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Asymptotic behaviour of certain zero-balanced hypergeometric series

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Abstract. In this paper an attempt has been made to give a very simple method of extending certain results of Ramanujan, Evans and Stanton on obtaining the asymptotic behaviour of a class of zero-balanced hypergeometric series. A more recent result of Saigo and Srivastava has also been used to obtain a Ramanujan type of result for a partial sum of a zero-balanced ${}_4F_3\,(1)$ and similar other partial series of higher order.

Keywords. Asymptotic behaviour; zero-balanced hypergeometric series; Kampé de Feriét function.

1. Introduction

A number of results exist in the literature which express the sum of the first n terms of an ordinary hypergeometric series, with unit argument, in terms of an infinite series of the type ${}_3F_2$. The earliest of such results are due to Hill and Whipple [6, 7, 8] but new interest was aroused in them by a theorem due to Ramanujan [2; p. 92 (10.4.1)]. Various generalizations of Ramanujan's result have been given from time to time by Whipple [2; p. 94 (10.4.4)], Hodgkinson [2, p. 94 (10.4.5)], Bailey [2, p. 93 (10.4.3)], Agarwal [1, p. 442 (2.5)] and others.

Ramanujan [9 Ch. XI Theorem 1] stated an asymptotic formula as $m \to \infty$ for the mth partial sums of a zero-balanced (i.e. when $d + e = a + b + c$) hypergeometric series ${}_3F_2\left[\begin{matrix} a, b, c; \\ d, e \end{matrix} \; 1\right]$. Evans and Stanton [5, p. 1017 (4)] have given a very elegant proof of this asymptotic formula and extended his method to obtain a similar result for a zero-balanced basic hypergeometric series ${}_3\Phi_2$. Evans [4, p. 556 (Theorem 22)] has also given a proof for the asymptotic formula for a zero-balanced ${}_5F_4$ series stated by Ramanujan [9; Entry 6] alongwith several other particular results. It would be interesting if one could extend Ramanujan's results to series of higher order series also.

More recently Saigo and Srivastava [10] have studied the behaviour of a zero-balanced hypergeometric series ${}_pF_{p-1}$ near the boundary point $z = 1$ of its region of convergence. They have investigated the behaviour by considering a transformation connecting a ${}_pF_{p-1}(x)$ with another ${}_{p-1}F_{p-2}(x)$ and a Kampé de Feriét type of series $F^{0:p-1;3}_{1:p-3;1}(x, 1)$. They have shown that special cases of their result yield the behaviour of zero-balanced ${}_3F_2(1)$ and ${}_4F_3(1)$ series.

Our approach in the present paper is markedly different. Our objective is to obtain a Ramanujan type of result [9; Chapter XI, Theorem I and Chapter X, Entry 6] for a partial sum of a zero-balanced ${}_4F_3\,(1)$ and similar other partial series of higher order.

In our analysis, incidently we get a direct generalization of the results of Ramanujan, Evans and Stanton [5, Theorem 3]. The present method has the advantage of giving on explicit expression for the order terms also [see (4.2), (4.3), (7.1)] which are not available as such in the results of Saigo and Srivastava.

By repeated iteration of our result one can lead to the behaviour of a zero-balanced ${}_{3r+3}F_{3r+2}\,(1)$ and ${}_{3r+5}F_{3r+4}\,(1)$ series.

In § 7 of paper the partial-sum method is applied to a transformation between two ${}_pF_{p-1}$'s and a Kampê de Feriêt function given by Saigo and Srivastava [10] to derive the behaviour of a zero-balanced ${}_pF_{p-1}\,(1)$.

2. Definitions and Notations

Let $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1),\ n>0\ (\alpha)_0 = 1.$

An ordinary hypergeometric series is then defined as

$$ {}_{p+1}F_p\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_{p+1};\\ \rho_1,\rho_2,\ldots,\rho_p\end{matrix}\ Z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_{p+1})_n Z^n}{(\rho_1)_n(\rho_2)_n\cdots(\rho_p)_n n!}. \tag{2.1} $$

The series converges when $|Z|<1$ and also when $Z=1$, provided that $\mathrm{Re}[\Sigma(\rho_p)-\Sigma(\alpha_{p+1})]>0$. The series (2.1) shall be called a zero-balanced series if $\Sigma(\rho_p)=\Sigma(\alpha_{p+1})$.

By

$$ {}_{p+1}F_p\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_{p+1};\\ \rho_1,\rho_2,\ldots,\rho_p\end{matrix}\ Z\right]_m $$

we shall denote the partial sum

$$ \sum_{n=0}^{m} \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_{p+1})_n Z^n}{(\rho_1)_n(\rho_2)_n\cdots(\rho_p)_n n!}. $$

A Kampê de Feriêt series is defined as

$$ F^{0:p-1;2}_{1:p-3;0}\left[\begin{matrix}\text{———}:(\alpha_{p-1});\beta_{p-2}-\alpha_p,\beta_{p-1}-\alpha_p;\\ \beta_{p-2}+\beta_{p-1}-\alpha_p:(\beta_{p-3});\text{———};\end{matrix}\ x,y\right] $$

$$ = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{[(\alpha_{p-1})]_n(\beta_{p-2}-\alpha_p)_m(\beta_{p-1}-\alpha_p)_m x^n y^m}{(\beta_{p-2}+\beta_{p-1}-\alpha_p)_{n+m}[(\beta_{p-3})]_n n!\,m!}\ (p \geqq 3) \tag{2.2} $$

where, for convergence, $\max\{|x|,|y|\}<1$.

