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Application of the absolute Euler method to some series related to Fourier series and its conjugate series

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Abstract. We study the absolute Euler summability problem of some series associated with Fourier series and its conjugate series generalizing some known results in the literature. Also, it is shown that absolute Euler summability of rth derived Fourier series and rth derived conjugate series can be ensured under local conditions.

Keywords. Absolute Euler summability; Fourier series; summability factor; local property.

1. Introduction

1.1. DEFINITION 1

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with the sequence of partial sums $\{U_n\}$ and let

$$v_n = \frac{1}{(q+1)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} u_k, \quad q > 0, \tag{1}$$

If $\sum v_n$ is convergent, we say that the series $\sum u_n$ of the sequence $\{U_n\}$ is summable (E, q), $q > 0$ ([7], [11]) and in short we write

$$\sum u_n (\text{or } \{U_n\}) \in (E, q), \quad q > 0$$

If $\sum v_n$ is absolutely convergent, we say that the series $\sum u_n$ or the sequence $\{U_n\}$ is absolutely summable (E, q) and in short we write

$$\sum u_n (\text{or } \{U_n\}) \in |E, q|, \quad q > 0.$$

1.2. Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. We assume without loss of generality that the constant term in the Fourier series of $f(t)$ is zero. Thus

$$\int_{-\pi}^{\pi} f(t)\mathrm{d}t = 0,$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \tag{2}$$

The series conjugate to the Fourier series of f at $t = x$ is

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x). \tag{3}$$

The rth derived series of (2) at $t = x$ and rth derived series of (3) are respectively

$$\sum_{n=1}^{\infty} \left[\left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^r A_n(t) \right]_{t=x} \equiv \sum_{n=1}^{\infty} A_{n.r}(x) \tag{4}$$

and

$$\sum_{n=1}^{\infty} \left[\left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^r B_n(t) \right]_{t=x} \equiv \sum_{n=1}^{\infty} B_{n.r}(x). \tag{5}$$

We write

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2S\},$$

where S is a function of x. In particular if f is a continuous function at $t = x$ then S is taken to be $f(x)$.

$$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}$$

$$g(t) = \phi(t)\left(\log\frac{2\pi}{t}\right)^{1-\delta}$$

$$h(t) = \psi(t)\left(\log\frac{2\pi}{t}\right)^{-\delta}$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)\mathrm{d}u, \quad \alpha > 0$$

$$\Phi_0(t) = \phi(t)$$

$$\phi_\alpha(t) = (\alpha+1)t^{-\alpha}\Phi_\alpha(t), \quad \alpha \geqslant 0$$

$$P(t) = \sum_{i=0}^{n-1} \frac{\theta_i t^i}{i!},$$

where θ_i for $i = 0, 1, \ldots, (r-1)$ are arbitrary.

$$g^*(t) = \frac{\{f(x+t) - P(t)\} + (-1)^r\{f(x-t) - P(-t)\}}{2t^r}$$

$$h^*(t) = \frac{\{f(x+t) - P(t)\} - (-1)^r\{f(x-t) - P(-t)\}}{2t^r}.$$

1.3. In 1968, Mohanty and Mohapatra [11] began the study of absolute Euler summability of Fourier series and its conjugate series in proving the following theorems:

Theorem A. *For* $0 < \delta < 1$

$$\phi(t)\log\frac{1}{t} \in BV(0,\delta) \Rightarrow \sum A_n(x) \in |E, q|, \quad q > 0.$$

Theorem B. *For* $0 < \delta < 1$

$$\psi(+0) = 0 \textit{ and } \int_0^\delta \log\frac{1}{t}|\mathrm{d}\psi(t)| < \infty \Rightarrow \sum B_n(x) \in |E, q|, \quad q > 0.$$

The above two theorems were also independently obtained by Kwee [9] in 1972. Further Kwee [9] proved

Theorem C. *The condition* $\phi(t)\log(1/t)\in BV(0,\delta)$, $0<\delta<1$ *in Theorem A cannot be replaced by the weaker assumption:*

$$\phi(t)\left(\log\frac{1}{t}\right)^{\eta}\in BV(0,\delta),\quad 0<\delta<1 \text{ for any } \eta<1.$$

Kwee [9] first studied the absolute Euler summability of derived Fourier series and his result reads as follows:

Theorem D ([9], Theorem 6). If $\psi(+0)=0$ *and*

$$\int_0^{\delta} u^{-2}|\mathrm{d}\psi(u)|<\infty \quad \text{for } 0<\delta<1 \tag{6}$$

then $\sum_{n=1}^{\infty} nB_n(x)\in|E,q|$, $q>0$. *Further (6) cannot be replaced by the weaker assumption*

$$\int_0^{\delta} u^{-\eta}|\mathrm{d}\psi(u)|<\infty \quad \text{for any } \eta<2.$$

Subsequently several authors such as Chandra [4], Chandra and Dikshit [5] have obtained number of results on the absolute Euler summability of Fourier series, conjugate series, first derived Fourier series and first derived conjugate series. Chandra and Dikshit [5] have obtained the following result which is an improvement of an earlier work due to Kwee (Theorem D of the present paper).

Theorem E ([5], Theorem 4(i)). *Let* $0<\delta<\pi$, $q>0$. *Then*

$$\frac{\psi(t)}{t^2}\in BV(0,\delta)\Rightarrow \sum_{n=1}^{\infty} nB_n(x)\in|E,q|.$$

Concerning the absolute Euler summability factors of Fourier series, Tripathy [15] and Chandra [4] have independently proved the following.

Theorem F. $\phi(t)\in BV(0,\pi)\Rightarrow \sum_{n=1}^{\infty}\frac{A_n(x)}{\log(n+1)}\in|E,q|,\quad q>0.$

Also Tripathy [16] proved the following

Theorem G. *If* $\phi_\alpha(t)\in BV(0,\pi)$, $0<\alpha<3/2$ *then*

$$\sum n^{-\lambda}A_n(x)\in|E,q|,\quad q>0$$

where

$$\lambda>\max(\alpha-1/2,\ 1/2).$$

Ray and Patra [13] improved upon Theorem G by proving the following:

Theorem H.

$$\phi_\alpha(t)\in BV(0,\delta)0<\delta<\pi\Rightarrow \sum_{n=1}^{\infty}\frac{A_n(x)}{n^{\alpha/2}}\in|E,q|,\quad q>0 \quad \text{for } 0<\alpha\leqslant 2.$$

Concerning the absolute Euler summability factors of conjugate series Chandra ([4], Theorem 3, 4) proved the following theorems:

Theorem I.

$$\psi(+0) = 0 \text{ and } \psi(t) \in BV(0,\pi) \Rightarrow \sum_{n=1}^{\infty} \frac{B_n(x)}{\log(n+1)} \in |E,q|, \quad q > 0.$$

Theorem J.

$$\psi(t) \in BV(0,\pi) \text{ and } \frac{\psi(t)}{t} \in L(0,\pi) \Rightarrow \sum_{n=1}^{\infty} \frac{B_n(x)}{\log(n+1)} \in |E,q|, \quad q > 0.$$

2. Theorems

2.1. The purpose of the present work is manifold as detailed below from (I) to (IV).

(I) The conditions taken on the respective generating functions in proving Theorem F. Theorem I and Theorem J are non-local in character. We wish to show that $|E,q|, q > 0$ summability of

$$\sum_{n=1}^{\infty} \frac{A_n(x)}{\{\log(n+1)\}^{\delta}}, \quad 0 \leqslant \delta \leqslant 1 \tag{7}$$

and

$$\sum_{n=1}^{\infty} \frac{B_n(x)}{\{\log(n+1)\}^{\delta}}, \quad 0 \leqslant \delta \leqslant 1 \tag{8}$$

can be ensured by imposing local conditions on the respective generating functions. To be more exact, we prove

Theorem 1. *Let $0 < c < 1$ and $0 \leqslant \delta \leqslant 1$. If $g(t) \in BV(0,c)$ then the series (7) is summable $|E,q|, q > 0$.*

Theorem 2. *Let $0 < c < 1$ and $0 \leqslant \delta \leqslant 1$. If $h(t)\log(2\pi/t) \in BV(0,c)$ and $h(t)/t \in L(0,c)$ then the series (8) is summable $|E,q|, q > 0$.*

(II) We wish to point out that Theorem I of Chandra [4] is not true. In this context, we prove

Theorem 3. *There exists a function $f(t)$ of the class L such that*

$$\psi(t) \log\log\frac{k}{t} \in BV(0,\pi), \quad \text{where } k > e\pi$$

but the series $\sum_{n=1}^{\infty} B_n(x)/\log(n+1)$ is not summable by any regular summability method and a fortiori not summable $|E,q|, q > 0$.

We observe that $\psi(+0) = 0$ and $\psi(t) \in BV(0,\pi)$ whenever $\psi(t)\log\log(k/t) \in BV(0,\pi)$ and hence Theorem I turns out to be false by an appeal to Theorem 3. The defect in the proof of Theorem I can be explained as follows.

