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Weak convergence and weak compactness in the space of almost periodic functions on the real line

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Abstract. We give necessary and sufficient conditions for sequences in the space AP(ℝ) of continuous almost periodic functions on the real line to converge in the weak topology. The abstract results are illustrated by a number of examples which show that weak convergence seems to be a rare phenomenon. We also characterize the weakly compact subsets in AP(ℝ). In particular, earlier statements made in the monograph by Dunford and Schwartz are refined and completed. We close with some open problems.

Keywords. Almost periodic functions; weak convergence and weak compactness in spaces of continuous functions.

1. Introduction

In the investigation of classical Banach spaces the characterizations of weakly convergent sequences and weakly compact subsets have always been considered an integral part of the desirable knowledge about these spaces [7, p. 374–379]. It is well-known that the investigation of weak compactness has led to some of the deepest results in functional analysis. In particular, spaces such as the space of the Bochner integrable functions, the problem has been under research for more than twenty years and it seems only now that a satisfactory answer has been obtained [4].

In this paper we call the reader's attention to the problems of characterizing weak convergence and weak compactness in the space AP(ℝ) of the complex valued continuous almost periodic functions on ℝ. In spite of the extensive literature about AP(ℝ) written over several decades these questions seem to have been addressed only in the treatise of Dunford and Schwartz [7, p. 379]. As for weak compactness, the reader is referred to a criterion for relatively weakly compact subsets in the space $B(S)$ of all bounded functions on an arbitrary set S [7, p. 280], which is not appropriate for AP(ℝ); weak convergence is characterized by one condition in an exercise [7, p. 345], which is not easy to apply. Earlier, W F Eberlein had termed sequential convergence in $C_b(\mathbb{R})$ "refractory" [8, p. 232]; this remark has retained its truth equally for the subspace AP(ℝ). In fact, a characterization of weak convergence or weak compactness in AP(ℝ) faces the following difficulties:

1. The reduction of the problem to $C(S)$ (S being a compact Háusdorff space) with the help of the isometric isomorphism $\mathrm{AP}(\mathbb{R}) \equiv C(\mathbb{R}^a)$ (where $\mathbb{R}^a$ is the Bohr compactification of ℝ) is not illuminating because of the involvedness of $\mathbb{R}^a$.
2. It seems to be impossible to make use of the representation of the dual AP(ℝ)* of AP(ℝ) as a space of finitely additive measures which was given by Hewitt [9]. In fact, obtaining a natural sufficient condition for weak convergence amounts to proving

a theorem for passing to a limit under the integral sign where the integral is taken with respect to a finitely additive measure, and the status of the theory of finitely additive measures [15] gives enough evidence that the question poses insurmountable difficulties.

With the intention to fill the gap in the literature we reinvestigate these problems. Our study shows, that, as for weak convergence, the main problem is to find conditions under which a bounded sequence (f_n) in AP($\mathbb{R}$) which converges pointwise on $\mathbb{R}$, converges to a limit f in AP($\mathbb{R}$). Considering the corresponding sequence $(\hat{f}_n)$ in $C(\mathbb{R}^a)$, we need to find conditions under which the pointwise convergence of the (f_n) on the dense subset $\mathbb{R}$ leads to pointwise convergence on the compact Hausdorff space $\mathbb{R}^a$ to a limit in $C(\mathbb{R}^a)$. Lemma 2.1 of § 2 answers this question in a more general situation. In Theorem 2.3, we characterize weak convergence in AP($\mathbb{R}$) by eight conditions. A tool which is used here is the concept of a Bohr net in $\mathbb{R}$.

We have made particular efforts to give conditions for sequences in AP($\mathbb{R}$) as functions on $\mathbb{R}$ rather than as sequences in $C(\mathbb{R}^a)$. According to Arzelà's Theorem [7, p. 268] the limit of a bounded pointwise converging sequence in $C_b(\mathbb{R})$ is continuous iff the convergence is quasiuniform on every compact subinterval. In a similar spirit, we characterize these sequences in AP($\mathbb{R}$) for which the convergence is quasiuniform and at the same time the pointwise limit belongs to AP($\mathbb{R}$); this is done in Lemma 2.2.

Section 3 contains the main theorem on weak compactness. In § 4 we illustrate the abstract results by a number of examples and counterexamples which may have interest of their own. Weak convergence is studied here also with a look at the ε-periods of the involved functions (the reader may compare Examples 4.6 and 4.7 with the somewhat surprising Proposition 4.8). Using the representation for the characters on $\mathbb{R}$ by v. Neumann and a theorem of Lindemann–Weierstrass–Kronecker on the linear independence of a certain class of real numbers over the rationals, we disprove weak convergence for some natural candidates of sequences. The given examples create the impression that weak convergence aside from trivial (e.g. norm convergent or uniformly periodic) examples seems to be a rare phenomenon.