We shall use the following two results of Ramanujan [9; Chapter XI Theorem I and Chapter X Entry 6; See also [3] Theorem I, p. 70 and Entry 6, p. 12] stated without proof. The asymptotic formula for a zero-balanced hypergeometric series ${}_3F_2$ as:

If $a+b+c=d+e$ and $\mathrm{Ri}(c)>0$, then for

$$ L = \sum_{k=0}^{\infty}\left\{\frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c+k)}{\Gamma(d+k)\Gamma(e+k)\Gamma(l+k)} - \frac{1}{k+1}\right\} \tag{2.3} $$

$$ = -2\gamma - \Psi(a) - \Psi(b) + \sum_{k=1}^{\infty} \frac{(d-c)_k(e-c)_k}{(a)_k(b)_k k}, \tag{2.4} $$

with γ, the Euler's constant and $\Psi(x)=\dfrac{\Gamma'(x)}{\Gamma(x)}$ we have, as $m\to\infty$

$$ \sum_{k=0}^{m-1} \frac{(a)_k(b)_k(c)_k}{(d)_k(e)_k k!} \sim \frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)\Gamma(c)}\{\log m + L + \gamma\} + O\left(\frac{1}{m}\right), \tag{2.5} $$

where the implied constant depends on a,b,c,d,e but not on m.

The asymptotic formula for a zero-balanced hypergeometric series ${}_5F_4$ as:
If $a+b+c\notin\{0,-1,-2,\dots\}$, then as the integer m tends to ∞,

$$\frac{\Gamma(a+b+c)\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(b+c)\Gamma(a+c)\Gamma(a+b)}\,{}_5F_4\left[\begin{matrix}\frac{a+b+c+1}{2}, & a+b+c-1, & a,b,c; \\ \frac{a+b+c-1}{2}, & b+c, & a+c, & a+b\end{matrix}\right]_m$$

$$\sim 2\log m-\gamma-\Psi(a)-\Psi(b)-\Psi(c)+O\left(\frac{\log m}{m}\right). \tag{2.6}$$

3. A general transformation theorem for partial hypergeometric series

We begin this section by proving the following transformation Theorem:
We have for arbitrary complex numbers $a,b,c,(a_s)$ and (p_s), that

$${}_{s+3}F_{s+2}\left[\begin{matrix}a,b,c, & a_1,a_2,\dots,a_s; & \\ 1+a-b, & 1+a-c, & p_1,p_2,\dots,p_s\end{matrix}\; z\right]_m$$

$$=\sum_{r=0}^{m}\frac{(-4)^r(1+a+b-c)_r\left(\frac{a}{2}\right)_r\left(\frac{a+1}{2}\right)_r(a_1)_r(a_2)_r\cdots(a_s)_r z^r}{r!(1+a-b)_r(1+a-c)_r(p_1)_r(p_2)_r\cdots(p_s)_r}$$

$$\times\,{}_{s+1}F_s\left[\begin{matrix}a+2r, & a_1+r, & a_2+r,\dots,a_s+r; \\ p_1+r, & p_2+r,\dots,p_s+r\end{matrix}\; z\right]_{(m-r)}. \tag{3.1}$$

Proof of the Theorem. Saalschütz's theorem is given by

$${}_3F_2\left[\begin{matrix}1+a-b-c, & a+n, & -n; \\ 1+a-b, & 1+a-c\end{matrix}\; 1\right]=\frac{(b)_n(c)_n}{(1+a-b)_n(1+a-c)_n}. \tag{3.2}$$

Now

$${}_{s+3}F_{s+2}\left[\begin{matrix}a,b,c, & a_1,a_2,\dots,a_s; & \\ 1+a-b, & 1+a-c, & p_1,p_2,\dots,p_s\end{matrix}\; z\right]_m$$

$$=\sum_{n=0}^{m}\frac{(a)_n(a_1)_n(a_2)_n\cdots(a_s)_n z^n(b)_n(c)_n}{(p_1)_n(p_2)_n\cdots(p_s)_n n!(1+a-b)_n(1+a-c)_n}$$

$$=\sum_{n=0}^{m}\frac{(a)_n(a_1)_n(a_2)_n\cdots(a_s)_n z^n}{(p_1)_n(p_2)_n\cdots(p_s)_n n!}\sum_{r=0}^{n}\frac{(1+a-b-c)_r(a+n)_r(-n)_r}{(1+a-b)_r(1+a-c)_r r!}$$

by (3.2).

Changing the order of summation, we get

$$=\sum_{r=0}^{m}\sum_{n=r}^{m}\frac{(a)_{n+r}(a_1)_n(a_2)_n\cdots(a_s)_n(1+a-b-c)_r(-n)_r z^n}{(p_1)_n(p_2)_n\cdots(p_s)_n(1+a-b)_r(1+a-c)_r n!\,r!}.$$

Putting $n=r+t$, we get

$$=\sum_{r=0}^{m}\sum_{t=0}^{m-r}\frac{(a)_{2r+t}(a_1)_{r+t}(a_2)_{r+t}\cdots(a_s)_{r+t}(1+a-b-c)_r(-r-t)_r z^{r+t}}{(p_1)_{r+t}(p_2)_{r+t}\cdots(p_s)_{r+t}(1+a-b)_r(1+a-c)_r\, r+t!\,r!}. \tag{3.3}$$

Writing the t-series as a F-series on the r.h.s., we get (3.1), after some simplification.

It may be remarked that we could obtain (3.1) directly [for $z=1$] from Bailey's known transformation Theorem [2, p. 24 (4.3.1)].