We shall adopt a technique already developed by Kuttner (see Chandra [3]). We know that (see [4], p. 1015) when $\psi(+0) = 0 = \psi(\pi)$

$$B_m(x) = \begin{cases} \dfrac{2}{m\pi}\displaystyle\int_0^\pi \cos mt \, \mathrm{d}\psi(t) & (9) \\ \dfrac{2}{m\pi}\displaystyle\int_0^\pi (\cos mt - 1)\mathrm{d}\psi(t). & (10) \end{cases}$$

Kuttner [3] while remarking on a paper [2] pointed out that if E_1 and E_2 are two disjoint intervals (or finite sums of intervals) whose union is $[0, \pi]$, it is not in general true that

$$B_m(x) = \frac{2}{m\pi}\int_{E_1} \cos mt \, \mathrm{d}\psi(t) + \frac{2}{m\pi}\int_{E_2} (\cos mt - 1)\mathrm{d}\psi(t). \tag{11}$$

On p. 1016 (Chandra [4]) after splitting the sum

$$\sum = \sum_{n=1}^{\infty} \left| \sum_{m=1}^{n} v_{m-1}^{p}(n-1) y_m B_m(x) \right| \equiv \sum_{n=1}^{T_1} + \sum_{n=T_1+1}^{\infty}$$

where $v_{m-1}^{p}(n-1) = (1+p)^{-n}\binom{n}{m} p^{n-m}$, $y_m = [\log(m+1)]^{-1}$ and $T_1 = [2\pi/t]$. Chandra has used (10) and (9) respectively in the sums $\sum_{n=1}^{T_1}$ and $\sum_{n=T_1+1}^{\infty}$ while expressing $B_m(x)$ as an integral. By doing so the integrand $(\cos mt - 1)$ has been taken when $n \leqslant 2\pi/t$ (i.e., $0 < t \leqslant 2\pi/n$) and $\cos mt$ when $n > 2\pi/t$ (i.e., $t > 2\pi/n$) while expressing $B_m(x)$ as an integral. This is equivalent to assuming (11) with $E_1 = (2\pi/n, \pi)$ and $E_2 = [0, 2\pi/n]$ and hence the proof of Theorem I is erroneous.

(III) It is known that ([7], p. 364) the method (E, q) is not Fourier effective, i.e., the Fourier series of a continuous function need not be summable (E, q), $q > 0$. We may therefore ask about $|E, q|$, $q > 0$ summability factors of Fourier series, that is to say, a sequence $\{\lambda_n\}$ to be obtained such that

$$\sum \lambda_n A_n(x) \in |E, q|, \quad q > 0$$

whenever $f(x)$ is continuous at $t = x$. In this connection, we prove

Theorem 4. *If*

$$\int_0^t |\phi(u)| \mathrm{d}u = O(t) \tag{12}$$

(in particular when $\phi(t)/t \in L(0, \pi)$) then $\sum_{n=1}^{\infty} \frac{A_n(x)}{n^{(1/2)+\varepsilon}}$ is summable $|E, q|$, $q > 0$. Clearly the mentioned series is summable $|E, q|$, $q > 0$ almost everywhere.

Theorem 5. *If*

$$\frac{\phi(t)}{t} \in L(0, c), \quad 0 < c < 1 \tag{13}$$

then

$$\sum_{n=1}^{\infty} \frac{A_n(x)}{n^{1/2}} \in |E, q|, \quad q > 0.$$

It is known that [6], if (12) holds then $\sum_{n=1}^{\infty} \frac{A_n(x)}{\{\log(n+1)\}^{(3/2)+\varepsilon}}$ is summable $|C, 1|$ for every $\epsilon > 0$. It was shown by Wang [14] that the factor $\{\log(n+1)\}^{-(3/2)-\varepsilon}$ cannot be replaced by $\{\log(n+1)\}^{-3/2}$. In Theorem 5 the factor $1/n^{1/2}$ is best possible in the sense that it cannot be replaced by $1/n^{(1/2)-\varepsilon}$ for $0 < \varepsilon \leqslant 1/2$. In this connection, we prove

Theorem 6. *There exists a function $f(t)$ of the class L such that $\phi(t)/t \in L(0, \pi)$ but the series $\sum_{n=1}^{\infty} (A_n(x)/n^{(1/2)-\varepsilon})$ is not summable $|E, q|$, $q > 0$ for $0 < \varepsilon < 1/2$.*

It is worth mentioning that Theorem 6 still holds when $\varepsilon = 1/2$ by a result of Kwee ([9], Theorem 3) who proves that the condition $\phi(t)/t^{\eta} \in L(0, \pi)$ where $\eta < 2$, does not ensure that $\sum A_n(x)$ is summable $|E, q|$, $q > 0$. As (13) implies (12), we may remark that the factor $1/(n^{(1/2)-\varepsilon})$ in Theorem 4 cannot be replaced by $1/(n^{(1/2)-\varepsilon})$, $0 < \varepsilon \leqslant 1/2$. It is natural to ask whether Theorem 4 remains valid when $\varepsilon = 0$. As we have not been able to answer this question it remains as an open problem.

(IV) It is known that ([9], also see Theorems D and E of the present paper) $|E, q|$, $q > 0$ summability of first derived Fourier series and first derived conjugate series hold under local conditions. We prove

Theorem 7. *The absolute Euler summability of rth derived Fourier series* (4) *is a local property of its generating function.*

Theorem 8. *The absolute Euler summability of rth derived conjugate series* (5) *is a local property of its generating function.*

3. Notations, order estimates and lemmas

3.1 *Notations.* Let

$$\tau = [t^{-1}],\ T = [t^{-2}],\ N = \left[\frac{n+1}{q+1}\right]$$

$$V(n, v) = (1+q)^{-n} \binom{n}{v} q^{n-v},\ v \leqslant n$$

$$S_v(t) = \sum_{k=1}^{v} V(n, k) \cos kt,\ v \leqslant n$$

$$\bar{S}_v(t) = \sum_{k=1}^{v} V(n, k) \sin kt,\ v \leqslant n$$

$$\rho(t) = \frac{(1+q^2+2q\cos t)^{1/2}}{1+q}, \quad \Phi = \tan^{-1} \frac{\sin t}{q + \cos t}$$

Let $\{\lambda_n\}$ be a non-increasing sequence of positive numbers such that

(i) $\{n^i \lambda_n\}$ is increasing for some non-negative integer i.
(ii) $\{n^j \Delta\lambda_n\}$ is increasing for some non-negative integer j,

where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

$$g_c(n, \lambda, t) = \sum_{v=1}^{n} V(n, v) \lambda_v \cos vt$$

$$g_s(n,\lambda,t)=\sum_{v=1}^{n} V(n,v)\lambda_v \sin vt$$

$$\xi(n,t)=\int_0^t \frac{g_s(n,n^{-1}(\log(n+1))^{-\delta},u)}{u(\log(2\pi/u))^{2-\delta}}\mathrm{d}u$$

$$\eta(n,t)=\int_t^c \frac{g_s(n,n^{-1}(\log(n+1))^{-\delta},u)}{u(\log(2\pi/u))^{2-\delta}}\mathrm{d}u$$

$$\xi^*(n,t)=\int_0^t \left(\log\frac{2\pi}{u}\right)^{\delta} g_s(n,(\log(\mathrm{n}+1))^{-\delta},u)\mathrm{d}u$$

$$\eta^*(n,t)=\int_t^c \left(\log\frac{2\pi}{u}\right)^{\delta} g_s(n,(\log(n+1))^{-\delta},u)\mathrm{d}u$$

3.2. *Estimates.* We need the following estimates:

Clearly

$$\sum_{m=0}^{n} V(n,m)=1 \quad \text{and} \quad V(n,m)=\frac{(m+1)(q+1)}{n+1}V(n+1,m+1) \tag{14}$$

$$V(n,m)=O\{V(n,N)\}=O(n^{-1/2}),\quad m\leqslant n \tag{15}$$

$$S_n(t)=\begin{cases} O\{\rho^n(t)\}+\left\{\left(\dfrac{q}{1+q}\right)^n\right\} & (16)\\ O(t^{-1}n^{-1/2}) & (17)\end{cases}$$

$$\bar{S}_n(t)=\begin{cases} O\{\rho^n(t)\} & (18)\\ O(t^{-1}n^{-1/2}) & (19)\end{cases}$$

$$g_c(n,\lambda,t)=\begin{cases} O(\lambda_n) & (20)\\ O(t^{-1}\Delta\lambda_n)+O(n^{-1/2}t^{-1}\lambda_n) & (21)\\ O(t^{-1}\Delta\lambda_n)+O(\lambda_n\rho^n(t))+O\left\{\left(\dfrac{q}{q+1}\right)^n\lambda_n\right\} & (22)\end{cases}$$

$$g_s(n,\lambda,t)=\begin{cases} O(nt\lambda_n) & (23)\\ O(t^{-1}\Delta\lambda_n)+O(n^{-1/2}t^{-1}\lambda_n) & (24)\\ O(t^{-1}\Delta\lambda_n)+O(\lambda_n\rho^n(t)) & (25)\end{cases}$$

$$g_c(n,n^{-(1/2)+\varepsilon},t)=S_n(t)(N+1)^{-(1/2)+\varepsilon}+O\{t^{-1}n^{-(3/2)+\varepsilon}\} \tag{26}$$

$$\xi(n,t)=O\left\{t\left(\log\frac{2\pi}{t}\right)^{\delta-2}(\log n)^{-\delta}\right\} \tag{27}$$

$$\begin{aligned}\eta(n,t)=&\,O\left\{t^{-2}\left(\log\frac{2\pi}{t}\right)^{\delta-2}n^{-3}(\log n)^{-\delta}\right\}\\ &+O\left\{\rho^n(t)t^{-1}\left(\log\frac{2\pi}{t}\right)^{\delta-2}n^{-2}(\log n)^{-\delta}\right\}\\ &+O\left\{\left(\frac{q}{q+1}\right)^n t^{-1}\left(\log\frac{2\pi}{t}\right)^{\delta-2}n^{-2}(\log n)^{-\delta}\right\}\end{aligned} \tag{28}$$

$$\xi^*(n,t) = O\left\{t^2\left(\log\frac{2\pi}{t}\right)^{\delta} n(\log n)^{-\delta}\right\} \tag{29}$$

$$\frac{1}{1-\rho(t)} = O(t^{-2}), \quad 0 < t \leqslant \pi \tag{30}$$

$$\frac{1}{1-\rho(t)} > 4t^{-2}, \quad 0 < t \leqslant \pi \tag{31}$$

It is known that ([7], p. 214) $V(n,m)$ attains its maximum value when $m = N$ and $V(n,N) = O(n^{-1/2})$ and hence (15) follows.

