}\right] f(t)\mathrm{d}t \quad (32)$$

then

$$\int_0^\infty x^l H[z_{r+1}x^{-u_{r+1}},\ldots,z_{r+s}x^{-u_{r+s}}]\phi(x)\mathrm{d}x$$

$$=\Gamma(1+\alpha)\sum_{k=0}^n \frac{(-n)_k}{k!}\binom{n+\zeta}{n}\frac{(\zeta+\beta+n+1)_k}{(\zeta+1)_k}\int_0^\infty x^l H^{o,N+N_1+1\,:**}_{P+P_1+1,Q+Q_1+1\,:**}$$

$$\left[\begin{matrix} z_1 \\ \vdots \\ z_r \\ z_{r+1}x^{-u_{r+1}} \\ \vdots \\ z_{r+s}x^{-u_{r+s}} \end{matrix}\;\middle|\; \begin{matrix} (1+l-\eta-k;u_1,\ldots,u_{r+s}), & \left(a_j;\alpha_j',\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,N}, \\ (l-\eta-\alpha-k;u_1,\ldots,u_{r+s}), & \left(b_j;\beta_j',\ldots,\beta_j^{(r)},\frac{o,\ldots,o}{s}\right)_{1,Q}, \end{matrix}\right.$$

$$\left.\begin{matrix} \left(a_j';\frac{o,\ldots,o}{r},\alpha_j^{(r+1)},\ldots,\alpha_j^{(r+s)}\right)_{1,P_1}, & \left(a_j;\alpha_j',\ldots,\alpha_j^{(r)},\frac{o,\ldots,o}{s}\right)_{N+1,P} & :** \\ \left(b_j';\frac{o,\ldots,o}{r},\beta_j^{(r+1)},\ldots,\beta_j^{(r+s)}\right)_{1,Q_1} & & :** \end{matrix}\right] f(x)\mathrm{d}x \quad (33)$$

where the asterisks (**) in (33) indicate that the parameters at these places are the same as the parameters of the H-function of $r+s$ complex variables occurring in (22) and the conditions easily obtainable from Theorem 6 are satisfied.

Corollaries similar to those obtained above can also be obtained for Theorem 7 but we do not record them here explicitly on account of the triviality of the analysis involved. A number of other corollaries of Theorems 6 and 7 involving various simpler functions and polynomials can also be obtained but we do not record them here for lack of space.

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