4. The asymptotic behaviour of certain zero-balanced partial hypergeometric series

We begin this section by proving the following two asymptotic formulae for certain zero-balanced hypergeometric series:

(A) For k, a fixed positive integer $< m$ and if $a_1+a_2+a_3+k=p_2+p_3$, $Rl(a_3+k)>0$, then as $m\to\infty$

$$
{}_6F_5\begin{bmatrix} b+c-k-1, & b, & c, & a_1, & a_2, & a_3; \\ b-k, & c-k, & b+c-1, & p_2, & p_3 & \end{bmatrix}_m
$$
$$
\sim\frac{(-1)^k(2-b-c)_k\Gamma(p_2)\Gamma(p_3)}{(1-b)_k(1-c)_k\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\{\log(m-k+1)+L'+\gamma\}+O\left(\frac{1}{m-k+1}\right), \tag{4.1}
$$

where γ is Euler's constant and

$$
L'=-2\gamma-\Psi(a_1+k)-\Psi(a_2+k)+\sum_{s=1}^{\infty}\frac{(p_2-a_3)_s(p_3-a_3)_s}{(a_1+k)_s(a_2+k)_s s}
$$
$$
+\frac{(-1)^k(1-b)_k(1-c)_k\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}{(2-b-c)_k\Gamma(p_2)\Gamma(p_3)}
$$
$$
\times\sum_{r=0}^{k-1}\frac{(-4)^r(-k)_r\left(\frac{b+c-k-1}{2}\right)_r\left(\frac{b+c-k}{2}\right)_r(a_1)_r(a_2)_r(a_3)_r}{r!(b-k)_r(c-k)_r(b+c-1)_r(p_2)_r(p_3)_r}
$$
$$
\times{}_4F_3\begin{bmatrix} b+c-k-1+2r, & a_1+r, & a_2+r, & a_3+r; \\ b+c-1+r, & p_2+r, & p_3+r & \end{bmatrix}. \tag{4.2}
$$

(B) If $a+b+c+3k\notin\{0,-1,-2,\ldots\}$, then as the integer m tends to ∞, we have

$$
{}_8F_7\begin{bmatrix} b_1+c_1-k-1, \; b_1, \; c_1, \; \frac{a+b+c+k+1}{2}, \; a+b+c+2k-1, \; a,b,c; \\ b_1-k, \; c_1-k, \; b_1+c_1-1, \; \frac{a+b+c+k-1}{2}, \; b+c+k, \; a+c+k, \; a+b+k \end{bmatrix}_m
$$
$$
\sim\frac{(-1)^{k+1}(a+b+c-1)_{3k+1}(2-b_1-c_1)_k\Gamma(b+c+k)\Gamma(a+c+k)\Gamma(a+b+k)}{(1-a-b-c-k)\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(a+b+c+3k)(1-b_1)_k(1-c_1)_k(a+b+c-1)_{2k}}
$$
$$
\times\{2\log(m-k)-\gamma-\Psi(a+k)-\Psi(b+k)-\Psi(c+k)\}
$$
$$
+\sum_{r=0}^{k-1}\frac{(-4)^r(-k)_r\left(\frac{b_1+c_1-k-1}{2}\right)_r\left(\frac{b_1+c_1-k}{2}\right)_r\left(\frac{a+b+c+k+1}{2}\right)_r(a+b+c+2k-1)_r}{r!(b_1-k)_r(c_1-k)_r(b_1+c_1-1)_r\left(\frac{a+b+c+k-1}{2}\right)_r(b+c+k)_r(a+c+k)_r}
$$
$$
\times\frac{(a)_r(b)_r(c)_r}{(a+b+k)_r}
$$

$$
{}_6F_5\left[\begin{matrix} b_1+c_1-k-1+2r, \dfrac{a+b+c+k+1}{2}+r, a+b+c-1+2k+r, a+r, b+r, c+r; \\ b_1+c_1-1+r, \dfrac{a+b+c+k-1}{2}+r, b+c+k+r, a+c+k+r, a+b+k+r \end{matrix}\right]
$$

$$
+O\left(\frac{\log(m-k)}{(m-k)}\right), \qquad (4.3)
$$

where γ is Euler's constant, k is a fixed positive integer $< m$.

Proof of (4.1). Taking $s=3$ in (3.1) and putting $a=b+c-k-1, z=1, p_1=b+c-1$, where k is a fixed positive integer $< m$, we get

$$
{}_6F_5\left[\begin{matrix} b+c-k-1, & b, & c, & a_1, a_2, a_3; \\ b-k, & c-k, & b+c-1, & p_2, & p_3 \end{matrix}\right]_m
$$

$$
=\sum_{r=0}^{k-1}\frac{(-4)^r(-k)_r\left(\dfrac{b+c-k-1}{2}\right)_r\left(\dfrac{b+c-k}{2}\right)_r(a_1)_r(a_2)_r(a_3)_r}{r!\,(b-k)_r(c-k)_r(b+c-1)_r(p_2)_r(p_3)_r}
$$

$$
\times\,{}_4F_3\left[\begin{matrix} b+c-k-1+2r, & a_1+r, & a_2+r, & a_3+r; \\ b+c-1+r, & p_2+r, & p_3+r \end{matrix}\right]_{(m-r)}
$$

$$
+\frac{(-4)^k(-k)_k\left(\dfrac{b+c-k-1}{2}\right)_k\left(\dfrac{b+c-k}{2}\right)_k(a_1)_k(a_2)_k(a_3)_k}{k!\,(b-k)_k(c-k)_k(b+c-1)_k(p_2)_k(p_3)_k}
$$

$$
\times\,{}_3F_2\left[\begin{matrix} a_1+k, & a_2+k, & a_3+k; \\ p_2+k, & p_3+k \end{matrix}\right]_{(m-k)}.
$$