Proof of (16) *and* (18). Taking $\rho(t)$ and Φ as in § 3.1, we get

$$\begin{aligned} S_n(t) + i\bar{S}_n(t) &= \sum_{\nu=0}^{n} V(n,\nu)e^{i\nu t} - \left(\frac{q}{q+1}\right)^n \\ &= \left(\frac{q+e^{it}}{q+1}\right)^n - \left(\frac{q}{q+1}\right)^n \\ &= \rho^n(t)(\cos n\Phi + i\sin n\Phi) - \left(\frac{q}{q+1}\right)^n, \end{aligned} \tag{32}$$

from which (16) and (18) follow at once.

Proof of (17) *and* (19). As $V(n,k)$ attains its maximum value when $k = N$, we get

$$\begin{aligned} |S_n(t)| &\leqslant V(n,N) \max_{1<L,L'<n} \left|\sum_{L}^{L'} \cos kt\right| \\ &= O\{V(n,N)t^{-1}\} \\ &= O\{t^{-1}n^{-1/2}\} \end{aligned}$$

using (15). Similarly (19) can be derived.

Proof of (20), (21) *and* (22). There exists a non-negative integer i such that $\{n^i\lambda_n\}$ is increasing. After this choice of i, we have by using (14)

$$\begin{aligned} g_c(n,\lambda,t) &= \sum_{k=1}^{n} V(n,k)\lambda_k \cos kt \\ &= \frac{(q+1)^i}{(n+1)(n+2)\ldots(n+i)} \sum_{k=1}^{n} V(n+i,k+i)(k+1)(k+2)\ldots \\ &\qquad\qquad (k+i)\lambda_k \cos kt \\ &= O(\lambda_n) \sum_{k=1}^{n} V(n+i,k+i), \text{ as } \{k^i\lambda_k\} \text{ is increasing} \\ &= O(\lambda_n) \sum_{k=-i}^{n} V(n+i,k+i) \\ &= O(\lambda_n), \end{aligned}$$

as the last sum is unity. By Abel's method of partial summation

$$
\begin{aligned}
g_c(n,\lambda,t) &= \sum_{\nu=1}^{n} V(n,\nu)\lambda_\nu \cos\nu t \\
&= \sum_{\nu=1}^{n-1} \Delta\lambda_\nu S_\nu(t) + \lambda_n S_n(t) \\
&= \sum_{\nu=1}^{N} \Delta\lambda_\nu S_\nu(t) + \sum_{\nu=N+1}^{n-1} \Delta\lambda_\nu S_\nu(t) + \lambda_n S_n(t) \\
&= \sum_{\nu=1}^{N} \Delta\lambda_\nu S_\nu(t) + \sum_{\nu=N+1}^{n-1} \Delta\lambda_\nu \left[S_n(t) - \sum_{k=\nu+1}^{n} V(n,k)\cos kt \right] + \lambda_n S_n(t) \\
&= \sum_{\nu=1}^{N} \Delta\lambda_\nu S_\nu(t) + S_n(t)(\lambda_{N+1} - \lambda_n) \\
&\quad - \sum_{\nu=N+1}^{n-1} \Delta\lambda_\nu \sum_{k=\nu+1}^{n} V(n,k)\cos kt + \lambda_n S_n(t) \\
&= \sum_{\nu=1}^{N} \Delta\lambda_\nu S_\nu(t) + \lambda_{N+1} S_n(t) - \sum_{\nu=N+1}^{n-1} \Delta\lambda_\nu \sum_{k=\nu+1}^{n} V(n,k)\cos kt \\
&= T_1 + T_2 - T_3. \qquad (33)
\end{aligned}
$$

As $V(n,k)$ is monotonic increasing for $k \leqslant N$, we get

$$
\begin{aligned}
|T_1| &\leqslant \sum_{\nu=1}^{N} \Delta\lambda_\nu \left| \sum_{k=1}^{\nu} V(n,k)\cos kt \right| \\
&\leqslant \sum_{\nu=1}^{N} \Delta\lambda_\nu \, V(n,\nu) \max_{1<L,L'<\nu} \left| \sum_{L}^{L'} \cos kt \right| \\
&= O(t^{-1}) \sum_{\nu=1}^{N} \Delta\lambda_\nu \, V(n,\nu) \\
&= O(t^{-1}\Delta\lambda_n),
\end{aligned}
$$

by employing argument similar to those used in proving (20). By (16) and (17)

$$
T_2 = \begin{cases} O\{\rho^n(t)\lambda_n\} + O\left\{\left(\dfrac{q}{q+1}\right)^n \lambda_n\right\} \\ O\{\lambda_n t^{-1} n^{-1/2}\}. \end{cases}
$$

As $V(n,k)$ is monotonic decreasing for $k > N$, we have

$$
\begin{aligned}
|T_3| &\leqslant \sum_{\nu=N+1}^{n-1} \Delta\lambda_\nu \, V(n,\nu) \max_{\nu+1<M,M'<n} \left| \sum_{M}^{M'} \cos kt \right| \\
&= O(t^{-1}) \sum_{\nu=N+1}^{n-1} \Delta\lambda_\nu \, V(n,\nu) \\
&= O(t^{-1}) \sum_{\nu=1}^{n} \Delta\lambda_\nu \, V(n,\nu) = O(t^{-1}\Delta\lambda_n),
\end{aligned}
$$

using the technique employed in proving (20). Now using the estimates for T_1, T_2 and T_3 in (33) we obtain (21) and (22).

Proof of (23), (24) *and* (25). We have

$$g_s(n,\lambda,t) = \sum_{k=1}^{n} V(n,k)\lambda_k \sin kt$$
$$= O\left(t\sum_{k=1}^{n} V(n,k)k\,\lambda_k\right) = O(tn\,\lambda_n),$$

using the technique employed in proving (20). The proof of (24) and (25) are similar to those of (21) and (22).

Proof of (26). Taking $\lambda_n = n^{-(1/2)+\varepsilon}$ in the proof of (21) and (22) we get

$$T_1 = O(t^{-1}n^{-(3/2)+\varepsilon}),\ T_2 = \frac{S_n(t)}{(N+1)^{(1/2)-\varepsilon}},\ T_3 = O(t^{-1}n^{-(3/2)+\varepsilon})$$

and hence using these results in (33) we obtain

$$g_c(n, n^{-(1/2)+\varepsilon}, t) = T_1 + T_2 - T_3$$
$$= \frac{S_n(t)}{(N+1)^{(1/2)-\varepsilon}} + O(t^{-1}n^{-(3/2)+\varepsilon}).$$

Proof of (27). Using (23), we get

$$\xi(n,t) = \int_0^t \frac{g_s(n,n^{-1}(\log(n+1))^{-\delta},u)}{u(\log(2\pi/u))^{2-\delta}}\mathrm{d}u$$
$$= O\left\{\int_0^t \frac{\mathrm{d}u}{(\log n)^{\delta}(\log(2\pi/u))^{2-\delta}}\right\} = O\left\{t(\log n)^{-\delta}\left(\log\frac{2\pi}{t}\right)^{\delta-2}\right\}.$$

Proof of (28). By mean value theorem for some $t < t' = t'(n) < c$

$$\eta(n,t) = \int_t^c \frac{g_s(n,n^{-1}(\log(n+1))^{-\delta},u)}{u(\log(2\pi/u))^{2-\delta}}\mathrm{d}u$$
$$= t^{-1}\left(\log\frac{2\pi}{t}\right)^{\delta-2}\int_t^{t'} g_s(n,n^{-1}(\log(n+1))^{-\delta},u)\mathrm{d}u$$
$$= t^{-1}\left(\log\frac{2\pi}{t}\right)^{\delta-2}\{g_c(n,n^{-2}(\log(n+1))^{-\delta},t)$$
$$- g_c(n,n^{-2}(\log(n+1))^{-\delta},t')\}.$$

Now using (22) we get (28) at once since $t'^{-1} < t^{-1}$ and $\rho(t') < \rho(t)$.