Let us now suppose that $a_1+a_2+a_3+k=p_2+p_3$, with $\mathrm{Rl}(a_3+k)>0$. Now apply Ramanujan's asymptotic formula (2.5) on r.h.s. of the above equation, to get for $m\to\infty$,

$$
{}_6F_5\left[\begin{matrix} b+c-k-1, & b, & c, & a_1, & a_2, & a_3; \\ b-k, & c-k, & b+c-1, & p_2, & p_3 \end{matrix}\right]_m
$$

$$
\sim\sum_{r=0}^{k-1}\frac{(-4)^r(-k)_r\left(\dfrac{b+c-k-1}{2}\right)_r\left(\dfrac{b+c-k}{2}\right)_r(a_1)_r(a_2)_r(a_3)_r}{r!\,(b-k)_r(c-k)_r(b+c-1)_r(p_2)_r(p_3)_r}
$$

$$
\times\,{}_4F_3\left[\begin{matrix} b+c-k-1+2r, & a_1+r, & a_2+r, & a_3+r; \\ b+c-1+r, & p_2+r, & p_3+r \end{matrix}\right]_{(m-r)}
$$

$$
+\frac{(+4)^k\left(\dfrac{b+c-k-1}{2}\right)_k\left(\dfrac{b+c-k}{2}\right)_k(a_1)_k(a_2)_k(a_3)_k}{(b-k)_k(c-k)_k(b+c-1)_k(p_2)_k(p_3)_k}
$$

$$
\times\left[\frac{\Gamma(p_2+k)\Gamma(p_3+k)}{\Gamma(a_1+k)\Gamma(a_2+k)\Gamma(a_3+k)}\{\log(m-k+1)+L'+\gamma\}\right.
$$

$$
\left.+O\left(\frac{1}{m-k+1}\right)\right].
$$

Simplifying we get (4.1).

Again applying the asymptotic formula (4.1) on r.h.s. in (3.1), for $s=5$, $z=1$ we get the asymptotic behaviour of a zero-balanced ${}_9F_8$ series. Similarly, by repeating this process we can get asymptotic behaviour of a zero-balanced ${}_{3r+3}F_{3r+2}$ series.

Proof of (4.3). Proceeding exactly as in the case of (4.1), we get (4.3).

Again applying the asymptotic formula (4.3) on r.h.s. in (3.1), we can get the asymptotic behaviour of a zero-balanced ${}_{11}F_{10}$ series. Similarly, by repeating this process, we can get the asymptotic behaviour of a zero-balanced ${}_{3r+5}F_{3r+4}$ series.

5. Special cases

(i) Putting $k=1$, $a_2=b-1$ and $c\to\infty$ in (4.1), we get back Ramanujan's result (2.5) for a zero-balanced ${}_3F_2$.

(ii) Putting $a_2=b-k$ in (4.1), we get

For k a fixed positive integer $<m$ and if $a_1+b+a_3=p_2+p_3$, $Rl(a_3+k)>0$, then as $m\to\infty$

$$
\begin{aligned}
&{}_5F_4\begin{bmatrix} b+c-k-1, & b, & c, & a_1, & a_3; \\ c-k, & b+c-1, & p_2, & p_3 & \end{bmatrix}_m \\
&\sim\frac{(-1)^k(2-b-c)_k\Gamma(p_2)\Gamma(p_3)}{(1-b)_k(1-c)_k\Gamma(a_1)\Gamma(b-k)\Gamma(a_3)}\{\log(m-k+1)+L''+\gamma\} \\
&+O\left(\frac{1}{m-k+1}\right),
\end{aligned}
\tag{5.1}
$$

where γ is Euler's constant and

$$
\begin{aligned}
L''=&-2\gamma-\Psi(a_1+k)-\Psi(b)+\sum_{s=1}^{\infty}\frac{(p_2-a_3)_s(p_3-a_3)_s}{(a_1+k)_s(b)_s s} \\
&+\frac{(-1)^k(1-b)_k(1-c)_k\Gamma(a_1)\Gamma(b-k)\Gamma(a_3)}{(2-b-c)_k\Gamma(p_2)\Gamma(p_3)} \\
&\times\sum_{r=0}^{k-1}\frac{(-4)^r(-k)_r\left(\dfrac{b+c-k-1}{2}\right)_r\left(\dfrac{b+c-k}{2}\right)_r(a_1)_r(a_3)_r}{r!(c-k)_r(b+c-1)_r(p_2)_r(p_3)_r} \\
&\times{}_4F_3\begin{bmatrix} b+c-k-1+2r, & a_1+r, & b-k+r, & a_3+r; \\ b+c-1+r, & p_2+r, & p_3+r & \end{bmatrix}.
\end{aligned}
\tag{5.2}
$$

(iii) Putting $a_1=c-k$ in (5.1), we get the asymptotic behaviour of a zero-balanced series ${}_4F_3$.

(iv) Putting $c=b_1-k$ in (4.3), we get

If $a+b+b_1+2k\notin\{0,-1,-2,\ldots\}$, then for the integer m tending to ∞

$$
{}_7F_6\begin{bmatrix} b_1+c_1-k-1, & b_1, & c_1, & \tfrac{1}{2}(a+b+b_1+1), & a+b+b_1+k-1, & a, & b; \\ c_1-k, & b_1+c_1-1, & \tfrac{1}{2}(a+b+b_1-1), & b+b_1, & a+b_1, & a+b+k & \end{bmatrix}_m
$$

$$\sim \frac{(-1)^k(2-b_1-c_1)_k\Gamma(b+b_1)\Gamma(a+b_1)\Gamma(a+b+k)}{\Gamma(a)\Gamma(b)\Gamma(b_1-k)\Gamma(a+b+b_1)(1-b_1)_k(1-c_1)_k(a+b+b_1-1)_k}$$

$$\times \{2\log(m-k)-\gamma-\Psi(a+k)-\Psi(b+k)-\Psi(b_1)\}$$

$$+\sum_{r=0}^{k-1}\frac{(-4)^r(-k)_r\left(\frac{b_1+c_1-k-1}{2}\right)_r\left(\frac{b_1+c_1-k}{2}\right)_r\left(\frac{a+b+b_1+1}{2}\right)_r}{r!(c_1-k)_r(b_1+c_1-1)_r\left(\frac{a+b+b_1-1}{2}\right)_r}$$