Proof of (29). Using (23), we get

$$\xi^*(n,t) = \int_0^t \left(\log\frac{2\pi}{u}\right)^{\delta} g_s(n,(\log(n+1))^{-\delta},u)\mathrm{d}u$$

$$= O\left\{n(\log n)^{-\delta}\int_0^t u\left(\log\frac{2\pi}{u}\right)^{\delta}\mathrm{d}u\right\}$$

$$= O\left\{t^2\left(\log\frac{2\pi}{t}\right)^{\delta} n(\log n)^{-\delta}\right\}.$$

Proof of (30) *and* (31). Using the expression for $\rho(t)$ as given in § 3.1, we get

$$[1-\rho^2(t)]^{-1} = \left[1 - \frac{1+2q\cos t + q^2}{(1+q)^2}\right]^{-1}$$

$$= \frac{(1+q)^2}{2q(1-\cos t)} = \frac{(1+q)^2}{4q\sin^2(1/2)t} \tag{34}$$

from which it follows that

$$\frac{(1+q)^2}{q}t^{-2} \leqslant \frac{1}{1-\rho^2(t)} \leqslant \frac{\pi^2(1+q)^2}{4q}t^{-2}, \quad 0<t\leqslant\pi \tag{35}$$

as $t/\pi \leqslant \sin(1/2)t \leqslant t/2$ whenever $0<t\leqslant\pi$. Writing $1/(1-\rho(t)) = (1+\rho(t))/(1-\rho^2(t))$ and making use of the fact that $1 < 1+\rho(t) < 2$ for all $0<t\leqslant\pi$, we obtain

$$\frac{(1+q)^2}{q}t^{-2} < \frac{1}{1-\rho(t)} < \frac{\pi^2(1+q)^2}{2q}t^{-2}, \quad 0<t\leqslant\pi. \tag{36}$$

Now (30) and (31) follow at once from (36) as q is a fixed positive constant and $(1+q)^2/q \geqslant 4$ for all $q>0$.

3.3. *Lemmas.* We need the following lemmas:

Lemma 1 [10]. *If* $\eta>0$, *then necessary and sufficient conditions that* (i) $h(t)\log(2\pi/t)$ *should be of bounded variation in* $(0,\eta)$ *and* (ii) $h(t)/t$ *should be integrable L in* $(0,\eta)$ *are that*

$$\int_0^{\eta}\log\frac{2\pi}{t}|\mathrm{d}h(t)| < \infty \text{ and } h(+0)=0.$$

Lemma 2. (i) $\displaystyle\int_0^c (\log(2\pi/t))^{\beta}\sin nt\,\mathrm{d}t \sim \frac{(\log n)^{\beta}}{n}$ *for all* $\beta>0$.

(ii) $\displaystyle\int_0^c\left(\log\frac{2\pi}{t}\right)^{\beta}\frac{\sin nt}{t}\mathrm{d}t \sim \frac{\pi}{2}(\log n)^{\beta}$ *for all* β.

The proof is similar (merely replace π by c) to the proof of Lemma 3(i) of [12].

Lemma 3. $\displaystyle\int_0^{\pi}\frac{\sin nt}{\log\log(k/t)}\mathrm{d}t = -\frac{(-1)^n}{n\log\log(k/\pi)} + \theta_n$

where

$$\theta_n \sim 1/(n\log\log n).$$

We omit the proof of Lemma 3 as it can be proved using arguments similar to those used in proving Lemma 3 of [12].

Lemma 4.

(i) $$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r S_n(t) = \begin{cases} -\left(\dfrac{q}{1+q}\right)^n + O(\rho^n(t)), & \text{for } r=0 \\ O(n^r\rho^{n-r}(t)), & \text{for } r=1,2,3,\ldots \end{cases}$$

(ii) $$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r \bar{S}_n(t) = O(n^r\rho^{n-r}(t)). \qquad \text{for } r=0,1,2,3,\ldots.$$

Proof. Put

$$w = S_n(t) + i\bar{S}_n(t) + \left(\frac{q}{q+1}\right)^n, \quad u = q + e^{it}$$

$$w_r = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r w, \quad u_r = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r u.$$

Using (32) we get

$$w = \left(\frac{q+e^{it}}{q+1}\right)^n = \rho^n(t)e^{in\Phi} = O(\rho^n(t)).$$

Clearly

$$w_1 = \frac{ni}{q+1}\left(\frac{q+e^{it}}{q+1}\right)^{n-1} e^{it} = O(n\,\rho^{n-1}(t)). \tag{37}$$

Let us assume that

$$w_r = O\{n^r\rho^{n-r}(t)\} \quad \text{for } r=0,1,2,\ldots,k. \tag{38}$$

From (32), we get

$$w_1 u = ni\,we^{it}. \tag{39}$$

Differentiating k times both sides of (39) with respect to t, we obtain

$$w_{k+1}u + \sum_{v=1}^{k}\binom{k}{v} w_{k+1-v}u_v = ni\sum_{v=0}^{k}\binom{k}{v} w_{k-v}(i)^v e^{it}.$$

Using (37), we get

$$w_{k+1}u = O\left(\sum_{v=1}^{k}\binom{k}{v} n^{k+1-v}\rho^{n-k-1+v}(t)\right) + O\left(n\sum_{v=0}^{k}\binom{k}{v} n^{k-v}\rho^{n-k+v}(t)\right)$$
$$= O(n^{k+1}\rho^{n-k}(t))$$

as the first term in each of the above sums dominates over the remaining terms. Since $|u| = (1+q)\rho(t)$, we have

$$w_{k+1} = O(n^{k+1}\rho^{n-k-1}(t)).$$

which is in conjunction with (38) gives

$$w_r = O(n^r\rho^{n-r}(t)) \quad \text{for } r=0,1,2,3,\ldots,(k+1).$$

Hence by induction principle

$$w_r = O(n^r\rho^{n-r}(t)) \quad \text{for } r=0,1,2,\ldots.$$

This completes the proof of the lemma.

Lemma 5. *For* $r = 1, 2, 3, \ldots$

$$\sum n^r(-1)^n \in |E, q|, \quad q > 0.$$

Proof. Case (I) Let r be odd, i.e., $r = 2m+1$, $m = 0, 1, 2, \ldots$. Clearly $n^r(-1)^n = [(\mathrm{d}/\mathrm{d}t)^{2m+1} \sin nt]_{t=\pi}$.
So the series $\sum n^r(-1)^n \in |E, q|$

$$\Leftrightarrow \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} V(n,k) \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m+1} \sin kt \right]_{t=\pi} \right| < \infty$$

$$\Leftrightarrow \sum_{n=1}^{\infty} \left| \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m+1} \bar{S}_n(t) \right]_{t=\pi} \right| < \infty. \tag{40}$$

By Lemma 4

$$\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m+1} \bar{S}_n(t) \right]_{t=\pi} = O(n^{2m+1} \rho^{n-2m-1}(\pi)) = O\left(n^r \left(\frac{|1-q|}{1+q} \right)^{n-r} \right),$$

and hence (40) follows at once.

Case (II). Let r be even, i.e., $r = 2m$, $m = 1, 2, 3, \ldots$. Clearly

$$n^r(-1)^n = \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m} \cos nt \right]_{t=\pi}$$

So the series $\sum n^r(-1)^n \in |E, q|$

$$\Leftrightarrow \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} V(n,k) \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m} \cos kt \right]_{t=\pi} \right| < \infty$$

$$\Leftrightarrow \sum_{n=1}^{\infty} \left| \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m} S_n(t) \right]_{t=\pi} \right| < \infty. \tag{41}$$

By Lemma 4

$$\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m} S_n(t) \right]_{t=\pi} = O(n^{2m} \rho^{n-2m}(\pi)) = O\left(n^r \left(\frac{|1-q|}{1+q} \right)^{n} \right),$$

which ensures the validity of (41).

This completes the proof of the lemma.

Lemma 6. (Bosanquet and Kestelman [1]). *Suppose that $f_n(x)$ is measurable in (a,b) where $b - a < \infty$, for $n = 1, 2, 3, \ldots$. Then a necessary and sufficient condition that for every $\lambda(x) \in L(a,b)$ the function $f_n(x)\lambda(x)$ should be $L(a,b)$ and*

$$\sum_{n=1}^{\infty} \left| \int_a^b \lambda(x) f_n(x) \mathrm{d}x \right| \leqslant K$$

is that

$$\sum_{n=1}^{\infty} |f_n(x)| \leqslant K$$

for almost every $x \in (a,b)$, where K is an absolute constant.

4. Proof of theorem 1

4.1. *Proof of theorem* 1. We have for $n \geqslant 1$

$$\frac{\pi}{2}A_n(x) = \int_0^c g(t)\frac{\cos nt}{(\log(2\pi/t))^{1-\delta}}\mathrm{d}t + \int_c^\pi \phi(t)\cos nt\,\mathrm{d}t$$
$$= \alpha_n + \beta_n. \tag{42}$$

The series $\sum_{n=1}^{\infty}\frac{\beta_n}{(\log(n+1))^\delta}\in|E,q|$

$$\Leftrightarrow \Sigma_1 = \sum_{n=1}^{\infty}\left|\sum_{k=1}^{n} V(n,k)\int_c^\pi \frac{\phi(t)\cos kt}{(\log(k+1))^\delta}\mathrm{d}t\right| < \infty$$

Now

$$\Sigma_1 \leqslant \int_c^\pi |\phi(t)|\sum_{n=1}^{\infty}\left|\sum_{k=1}^{n} V(n,k)\frac{\cos kt}{(\log(k+1))^\delta}\right|\mathrm{d}t$$
$$= \int_c^\pi |\phi(t)|\sum_{n=1}^{\infty}|g_c(n,\log(n+1))^{-\delta},t)|\mathrm{d}t. \tag{43}$$

In the case $\delta > 0$, using (22) and (30)

$$\sum_{n=1}^{\infty}|g_c(n,\log(n+1))^{-\delta},t)| = O\left(t^{-1}\sum_{n=2}^{\infty}\frac{1}{n(\log n)^{\delta+1}}\right) + O\left(\sum_{n=2}^{\infty}\frac{\rho^n(t)}{(\log n)^\delta}\right)$$
$$+ O\left(\sum_{n=2}^{\infty}\left(\frac{q}{q+1}\right)^n(\log n)^{-\delta}\right)$$
$$= O(t^{-1}) + O\left(\frac{1}{1-\rho(t)}\right) + O(1) = O(t^{-2}).$$

For $\delta = 0$, using (16) and (30)

$$\sum_{n=1}^{\infty}\left|g_c(n,(\log(n+1))^{-\delta},t)\right| = O\left(\sum_{n=1}^{\infty}\rho^n(t)\right) + O\left(\sum_{n=1}^{\infty}\left(\frac{q}{q+1}\right)^n\right)$$
$$= O\left(\frac{1}{1-\rho(t)}\right) + O(1) = O(t^{-2}).$$

Using these results in (43), we get

$$\Sigma_1 = O\left(\int_c^\pi \frac{|\phi(t)|}{t^2}\mathrm{d}t\right) = O(1),$$

which ensures that $\sum_{n=1}^{\infty}\frac{\beta_n}{(\log(n+1))^\delta}\in|E,q|$.

Next, integrating by parts

$$\alpha_n = \frac{g_c}{(\log(2\pi/c))^{1-\delta}}\frac{\sin nc}{n} - \frac{1-\delta}{n}g(c)\int_0^c \frac{\sin nt}{t(\log(2\pi/t))^{2-\delta}}\mathrm{d}t$$
$$-\frac{1}{n}\int_0^{c} \frac{\sin nt}{(\log(2\pi/t))^{1-\delta}}\mathrm{d}g(t) + \frac{1-\delta}{n}\int_0^c\left(\int_0^t \frac{\sin nu}{u(\log(2\pi/u)^{2-\delta}}\mathrm{d}u\right)\mathrm{d}g(t)$$
$$= \alpha_n^{(1)} - \alpha_n^{(2)} - \alpha_n^{(3)} + \alpha_n^{(4)}. \tag{44}$$

When $\delta = 1$, the terms $\alpha_n^{(2)}$ and $\alpha_n^{(4)}$ drop out and hence it is enough to study the absolute Euler summability of

$$\sum \frac{\alpha_n^{(1)}}{(\log(n+1))^\delta} \quad \text{and} \quad \sum \frac{\alpha_n^{(3)}}{(\log(n+1))^\delta} \quad \text{for } 0 \leqslant \delta \leqslant 1$$

and that of

$$\sum \frac{\alpha_n^{(2)}}{(\log(n+1))^\delta} \quad \text{and} \quad \sum \frac{\alpha_n^{(4)}}{(\log(n+1))^\delta} \quad \text{for } 0 \leqslant \delta < 1.$$

For $0 \leqslant \delta \leqslant 1$, $\sum \frac{\alpha_n^{(1)}}{(\log(n+1))^\delta} \in |E, q|$ if and only if

$$\sum_{n=1}^{\infty} |g_s(n, n^{-1}(\log(n+1))^{-\delta}, c)| < \infty. \tag{45}$$

By (24), $g_s(n, n^{-1}(\log(n+1))^{-\delta}, c) = O(n^{-3/2}(\log n)^{-\delta})$ and hence (45) follows. The series $\sum \frac{\alpha_n^{(3)}}{\{\log(n+1)\}^\delta} \in |E, q|$, if and only if

$$\sum_{n=1}^{\infty} \left| \int_0^c \frac{dg(t)\, g_s(n, n^{-1}(\log(n+1))^{-\delta}, t)}{(\log(2\pi/t))^{1-\delta}} \right| < \infty.$$

Since $\int_0^c |dg(t)|$ is finite, it remains to show that

$$\sum_{n=1}^{\infty} |g_s(n, n^{-1}(\log(n+1))^{-\delta}, t)| = O\left\{\left(\log \frac{2\pi}{t}\right)^{1-\delta}\right\} \quad \text{in } 0 < t < c. \tag{46}$$

Using (23), (25) and (30)

$$\begin{aligned}
\sum_{n=1}^{\infty} |g_s(n, n^{-1}(\log(n+1))^{-\delta}, t)| &= \sum_{n \leqslant \tau} |g_s(n, n^{-1}(\log(n+1))^{-\delta}, t)| \\
&\quad + \sum_{n > \tau} |g_s(n, n^{-1}(\log(n+1))^{-\delta}, t)| \\
&= O\left(t \sum_{n \leqslant \tau} (\log n)^{-\delta}\right) + O\left(\sum_{n > \tau} \frac{\rho^n(t)}{n(\log n)^\delta}\right) \\
&\quad + O\left(t^{-1} \sum_{n > \tau} \frac{1}{n^2(\log n)^\delta}\right) \\
&= O\left\{\left(\log \frac{2\pi}{t}\right)^{1-\delta}\right\},
\end{aligned} \tag{47}$$

since

$$\sum_{n=1}^{\infty} \frac{\rho^n(t)}{n} = \log \frac{1}{1-\rho(t)} = O\left(\log \frac{2\pi}{t}\right).$$

For $0 \leqslant \delta < 1$, by Lemma 2(ii)

$$\frac{\alpha_n^{(2)}}{(\log(n+1))^\delta} \sim (1-\delta) g(c) \frac{\pi}{2} \frac{1}{n(\log n)^2}.$$

So the series $\sum \frac{\alpha_n^{(2)}}{(\log(n+1))^\delta}$ is absolutely convergent and hence *a fortiori* it is summable $|E,q|$.

For $0 \leqslant \delta < 1$ the series $\sum \frac{\alpha_n^{(4)}}{(\log(n+1))^\delta} \in |E,q|$, if and only if

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} \frac{V(n,k)}{k\{\log(k+1)\}^\delta} \int_0^c \mathrm{d}g(t) \int_0^t \frac{\sin ku}{u(\log(2\pi/u))^{2-\delta}} \mathrm{d}u \right| < \infty$$

i.e.,

$$\sum_{n=1}^{\infty} \left| \int_0^c \mathrm{d}g(t)\, \xi(n,t) \right| < \infty. \tag{48}$$

Since $\int_0^c |\mathrm{d}g(t)|$ is finite for the validity of (48) it is enough to show that in $0 < t < c$

$$\sum_{n=1}^{\infty} |\xi(n,t)| = O(1). \tag{49}$$

Using (27)

$$\sum_{n \leqslant \tau} |\xi(n,t)| = O\left\{ t \left(\log \frac{2\pi}{t} \right)^{\delta-2} \sum_{n \leqslant \tau} (\log n)^{-\delta} \right\} = O\left(\left(\log \frac{2\pi}{t} \right)^{-2} \right). \tag{50}$$

Now

$$\begin{aligned} \sum_{n > \tau} |\xi(n,t)| &= \sum_{n > \tau} |\xi(n,c) - \eta(n,t)| \\ &\leqslant \sum_{n > \tau} |\xi(n,c) + \sum_{n > \tau} |\eta(n,t)|. \end{aligned} \tag{51}$$

By simple verification, we have

$$\sum \frac{\alpha_n^{(2)}}{(\log(n+1))^\delta} \in |E,q| \text{ if and only if } \sum |\xi(n,c)| < \infty.$$

The series $\sum \frac{\alpha_n^{(2)}}{\{\log(n+1)\}^\delta}$ is known to be summable $|E,q|$ and hence

$$\sum_{n > \tau} |\xi(n,c)| \leqslant \sum_{n=1}^{\infty} |\xi(n,c)| = O(1). \tag{52}$$

Using (28), we have

$$\begin{aligned} \sum_{n > \tau} |\eta(n,t)| &= O\left(t^{-2} \left(\log \frac{2\pi}{t} \right)^{\delta-2} \sum_{n > \tau} n^{-3} (\log n)^{-\delta} \right) \\ &\quad + O\left(t^{-1} \left(\log \frac{2\pi}{t} \right)^{\delta-2} \sum_{n > \tau} \frac{\rho^n(t)}{n^2 (\log n)^\delta} \right) \\ &\quad + O\left\{ t^{-1} \left(\log \frac{2\pi}{t} \right)^{\delta-2} \sum_{n > \tau} \left(\frac{q}{q+1} \right)^n n^{-2} (\log n)^{-\delta} \right\} \\ &= O\left\{ \left(\log \frac{2\pi}{t} \right)^{-2} \right\} + O\left\{ t^{-1} \left(\log \frac{2\pi}{t} \right)^{\delta-2} \tau^{-2} (\log \tau)^{-\delta} \sum_{n=1}^{\infty} \rho^n(t) \right\} \end{aligned}$$

$$+O\left\{t^{-1}\left(\log\frac{2\pi}{t}\right)^{\delta-2}\tau^{-2}(\log\tau)^{-\delta}\sum_{n=1}^{\infty}\left(\frac{q}{q+1}\right)^n\right\}$$

$$=O\left\{\left(\log\frac{2\pi}{t}\right)^{-2}\right\}, \tag{53}$$

as $\sum_{n=1}^{\infty}\rho^n(t)=1/(1-\rho(t))=O(t^{-2})$ by (30).