$$\times\frac{(a+b+b_1+k-1)_r(a)_r(b)_r}{(b+b_1)_r(a+b_1)_r(a+b+k)_r}\cdot$$

$$ {}_6F_5\left[\begin{matrix} b_1+c_1-k-1+2r,\ \frac{a+b+b_1+1}{2}+r,\ a+b+b_1-1+k+r,\ a+r,\\ b+r,\ \ b_1-k+r;\\ b_1+c_1-1+r,\ \frac{a+b+b_1-1}{2}+r,\ b+b_1+r,\ a+b_1+r,\ a+b+k+r\end{matrix}\right]$$

$$+O\left(\frac{\log(m-k)}{(m-k)}\right), \tag{5.3}$$

where γ is Euler's constant, k is fixed positive integer $< m$.

(v) Putting $b = c_1 - k$ in (5.3), we get the asymptotic behaviour of a zero-balanced series ${}_6F_5$.

(vi) Putting $b = c_1 - k$, $a = b_1 + c_1 - 1$ in (5.3), we get asymptotic behaviour of a zero-balanced series ${}_5F_4$ different from Evan's ${}_5F_4$ function.

6. Certain special cases of (3.1)

In this section we shall discuss certain particular cases of (3.1) to obtain transformations for special types of partial sums of hypergeometric series.

(i) Putting $c = -m$ in Vandermonde's theorem [11, p. 28 (1.7.7)], we get

$${}_1F_0\left[\begin{matrix}a;\\ -\end{matrix}\right]_m = \frac{(1+a)_m}{m!}. \tag{6.1}$$

Taking $s=0$, $z=1$ in (3.1) and then summing r.h.s. of (3.1) by (6.1), we get

$${}_3F_2\left[\begin{matrix}a,\ b,\ c;\\ 1+a-b,\ 1+a-c\end{matrix}\right]_m = \frac{(1+a)_m}{m!}$$

$${}_4F_3\left[\begin{matrix}1+a-b-c,\ \frac{a}{2},\ 1+a+m,\ -m;\\ 1+a-b,\ 1+a-c,\ 1+\frac{a}{2}\end{matrix}\right]_m \tag{6.2}$$

Again applying (6.2) on r.h.s. of (3.1), we get the following transformation for a new partial sum of a well-poised ${}_5F_4$ series:

$${}_5F_4\left[\begin{matrix}a,\ b,\ c,\ a_1,\ a_2;\\ 1+a-b,\ \ 1+a-c,\ \ 1+a-a_1,\ \ 1+a-a_2\end{matrix}\right]_m$$

$$= \frac{(1+a)_m}{m!} \sum_{r=0}^{m} \frac{(1+a-b-c)_r (a/2)_r (a_1)_r (a_2)_r (1+a+m)_r (-m)_r}{r!(1+a-b)_r (1+a-c)_r (1+a/2)_r (1+a-a_1)_r (1+a-a_2)_r}$$

$$\times {}_4F_3 \left[\begin{matrix} 1+a-a_1-a_2, & a/2+r, & 1+a+m+r, & -m+r; \\ 1+a-a_1+r, & 1+a-a_2+r, & 1+a/2+r & \end{matrix} \right]_{(m-r)}. \tag{6.3}$$

Similarly, by repeating this process we can get transformations for partial sums of series of higher orders.

(ii) Putting $1+a-c=-m$ in Whipple's transformation between a terminating well-poised ${}_7F_6$ and a terminating Saalschützian ${}_4F_3$ [see 2(4.3.4)], we get

$${}_5F_4 \left[\begin{matrix} a, & 1+\frac{a}{2}, & b, & d, & e; \\ \frac{a}{2}, & 1+a-b, & 1+a-d, & 1+a-e & \end{matrix} \right]_m$$

$$= \frac{(1+a)_m (1+a-d-e)_m}{(1+a-d)_m (1+a-e)_m} \; {}_3F_2 \left[\begin{matrix} -b-m, & d, e; \\ 1+a-b, & d+e-a-m \end{matrix} \right]_m. \tag{6.4}$$

Taking $s=4$ and putting $a_1 = 1+(a/2)$, $a_2 = d$, $a_3 = e$, $a_4 = f$, $z=1$, $p_1 = a/2$, $p_2 = 1+a-d$, $p_3 = 1+a-e$, $p_4 = 1+a-f$ in (3.1) and then transforming r.h.s. of (3.1) by (6.4), we get

$${}_7F_6 \left[\begin{matrix} a, & 1+\frac{a}{2}, & b, & c, & d, & e, & f; \\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f & \end{matrix} \right]_m$$

$$= \frac{(1+a)_m (1+a-e-f)_m}{(1+a-e)_m (1+a-f)_m}$$

$$\times \sum_{r=0}^{m} \frac{(1+a-b-c)_r (d)_r (e)_r (f)_r (1+a+m)_r}{r!(1+a-b)_r (1+a-c)_r (1+a-d)_r (e+f-a-m)_r}$$

$$\times {}_3F_2 \left[\begin{matrix} -d-m, & e+r, & f+r; \\ 1+a-d+r, & e+f-a-m+r & \end{matrix} \right]_{(m-r)} \tag{6.5}$$

Applying (6.5), again on r.h.s. in (3.1), we get the following transformation for partial sum of a ${}_9F_8$ series

$${}_9F_8 \left[\begin{matrix} a, & 1+\frac{a}{2}, & b, & c, & a_2, & a_3, & a_4, & a_5, & a_6; \\ 1+a-b, & 1+a-c, & \frac{a}{2}, & 1+a-a_2, & 1+a-a_3, & 1+a-a_4, & 1+a-a_5, & 1+a-a_6 & \end{matrix} \right]_m$$

$$= \frac{(1+a)_m (1+a-a_5-a_6)_m}{(1+a-a_5)_m (1+a-a_6)_m}$$

$$\times \sum_{r=0}^{m} \frac{(1+a-b-c)_r (a_2)_r (a_3)_r (a_4)_r (a_5)_r (a_6)_r}{r!(1+a-b)_r (1+a-c)_r (1+a-a_2)_r (1+a-a_3)_r (1+a-a_4)_r} \times \frac{(1+a+m)_r}{(a_5+a_6-a-m)_r}$$

$$\times \sum_{k=0}^{m-r} \frac{(1+a-a_2-a_3)_k (a_4+r)_k (a_5+r)_k (a_6+r)_k (1+a+m+r)_k}{k!(1+a-a_2+r)_k (1+a-a_3+r)_k (1+a-a_4+r)_k (a_5+a_6-a-m+r)_k}$$

$$\times {}_3F_2\left[\begin{matrix} -a_4-m, & a_5+r+k, & a_6+r+k; \\ 1+a-a_4+r+k, & a_5+a_6-a-m+r+k \end{matrix}\right]_{(m-r-k)} \tag{6.6}$$

Similarly, by repeating this process we get transformations for partial sums of series of higher orders.

(iii) Putting $1+a-c=-m$, in Whipple's transformation [2; (4.5.1)] we get a transformation for the partial sum of a nearly-poised, ${}_3F_2$ in terms of a partial Saalschützian ${}_4F_3$, namely

$${}_3F_2\left[\begin{matrix} a, & b, & 1+a+m; \\ 1+a-b, & w & \end{matrix}\right]_m$$

$$= \frac{(w-a)_m}{(w)_m}\, {}_4F_3\left[\begin{matrix} 1+a-w, & \frac{a}{2}, & \frac{1+a}{2}, & -b-m; \\ 1+a-b, & \frac{1+a-w-m}{2}, & 1+\frac{a-w-m}{2} & \end{matrix}\right]_m. \tag{6.7}$$

Taking $s=2$ and putting $a_1=d, a_2=1+a+m, p_1=1+a-d, p_2=w, z=1$, in (3.1) and then transforming the r.h.s. of (3.1) by (6.7), we get a transformation for a partial nearly-poised ${}_5F_4$ series

$${}_5F_4\left[\begin{matrix} a,b,c,d, & 1+a+m; & & \\ 1+a-b, & 1+a-c, & 1+a-d, & w \end{matrix}\right]_m = \frac{(w-a)_m}{(w_m)}$$

$$\times \sum_{r=0}^{m} \frac{(1+a-b-c)_r \left(\frac{a}{2}\right)_r \left(\frac{a+1}{2}\right)_r (d)_r (1+a+m)_r (1+a-w)_r}{r!(1+a-b)_r (1+a-c)_r (1+a-d)_r \left(\frac{1+a-w-m}{2}\right)_r \left(\frac{2+a-w-m}{2}\right)_r}$$

$$\times {}_4F_3\left[\begin{matrix} 1+a-w+r, & \frac{a}{2}+r, & \frac{1+a}{2}+r, & -d-m; \\ 1+a-d+r, & \frac{1+a-w-m}{2}+r, & 1+\frac{a-w-m}{2}+r & \end{matrix}\right]_{(m-r)} \tag{6.8}$$

Applying (6.8) again on r.h.s. in (3.1) we get the following transformation for the partial sum of ${}_7F_6$ series

$${}_7F_6\left[\begin{matrix} a,b,c, & a_1,a_2,a_3, & 1+a+m; & & & \\ 1+a-b, & 1+a-c, & 1+a-a_1, & 1+a-a_2, & 1+a-a_3, & p_4 \end{matrix}\right]_m = \frac{(p_4-a)_m}{(p_4)_m}$$

$$\times \sum_{r=0}^{m} \frac{4^r (1+a-b-c)_r \left(\frac{a}{2}\right)_r \left(\frac{a+1}{2}\right)_r (a_1)_r (a_2)_r (a_3)_r (1+a+m)_r (1-p_4+a)_r}{r!(1+a-b)_r (1+a-c)_r (1+a-a_1)_r (1+a-a_2)_r (1+a-a_3)_r (1-p_4+a-m)_{2r}}$$

$$\times \sum_{k=0}^{m-r} \frac{(1+a-a_1-a_2)_k\left(\frac{a}{2}+r\right)_k\left(\frac{a+2r+1}{2}\right)_k(a_3+r)_k(1+a+m+r)_k(1+a+r-p_4)_k}{r!(1+a-a_1+r)_k(1+a-a_2+r)_k(1+a-a_3+r)_k \times\left(\frac{1+a-p_4-m+2r}{2}\right)_k\left(\frac{2+a-p_4-m+2r}{2}\right)_k}$$

$${}_4F_3\left[\begin{matrix}1+a-p_4+r+k, & \frac{a}{2}+r+k, & \frac{1+a}{2}+r+k, & -a_3-m; \\ 1+a-a_3+r+k, & \frac{1+a-p_4-m+2r}{2}+k, & 1+\frac{a-p_4-m+2r}{2}+k & \end{matrix}\right]_{(m-r-k)} \tag{6.9}$$

Similarly by repeating this process we get transformations for partial sums of series of higher orders.