Now (49) follows immediately from (50), (51), (52) and (53). This completes the proof of Theorem 1.

5. Proof of theorem 2

5.1. By Lemma 1, Theorem 2 is equivalent to

Theorem 2(a). *Let* $0<c<1$. *If* $h(+0)=0$ *and* $\int_0^c|\mathrm{d}h(t)|\log\frac{2\pi}{t}<\infty$ *then* $\sum\frac{B_n(x)}{\{\log(n+1)\}^\delta}\in|E,q|$, $q>0$ *for* $0\leqslant\delta\leqslant 1$.

Proof of theorem 2(*a*). For $n\geqslant 1$, we write

$$\frac{\pi}{2}\frac{B_n(x)}{\{\log(n+1)\}^\delta}=\frac{1}{\{\log(n+1)\}^\delta}\left[\int_0^c\psi(t)\sin nt\,\mathrm{d}t+\int_c^\pi\psi(t)\sin nt\,\mathrm{d}t\right]$$

$$=\alpha_n+\beta_n. \tag{54}$$

$\sum\beta_n\in|E,q|$, $q>0$ if and only if

$$\Sigma_1=\sum_{n=1}^{\infty}\left|\sum_{k=1}^{n}V(n,k)\int_c^\pi\frac{\psi(t)\sin kt}{\{\log(k+1)\}^\delta}\mathrm{d}t\right|<\infty$$

Now

$$\Sigma_1\leqslant\int_c^\pi|\psi(t)|\sum_{n=1}^{\infty}\left|\sum_{k=1}^{n}V(n,k)\frac{\sin kt}{\{\log(k+1)\}^\delta}\right|\mathrm{d}t$$

$$=\int_c^\pi|\psi(t)|\sum_{n=1}^{\infty}|g_s(n,(\log(n+1))^{-\delta},t)|\mathrm{d}t \tag{55}$$

by (24) and (30)

$$\sum_{n=1}^{\infty}|g_s(n,(\log(n+1))^{-\delta},t)|=O\left\{t^{-1}\sum_{n=2}^{\infty}\frac{1}{n(\log n)^{1+\delta}}\right\}+O\left\{\sum_{n=2}^{\infty}\frac{\rho^n(t)}{(\log n)^\delta}\right\}$$

$$=O(t^{-1})+O\left(\sum_{n=1}^{\infty}\rho^n(t)\right)=O(t^{-2}).$$

Using this estimate in (55), we get

$$\Sigma_1=O\left(\int_c^\pi\frac{|\psi(t)|}{t^2}\mathrm{d}t\right)=O(1),$$

which ensures that $\sum\beta_n\in|E,q|$.

We have

$$\alpha_n = \int_0^c h(t)\left(\log\frac{2\pi}{t}\right)^\delta \sin nt\,\mathrm{d}t$$
$$= \int_0^c \mathrm{d}h(t)\cdot\int_t^c \left(\log\frac{2\pi}{u}\right)^\delta \sin nu\,\mathrm{d}u,$$

the integrated part vanishes as $h(+0)=0$ and $\int_t^c (\log(2\pi/u))^\delta \sin nu\,\mathrm{d}u = O(1)$. The series $\sum\alpha_n \in |E, q|$, if and only if

$$\Sigma_2 = \sum_{n=1}^\infty \left|\sum_{k=1}^n V(n,k)\int_0^c \mathrm{d}h(t)\int_t^c\left(\log\frac{2\pi}{u}\right)^\delta \frac{\sin ku}{\{\log(k+1)\}^\delta}\mathrm{d}u\right| < \infty. \tag{56}$$

Clearly

$$\Sigma_2 = \sum_{n=1}^\infty \left|\int_0^c \mathrm{d}h(t)\int_t^c\left(\log\frac{2\pi}{u}\right)^\delta g_s(n,(\log(n+1))^{-\delta},u)\mathrm{d}u\right|$$
$$= \sum_{n=1}^\infty \left|\int_0^c \mathrm{d}h(t)\,\eta^*(n,t)\right|$$
$$\leqslant \int_0^c |\mathrm{d}h(t)| \sum_{n=1}^\infty |\eta^*(n,t)|.$$

Since $\int_0^c |\mathrm{d}h(t)|\log(2\pi/t)$ is finite for the validity of (56) it is enough to show that in $0<t<c$

$$\sum_{n=1}^\infty |\eta^*(n,t)| = O\left(\log\frac{2\pi}{t}\right). \tag{57}$$

By Lemma 2(i), we get

$$\xi^*(n,c) = \sum_{k=1}^n V(n,k)\{\log(k+1)\}^{-\delta}\int_0^c\left(\log\frac{2\pi}{t}\right)^\delta \sin ku\,\mathrm{d}u$$
$$= O\left(\sum_{k=1}^n \frac{V(n,k)}{k}\right) = O(n^{-1}), \tag{58}$$

using the technique used in proving (20).

Now by (58) and (29)

$$\sum_{n\leqslant\tau}|\eta^*(n,t)| = \sum_{n\leqslant\tau}|\xi^*(n,c)-\xi^*(n,t)|$$
$$\leqslant \sum_{n\leqslant\tau}|\xi^*(n,c)| + \sum_{n\leqslant\tau}|\xi^*(n,t)|$$
$$= O\left(\sum_{n\leqslant\tau}\frac{1}{n}\right) + O\left\{t^2\left(\log\frac{2\pi}{t}\right)^\delta \sum_{n\leqslant\tau}\frac{n}{(\log n)^\delta}\right\}$$
$$= O\left(\log\frac{2\pi}{t}\right). \tag{59}$$

Now for some $t' = t'(n)$ with $t < t' < t'(n) < c$, we have

$$\eta^*(n,t) = \int_t^c \left(\log\frac{2\pi}{u}\right)^\delta g_s(n,(\log(n+1))^{-\delta},u)\mathrm{d}u$$
$$= \left(\log\frac{2\pi}{t}\right)^\delta \{g_c(n,n^{-1}(\log(n+1))^{-\delta},t) - g_c(n,n^{-1}(\log(n+1))^{-\delta},t')\}.$$

Using (22) for the estimates of $g_c(n,n^{-1}(\log(n+1))^{-\delta},t)$ and $g_c(n,n^{-1}(\log(n+1))^{-\delta},t')$ and taking note of the fact that $t'^{-1} < t^{-1}$, we obtain

$$\eta^*(n,t) = O\left\{t^{-1}\left(\log\frac{2\pi}{t}\right)^\delta n^{-2}(\log n)^{-\delta}\right\} + O\left\{\left(\log\frac{2\pi}{t}\right)^\delta n^{-1}(\log n)^{-\delta}\rho^n(t)\right\}$$
$$+ O\left\{\left(\frac{q}{q+1}\right)^n\left(\log\frac{2\pi}{t}\right)^\delta n^{-1}(\log n)^{-\delta}\right\}$$
$$+ O\left\{\left(\log\frac{2\pi}{t}\right)^\delta n^{-1}(\log n)^{-\delta}\rho^n(t')\right\}. \tag{60}$$

Using (60), we get

$$\sum_{n>\tau}|\eta^*(n,t)| = O\left\{t^{-1}\left(\log\frac{2\pi}{t}\right)^\delta \sum_{n>\tau} n^{-2}(\log n)^{-\delta}\right\}$$
$$+ O\left\{\left(\log\frac{2\pi}{t}\right)^\delta \sum_{n>\tau}\frac{\rho^n(t)}{n(\log n)^\delta}\right\}$$
$$+ O\left\{\left(\log\frac{2\pi}{t}\right)^\delta \sum_{n>\tau}\left(\frac{q}{q+1}\right)^n\frac{1}{n(\log n)^\delta}\right\}$$
$$+ O\left\{\left(\log\frac{2\pi}{t}\right)^\delta \sum_{n>\tau}\frac{\rho^n(t')}{n(\log n)^\delta}\right\}$$
$$= O(1) + O\left(\log\frac{1}{1-\rho(t)}\right) + O\left(\log\frac{1}{1-\rho(t')}\right)$$
$$= O\left(\log\frac{2\pi}{t}\right), \tag{61}$$

since $(1-\rho(t))^{-1} = O(t^{-2})$, $(1-\rho(t'))^{-1} = O(t'^{-2})$ and $t'^{-1} < t^{-1}$. Now (57) follows immediately from (59) and (61) and this completes the proof of Theorem 2(a).