(iv) Putting $1+a-c=-m$ in a transformation due to Bailey giving a nearly-poised ${}_5F_4$ in terms of Saalschützian ${}_5F_4$ [2, (4.5.2)], we get

$${}_4F_3\left[\begin{matrix}a, & 1+\frac{a}{2}, & b, & 1+a+m; \\ \frac{a}{2}, & 1+a-b, & w & \end{matrix}\right]_m = \frac{(w-a-1-m)(w-a)_{m-1}}{(w)_m}$$

$$\times {}_4F_3\left[\begin{matrix}1+\frac{a}{2}, & \frac{1}{2}+\frac{a}{2}, & -b-m, & 1+a-w; \\ \frac{3+a-w-m}{2}, & 1+\frac{a-w-m}{2}, & 1+a-b & \end{matrix}\right]_m \tag{6.10}$$

Taking $s=3$ and putting $a_1=1+(a/2)$, $a_2=d$, $a_3=1+a+m$, $z=1$, $p_1=(a/2)$, $p_2=1+a-d$, $p_3=w$ in (3.1) and then transforming r.h.s. of (3.1) by (6.10), we get

$${}_6F_5\left[\begin{matrix}a, & 1+\frac{a}{2}, & b, & c, & d, & 1+a+m; \\ \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & w & \end{matrix}\right]_m$$

$$= \frac{(w-a-1-m)(w-a)_{m-1}}{(w)_m}$$

$$\sum_{r=0}^{m} \frac{(1+a-b-c)_r\left(\frac{a+1}{2}\right)_r\left(1+\frac{a}{2}\right)_r(d)_r(1+a+m)_r(1+a-w)_r}{r!(1+a-b)_r(1+a-c)_r(1+a-d)_r\left(\frac{2+a-w-m}{2}\right)_r\left(\frac{3+a-w-m}{2}\right)_r}$$

$$\times {}_4F_3\left[\begin{matrix}1+\frac{a}{2}+r, & \frac{1}{2}+\frac{a}{2}+r, & -d-m, & 1+a-w+r; \\ \frac{3+a-w-m}{2}+r, & 1+\frac{a-w-m}{2}+r, & 1+a-d+r & \end{matrix}\right]_{(m-r)} \tag{6.11}$$

Applying (6.11) again on r.h.s. in (3.1), we get the following transformation for a partial sum of ${}_8F_7$ series

$$
{}_8F_7\left[\begin{matrix} a,\ b,\ c,\ 1+\frac{a}{2},\ a_2,\ a_3,\ a_4,\ 1+a+m; \\ 1+a-b,\ 1+a-c,\ \frac{a}{2},\ 1+a-a_2,\ 1+a-a_3,\ 1+a-a_4,\ p_5 \end{matrix}\right]_m
$$

$$
= \frac{(p_5-a-m-1)(p_5-a)_{m-1}}{(p_5)_m}
$$

$$
\times \sum_{r=0}^{m} \frac{4^r(1+a-b-c)_r\left(\frac{a+1}{2}\right)_r\left(1+\frac{a}{2}\right)_r(a_2)_r(a_3)_r(a_4)_r(1+a+m)_r(1-p_5+a)_r}{r!(1+a-b)_r(1+a-c)_r(1+a-a_2)_r(1+a-a_3)_r(1+a-a_4)_r(2+a-p_5-m)_{2r}}
$$

$$
\times \sum_{k=0}^{m-r} \frac{(1+a-a_2-a_3)_r\left(\frac{1+a+2r}{2}\right)_k\left(1+\frac{a+2r}{2}\right)_k(a_4+r)_k(1+a+m+r)_k}{k!(1+a-a_2+r)_k(1+a-a_3+r)_k(1+a-a_4+r)_k\left(\frac{2+a+2r-p_5-m}{2}\right)_k}
$$

$$
\times \frac{(1+a-p_5+r)_k}{\left(\frac{3+a-p_5-m+2r}{2}\right)_k}
$$

$$
\times\, {}_4F_3\left[\begin{matrix} 1+\frac{a}{2}+r+k,\ \frac{1}{2}+\frac{a}{2}+r+k,\ -a_4-m,\ 1+a-p_5+r+k; \\ \frac{3+a+2r-p_5-m}{2}+k,\ \frac{2+a+2r-p_5-m}{2}+k,\ 1+a-a_4+r+k \end{matrix}\right]_{(m-r-k)}
$$

Similarly, by repeating this process we get transformations for partial sums of series of higher orders.

7. The asymptotic behaviour of zero-balanced ${}_4F_3$ [1] and ${}_6F_5$ [1]

We begin this section by proving the following two asymptotic formulae for certain zero-balanced hypergeometric series:

(A) If $\alpha_1+\alpha_2+\alpha_3+\alpha_5=\beta_1+\beta_3+\beta_4$, $\mathrm{Rl}(\alpha_3)>0$, $\mathrm{Rl}(\alpha_5)>0$ then as $m\to\infty$

$$
{}_4F_3\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_5; \\ \beta_1,\beta_3,\beta_4\end{matrix}\ 1\right]_{m-1} \sim \frac{\Gamma(\beta_1)\Gamma(\beta_3)\Gamma(\beta_4)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_5)} \{\log m + L''' + \gamma\} + O\left(\frac{1}{m}\right), \tag{7.1}
$$

where γ is Euler's constant and

$$
L''' = -2\gamma - \Psi(\alpha_1) - \Psi(\alpha_2) + \sum_{k=1}^{\infty} \frac{(\beta_1-\alpha_3)_k(\beta_3+\beta_4-\alpha_3-\alpha_5)_k}{(\alpha_1)_k(\alpha_2)_k\, k}
$$

$$
+ \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)(\beta_3-\alpha_5)(\beta_4-\alpha_5)}{\Gamma(\beta_1)\Gamma(\beta_3+\beta_4-\alpha_5+1)}
$$

$$\times F^{0\;3;3}_{1:1;1}\left[\begin{matrix}\text{———}:\alpha_1,\alpha_2,\alpha_3;\beta_3-\alpha_5+1,\beta_4-\alpha_5+1,1;\\ \beta_3+\beta_4-\alpha_5+1:\beta_1;\quad 2\end{matrix}\quad 1,1\right] \tag{7.2}$$