6. Proof of theorem 3

6.1. *Proof of theorem* 3. Let ψ be odd and 2π-periodic function defined by

$$\psi(t) = \begin{cases} \dfrac{\pi}{2}\left(\log\log\dfrac{k}{t}\right)^{-1}, & 0 < t < \pi \\ 0, & t \in \{0,\pi\} \end{cases}$$

where $k > \pi e$. By Lemma 3

$$B_n(x) = \int_0^\pi \frac{\sin nt}{\log\log(k/t)}\mathrm{d}t = -\frac{(-1)^n}{n\log\log(k/\pi)} + 0_n,$$

where

$$\theta_n \sim \frac{1}{n\log\log n}.$$

The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\log(n+1)} \in |E,q|$, if and only if

$$\sum_{n=1}^{\infty} |g_c(n, n^{-1}(\log(n+1))^{-1}, \pi)| = \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} V(n,k)\frac{\cos k\pi}{k\log(k+1)} \right| < \infty \tag{62}$$

Using (21), we have $g_c(n, n^{-1}(\log(n+1))^{-1}, \pi) = O(n^{-3/2}(\log n)^{-1})$ and hence (62) follows at once.

Lastly $\sum \frac{\theta_n}{\log(n+1)}$ is not summable by any totally regular method as

$$\frac{\theta_n}{\log(n+1)} \sim \frac{1}{n\log n\log\log n}.$$

This completes the proof of Theorem 3.

7. Proof of theorems 4, 5 and 6

7.1 *Proof of theorem* 4. For $n \geqslant 1$, we write

$$\begin{aligned}\frac{\pi}{2}A_n(x) &= \int_0^{1/n} \phi(t)\cos nt\,\mathrm{d}t + \int_{1/n}^{\pi} \phi(t)\cos nt\,\mathrm{d}t\\ &= \alpha_n + \beta_n.\end{aligned}$$

$\sum \frac{\alpha_n}{n^{(1/2)+\varepsilon}} \in |E,q|$, if and only if

$$\sum_{n=1}^{\infty} \left| \int_0^{1/n} \phi(t) g_c(n, n^{-(1/2)-\varepsilon}, t)\mathrm{d}t \right| < \infty.$$

Using (20), we have

$$\begin{aligned}&\sum_{n=1}^{\infty} \left| \int_0^{1/n} \phi(t) g_c(n, n^{-(1/2)-\varepsilon}, t)\mathrm{d}t \right|\\ &= O\left(\sum_{n=1}^{\infty} n^{-(1/2)-\varepsilon} \int_0^{1/n} |\phi(t)|\mathrm{d}t \right)\\ &= O\left(\sum_{n=1}^{\infty} n^{-(3/2)-\varepsilon} \right) = O(1),\end{aligned}$$

which ensures that $\sum \alpha_n/(n^{(1/2)+\varepsilon}) \in |E,q|$.

Next $\sum \beta_n/(n^{(1/2)+\varepsilon}) \in |E,q|$, if and only if

$$\sum_{n=1}^{\infty} \left| \int_{1/n}^{\pi} \phi(t) g_c(n, n^{-(1/2)-\varepsilon}, t)\mathrm{d}t \right| < \infty. \tag{63}$$

Using (21), we get

$$\sum_{n=1}^{\infty}\left|\int_{1/n}^{\pi}\Phi(t)g_c(n,n^{-(1/2)-\varepsilon},t)\mathrm{d}t\right| = O\left(\sum_{n=1}^{\infty} n^{-1-\varepsilon}\int_{1/n}^{\pi}\frac{|\phi(t)|}{t}\mathrm{d}t\right)$$

$$= O\left(\sum_{n=1}^{\infty} n^{-1-\varepsilon}\log n\right) = O(1).$$

This completes the proof of Theorem 4.

7.2. *Proof of theorem* 5. For $n \geqslant 1$, we have

$$\frac{\pi}{2}A_n(x) = \int_0^{\pi}\phi(t)\cos nt\,\mathrm{d}t.$$

The series $\sum \frac{A_n(x)}{n^{1/2}} \in |E, q|$, if and only if

$$\sum_{n=1}^{\infty}\left|\int_0^{\pi}\phi(t)\,g_c(n,n^{-1/2},t)\mathrm{d}t\right| < \infty. \tag{64}$$

Since $\phi(t)/t \in L(0,\pi)$, for the validity of (64) it suffices to show that in $0 < t < \pi$

$$\sum_{n=1}^{\infty}|g_c(n,n^{-1/2},t)| = O(t^{-1}). \tag{65}$$

Using (20) and (22), we get for $T = [t^{-2}]$

$$\sum_{n=1}^{\infty}|g_c(n,n^{-1/2},t)| = \sum_{n\leqslant T}|g_c(n,n^{-1/2},t) + \sum_{n>T}|g_c(n,n^{-1/2},t)|$$

$$= O\left(\sum_{n\leqslant T} n^{-1/2}\right) + O\left(t^{-1}\sum_{n>T} n^{-3/2}\right)$$

$$+ O\left(\sum_{n>T}\frac{\rho^n(t)}{\sqrt{n}}\right) + O\left(\sum_{n>T}\left(\frac{q}{q+1}\right)^n n^{-1/2}\right)$$

$$= O(t^{-1})$$

which ensure (65) and this completes the proof of the theorem.

7.3. *Proof of theorem* 6. The series

$$\sum \frac{A_n(x)}{n^{(1/2)-\varepsilon}} \in |E, q|$$

if and only if

$$\sum_{n=1}^{\infty}\left|\int_0^{\pi}\phi(t)\,g_c(n,n^{-1/2+\varepsilon},t)\mathrm{d}t\right| < \infty. \tag{66}$$

By an appeal to Lemma 6, we note that necessary and sufficient condition for (66) to hold whenever $O(t)/t \in L(0,\pi)$ is that

$$\operatorname{ess}\overline{bd}\sum_n t|g_c(n,n^{-(1/2)+\varepsilon},t)| \leqslant K \quad \text{for } 0 < t \leqslant \pi \tag{67}$$

where K is an absolute constant.

Thus for the proof of our theorem, it suffices to show that

$$t\sum_{n=1}^{\infty}|g_c(n,n^{-(1/2)+\varepsilon},t)|\to+\infty \text{ at } t\to 0+ \tag{68}$$

From (26), we get

$$\begin{aligned} tg_c(n,n^{-(1/2)+\varepsilon},t) &= \frac{tS_n}{(N+1)^{(1/2)-\varepsilon}}+O(n^{-(3/2)+\varepsilon}) \\ &= \frac{t}{(N+1)^{(1/2)-\varepsilon}}\left[\rho^n(t)\cos n\phi-\left(\frac{q}{q+1}\right)^n\right]+O(n^{-(3/2)+\varepsilon}) \\ &= \frac{t\rho^n(t)\cos n\phi}{(N+1)^{(1/2)-\varepsilon}}+O\left\{tn^{\varepsilon-1/2}\left(\frac{q}{q+1}\right)^n\right\}+O(n^{-(3/2)+\varepsilon}). \end{aligned}$$

As the series $\sum n^{\varepsilon-1/2}(q/q+1)^n$ and $\sum n^{-(3/2)+\varepsilon}$, $0<\varepsilon<1/2$ are both convergent, we need only show that

$$\Sigma^*=t\sum_{n=1}^{\infty}\frac{\rho^n(t)|\cos n\phi|}{(N+1)^{(1/2)-\varepsilon}}\to+\infty \text{ as } t\to 0+ \tag{69}$$

We have

$$\begin{aligned} \Sigma^* &\geqslant t\sum_{n=1}^{\infty}\frac{\rho^n(t)\cos^2 n\phi}{n^{(1/2)-\varepsilon}} \\ &= \frac{t}{2}\sum_{n=1}^{\infty}\frac{\rho^n(t)}{n^{(1/2)-\varepsilon}}+\frac{t}{2}\sum_{n=1}^{\infty}\frac{\rho^n(t)\cos 2n\phi}{n^{(1/2)-\varepsilon}} \\ &= \Sigma_1^*+\Sigma_2^*. \end{aligned} \tag{70}$$

As $\dfrac{\rho^n(t)}{n^{(1/2)-\varepsilon}}$ is a monotonic decreasing function of n for fixed t

$$\begin{aligned} \Sigma_2^* &\leqslant \frac{t\rho(t)}{2}\max_{1<M,M'<\infty}\left|\sum_{M}^{M'}\cos 2n\phi\right| \\ &\leqslant \frac{t\rho(t)}{2\sin\phi}=O(1) \end{aligned} \tag{71}$$

since $\sin t=(1+q)\rho(t)\sin\phi$.

Using (31), we obtain as $t\to 0+$

$$\Sigma_1^*\sim\frac{t}{2}\frac{\Gamma((1/2)+\varepsilon)}{(1-\rho(t))^{(1/2)+\varepsilon}}>Ct^{-2\varepsilon}$$

where C is some positive constant. Therefore

$$\Sigma_1^*\to+\infty \text{ as } t\to 0+. \tag{72}$$

Now (69) follows immediately from (70), (71) and (72) and this completes the proof of the theorem.

8. Proof of theorems 7 and 8

8.1. *Proof of theorem* 7. We assume that $g^*(t) \in L(0,\pi)$.

For $r \geqslant 1$, we have ([8], Theorem 1)

$$\begin{aligned} A_{n,r}(x) &= \frac{2}{\pi}(-1)^r \int_0^\pi \frac{1}{2}\{f(x+t)+(-1)^r f(x-t)\}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r \cos nt\,\mathrm{d}t \\ &= \frac{2}{\pi}(-1)^r \int_0^\pi \frac{1}{2}\{P(t)+(-1)^r P(-t)\}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r \cos nt\,\mathrm{d}t \\ &\quad + \frac{2}{\pi r!}(-1)^r \int_0^\pi g^*(t)t^r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r \cos nt\,\mathrm{d}t \\ &= \alpha_n + \beta_n. \end{aligned} \tag{73}$$

We first proceed to show that $\sum \alpha_n \in |E,q|, q > 0$.

Case I. Let r be odd, i.e., $r = 2m+1$ $(m = 0,1,2,\ldots)$, then

$$\begin{aligned} \alpha_n &= -\frac{2}{\pi}\int_0^\pi \frac{1}{2}\{P(t)-P(-t)\}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m+1} \cos nt\,\mathrm{d}t \\ &= \frac{2}{\pi}(-1)^m n^{2m+1}\sum_{\mu=1}^{m}\frac{\theta_{2\mu-1}}{(2\mu-1)!}\int_0^\pi t^{2\mu-1}\sin nt\,\mathrm{d}t \\ &= \frac{2}{\pi}(-1)^m n^{2m+1}\sum_{\mu=1}^{m}\frac{\theta_{2\mu-1}}{(2\mu-1)!}(-1)^n\sum_{\rho=1}^{\mu}(-1)^\rho \\ &\quad \left(\frac{(2\mu-1)!}{(2\mu-\rho)!}\pi^{2\mu-2\rho+1}n^{-2\rho+1}\right) \\ &= \frac{2}{\pi}(-1)^n\sum_{\rho=1}^{m}(-1)^{m+\rho}n^{2m-2\rho+2}\sum_{\mu=\rho}^{m}\frac{\theta_{2\mu-1}}{(2\mu-\rho)!}\pi^{2\mu-2\rho+1}. \end{aligned} \tag{74}$$

Case II. Let r be even, i.e., $r = 2m$ $(m = 1,2,\ldots)$, then

$$\begin{aligned} \alpha_n &= \frac{2}{\pi}\int_0^\pi \frac{1}{2}\{P(t)+P(-t)\}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m} \cos nt\,\mathrm{d}t \\ &= \frac{2}{\pi}(-1)^m n^{2m}\sum_{\mu=1}^{m-1}\frac{\theta_{2\mu}}{(2\mu)!}\int_0^\pi t^{2\mu}\cos nt\,\mathrm{d}t \\ &= \frac{2}{\pi}(-1)^m n^{2m}\sum_{\mu=1}^{m-1}\frac{\theta_{2\mu}}{(2\mu)!}(-1)^n\sum_{\rho=1}^{\mu}(-1)^{\rho-1}\frac{(2\mu)!}{(2\mu-2\rho+1)!}\pi^{2\mu-2\rho+1}n^{-2\rho} \\ &= \frac{2}{\pi}(-1)^n\sum_{\rho=1}^{m-1}(-1)^{m+\rho-1}n^{2m-2\rho}\sum_{\mu=\rho}^{m-1}\frac{\theta_{2\mu}}{(2\mu-2\rho+1)!}\pi^{2\mu-2\rho+1}. \end{aligned} \tag{75}$$

Taking α_n as given in (74) and (75), we see that by Lemma 5 the series $\sum \alpha_n \in |E,q|, q > 0$. Next, we write for any positive constant c, however small

$$\begin{aligned} \beta_n &= \frac{2}{\pi r!}(-1)^r\left\{\int_0^c + \int_c^\pi\right\} g^*(t)t^r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r \cos nt\,\mathrm{d}t \\ &= \beta_{n,1} + \beta_{n,2}. \end{aligned}$$

The absolute Euler summability of rth derived Fourier series becomes a local property of its generating function, if

$$g^*(t)\in L(0,\pi)\Rightarrow \sum_{n=1}^{\infty}\beta_{n,2}\in|E,q|,\quad q>0. \tag{76}$$

Now $\sum\beta_{n,2}\in|E,q|$, if and only if

$$\sum\equiv\sum_{n=1}^{\infty}\left|\sum_{k=1}^{n}V(n,k)\int_c^\pi g^*(t)t^r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r\cos kt\,\mathrm{d}t\right|<\infty.$$

We have

$$\begin{aligned}\sum&=\sum_{n=1}^{\infty}\left|\int_c^\pi g^*(t)t^r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r S_n(t)\mathrm{d}t\right|\\
&=O\left\{\int_c^\pi|g^*(t)|t^r\mathrm{d}t\sum_{n=1}^{\infty}n^r\rho^{(n-r)}(t)\right\},\text{ by Lemma 4}\\
&=O\left\{\int_c^\pi|g^*(t)|t^r\frac{1}{(1-\rho(t))^{r+1}}\mathrm{d}t\right\}\\
&=O\left\{\int_c^\pi\frac{|g^*(t)|}{t^{r+2}}\mathrm{d}t\right\}=O(1),\end{aligned}$$

since $g^*\in L(0,\pi)$ and this completes the proof of the theorem.

8.2. *Proof of theorem* 8. We assume that $h^*(t)\in L(0,\pi)$.
For $r\geqslant 1$ ([8], Theorem 2)

$$\begin{aligned}B_{n,r}(x)&=\frac{2}{\pi}\int_0^\pi(-1)^r\frac{1}{2}\{f(x+t)-(-1)^rf(x-t)\}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r\sin nt\,\mathrm{d}t\\
&=\frac{2}{\pi}(-1)^r\int_0^\pi\frac{1}{2}\{P(t)-(-1)^rP(-t)\}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r\sin nt\,\mathrm{d}t\\
&\quad+\frac{2}{\pi}(-1)^r\int_0^\pi h^*(t)t^r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r\sin nt\,\mathrm{d}t\\
&=\alpha_n+\beta_n.\end{aligned}\tag{77}$$

We claim that $\sum\alpha_n\in|E,q|$, $q>0$.

Case I. Let $r=2m+1$ $(m=0,1,2,\ldots)$, then

$$\begin{aligned}\alpha_n&=-\frac{2}{\pi}\int_0^\pi\frac{1}{2}\{P(t)+P(-t)\}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m+1}\sin nt\,\mathrm{d}t\\
&=\frac{2}{\pi}(-1)^{m+1}\sum_{\mu=1}^{m}\frac{\theta_{2\mu}}{(2\mu)!}\int_0^\pi t^{2\mu}n^{2m+1}\cos nt\,\mathrm{d}t\\
&=\frac{2}{\pi}(-1)^{m+1}\sum_{\mu=1}^{m}\frac{\theta_{2\mu}}{(2\mu)!}(-1)^n n^{2m+1}\sum_{\nu=1}^{\mu}\frac{(-1)^{\nu-1}(2\mu)!}{(2\mu-2\nu+1)!}\frac{\pi^{2\mu-2\nu+1}}{n^{2\nu}}\\
&=\frac{2}{\pi}(-1)^n\sum_{\nu=1}^{n}(-1)^{m+\nu}n^{2m-2\nu+1}\sum_{\mu=\nu}^{m}\frac{\theta_{2\mu}}{(2\mu-2\nu+1)!}\pi^{2\mu-2\nu+1}.\end{aligned}\tag{78}$$

Case II. Let $r = 2m$ $(m = 1, 2, 3, \ldots)$, then

$$\begin{aligned}\alpha_n &= \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}(P(t) - P(-t))\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m}\sin nt\,\mathrm{d}t\\ &= \frac{2}{\pi}(-1)^m n^{2m}\sum_{\mu=1}^{m}\frac{\theta_{2\mu-1}}{(2\mu-1)!}\int_0^{\pi} t^{2\mu-1}\sin nt\,\mathrm{d}t\\ &= \frac{2}{\pi}(-1)^m n^{2m}\sum_{\mu=1}^{m}\frac{\theta_{2\mu-1}}{(2\mu-1)!}(-1)^n\sum_{\nu=1}^{\mu}\frac{(2\mu-1)!}{(2\mu-2\nu+1)!}\frac{\pi^{2\mu-2\nu+1}}{n^{2\nu-1}}\\ &= \frac{2}{\pi}(-1)^n\sum_{\nu=1}^{m}(-1)^{m+\nu+1}n^{2m-2\nu+1}\sum_{\mu=1}^{m}\frac{\theta_{2\mu-1}}{(2\mu-2\nu+1)!}\pi^{2\mu-2\nu+1}. \qquad (79)\end{aligned}$$

We observe that for both odd and even n the series $\sum\alpha_n \in |E, q|$, $q > 0$ by Lemma 4.

Next, we write for $0 < c < \pi$

$$\begin{aligned}\beta_n &= \frac{2}{\pi r!}(-1)^r\left(\int_0^c + \int_c^{\pi}\right)h^*(t)t^r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^r\sin nt\,\mathrm{d}t\\ &= \beta_{n,1} + \beta_{n,2}.\end{aligned}$$

Using the technique similar to those used in the proof of Theorem 7, we can show that

$$h^*(t) \in L(0, \pi) \Rightarrow \sum\beta_{n,2} \in |E, q|, \quad q > 0.$$

This shows that absolute Euler summability of rth derived conjugate series is a local property of its generating function.

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