(**B**) If $a+b+c\notin\{0,-1,-2,\ldots\}$ and $\mathrm{Rl}(\beta_5+\beta_6-a-b)>0$, then as the integer m tends to ∞,

$$ {}_6F_5\left[\begin{matrix}\frac{a+b+c+1}{2}, & a+b+c-1, & a,b,c,\beta_5+\beta_6-a-b;\\ \frac{a+b+c-1}{2}, & b+c, \quad a+c, & \beta_5,\quad \beta_6\end{matrix}\right]_m $$

$$\sim\frac{\Gamma(b+c)\Gamma(a+c)\Gamma(\beta_5)\Gamma(\beta_6)}{\Gamma(a+b+c)\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(\beta_5+\beta_6-a-b)}$$

$$\times\{2\log m-\gamma-\Psi(a)-\Psi(b)-\Psi(c)\}+\frac{(a+b-\beta_6)(a+b-\beta_5)\Gamma(\beta_5)\Gamma(\beta_6)}{\Gamma(\beta_5+\beta_6-a-b)\Gamma(a+b+1)}$$

$$\times F^{0\;5;3}_{1\;3;1}\left[\begin{matrix}\text{——}:\frac{a+b+c+1}{2},\ a+b+c-1,\ a,b,c;a+b-\beta_6,\ a+b-\beta_5,1;\\ a+b+1:\frac{a+b+c-1}{2},\ b+c,\ a+c;\quad 2\end{matrix}\quad 1,1\right]$$

$$+O\left(\frac{\log m}{m}\right). \tag{7.3}$$

Proof. We have that [See 10; p. 73 (7)]

$$ {}_pF_{p-1}\left[\begin{matrix}(\alpha_p);\\ (\beta_{p-1})\end{matrix}\quad x\right]=\frac{\Gamma(\beta_{p-2})\Gamma(\beta_{p-1})}{\Gamma(\alpha_p)\Gamma(\beta_{p-2}+\beta_{p-1}-\alpha_p)}$$

$$\times\,{}_{p-1}F_{p-2}\left[\begin{matrix}(\alpha_{p-1});\\ (\beta_{p-3}),\beta_{p-2}+\beta_{p-1}-\alpha_p\end{matrix}\quad x\right]$$

$$+\frac{(\beta_{p-2}-\alpha_p)(\beta_{p-1}-\alpha_p)\Gamma(\beta_{p-2})\Gamma(\beta_{p-1})}{\Gamma(\alpha_p)\Gamma(\beta_{p-2}+\beta_{p-1}-\alpha_p+1)}$$

$$\times F^{0:p-1;3}_{1:p-3;1}\left[\begin{matrix}\text{———}:(\alpha_{p-1});\beta_{p-2}-\alpha_p+1,\beta_{p-1}-\alpha_p+1,1;\\ \beta_{p-2}+\beta_{p-1}-\alpha_p+1:(\beta_{p-3});\quad 2\end{matrix}\quad x,1\right].$$

$$(p\geqq 3;\quad \mathrm{Re}(\alpha_p)>0) \tag{7.4}$$

One easily gets

$$ {}_{p-1}F_{p-2}\left[\begin{matrix}(\alpha_{p-2}),\alpha_p;\\ (\beta_{p-4}),\beta_{p-2},\beta_{p-1}\end{matrix}\quad x\right]_{m-1}=\frac{\Gamma(\beta_{p-2})\Gamma(\beta_{p-1})}{\Gamma(\alpha_p)\Gamma(\beta_{p-2}+\beta_{p-1}-\alpha_p)}$$

$$\times\,{}_{p-2}F_{p-3}\left[\begin{matrix}(\alpha_{p-2});\\ (\beta_{p-4}),\beta_{p-2}+\beta_{p-1}-\alpha_p\end{matrix}\quad x\right]_{m-1}$$

$$+\frac{(\beta_{p-2}-\alpha_p)(\beta_{p-1}-\alpha_p)\Gamma(\beta_{p-2})\Gamma(\beta_{p-1})}{\Gamma(\alpha_p)\Gamma(\beta_{p-2}+\beta_{p-1}-\alpha_p+1)}$$

$$\times\sum_{k=0}^{m-1}\sum_{n=0}^{\infty}\frac{[(\alpha_{p-2})]_k(\beta_{p-2}-\alpha_p+1)_n(\beta_{p-1}-\alpha_p+1)_n x^k}{(\beta_{p-2}+\beta_{p-1}-\alpha_p+1)_{k+n}[(\beta_{p-4})]_k(2)_n k!}$$

$$\mathrm{Re}(\alpha_p)>0 \tag{7.5}$$

by taking $\alpha_{p-1} = -m+1$ and $\beta_{p-3} = -m+1$, respectively.

Taking $p=5$, $x=1$ and applying Ramanujan's asymptotic formula (2.5) on r.h.s. of the above equation, we get (7.1).

Repeating this process for $p = 6, 7, \ldots$ we can get the asymptotic behaviour of a zero-balanced ${}_rF_{r-1}$ (1) series.

Proof of (7.3). Proceeding exactly as in the case of (7.1), we can generalize (2.6) also.

8. Concluding remarks

Proceeding exactly as in the case of (4.1) and (7.1) we can obtain the generalization of a result of Evans and Stanton [5, Theorem 3, p. 1011, (1.11)] namely

If $a+b+c=d+e$ and $\mathrm{Re}\,(c)>0$, then as $u \to 1$ with $0<u<1$

$$\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)}\ {}_3F_2\left[\begin{matrix} a, & b, & c; \\ d, & e & \end{matrix}\ u\right] = -\log(1-u) + L + O(1-u)\log(1-u)) \tag{8.1}$$

where L is defined by (2.4).

On particularising parameters in this generalization, we get Saigo and Srivastava's result [10; p. 74, 75 (10), (11), (12)].

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