In § 5 we collect some open questions which we hope might appeal to readers interested in classical real analysis.

The standard notation and terminology of [7] and of the literature on almost periodic functions (see e.g. [1], [3], [5], [11]–[13], [16]) has been used. In particular, AP($\mathbb{R}$) is the Banach algebra of complex valued continuous almost periodic functions on the real line $\mathbb{R}$ and $\mathbb{R}^a$ is the Bohr compactification of $\mathbb{R}$. For $f \in$ AP($\mathbb{R}$), $\hat{f}$ is its continuous extension to $\mathbb{R}^a$ and for $t \in \mathbb{R}$, $f_t(x) := f(x + t)$ will be its translate. $C_b(\mathbb{R})$ is the Banach space of bounded continuous functions on $\mathbb{R}$. For $f \in C_b(\mathbb{R})$ $\|f\|$ will be $\|f\|_\infty$.

2. Weak convergence

It follows from the characterization of weak convergence in $C(S)$ for a compact Hausdorff space S that a sequence $(f_n) \subset$ AP($\mathbb{R}$) tends to a limit $f \in$ AP($\mathbb{R}$) weakly if and only if it is bounded and $\hat{f}_n \longrightarrow \hat{f}$ pointwise on $\mathbb{R}^a$, which implies $f_n \longrightarrow f$ pointwise on a dense subset of $\mathbb{R}^a$. In the following lemma we treat a more general situation. One may compare the proof with that of [7, Theorem 14, p. 269].

Lemma 2.1. *Let T be a dense subset of a compact Hausdorff space S and let $(h_n) \subset C(S)$ be a bounded sequence converging pointwise on T to a limit function h: $T \longrightarrow \mathbb{C}$. Let e_s be the* *evaluation functional at* $s\in S$. *Then the following conditions are equivalent:*

(a) *h has an extension* $\hat{h}\in C(S)$ *and* $h_n \longrightarrow \hat{h}$ *pointwise on S.*
(b) *h has an extension* $\hat{h}\in C(S)$ *and for all nets* $(t_\alpha)\subset T$ *with* $t_\alpha\to s$ *for some* $s\in S$ *the following double limits exist and are equal:*

$$\lim_n \lim_\alpha h_n(t_\alpha)=\lim_\alpha \lim_n h_n(t_\alpha).$$

(c) *For all nets* $(t_\alpha)\subset T$ *with* $t_\alpha \longrightarrow s$ *for some* $s\in S$ *one has*

$$e_{t_\alpha} \longrightarrow e_s \text{ quasiuniformly on } \{h_n : n\in\mathbb{N}\}.$$

(d) *For all sequences* $(t_r)\subset T$ *such that* $\lim_{r\to\infty} h_n(t_r)=:L_n$ *exists for* $n\in\mathbb{N}$ *one has* $e_{t_r}\longrightarrow e_s$ *quasiuniformly on* $\{h_n : n\in\mathbb{N}\}$ *for all s such that* $L_n=h_n(s)$, $n\in\mathbb{N}$ *(and such s exist).*
(e) *For all sequences* $(t_r)\subset T$ *and all* $s\in S$ *such that*

$$\lim_{r\to\infty} h_n(t_r)=h_n(s),\quad n\in\mathbb{N}$$

the limit $\lim_{r\to\infty} h(t_r)$ *exists and*

$$\lim_{r\to\infty} h(t_r)=\lim_{n\to\infty} h_n(s).$$

(f) *Every sequence* $(t_r)\subset T$ *contains a subsequence* (t'_r) *such that*

$$\lim_{r\to\infty} h(t'_r)=\lim_{n\to\infty}\lim_{r\to\infty} h_n(t'_r).$$

Proof. (a)$\Leftrightarrow$(b) is obvious and (a)$\Rightarrow$(c) follows from Arzelà's Theorem [7, p. 268].

(c)$\Rightarrow$(d): Let $\mathscr{E}_0$ be the filter generated by the sets $E_r=\{t_r,t_{r+1},\ldots\}$, $r\in\mathbb{N}$, in S and $\mathscr{E}=\{K_\alpha:\alpha\in A\}$ be the ultrafilter refining $\mathscr{E}_0$. There exists $s\in S$ with $\mathscr{E}\longrightarrow s$. For all $\alpha\in A$, $E_1\cap K_\alpha\neq\phi$. Hence there exists some $r(\alpha)\in\mathbb{N}$ with $t_{r(\alpha)}\in K_\alpha$. If we define $\alpha\leqslant\beta$ to mean $K_\alpha\supset K_\beta$ then $(t_{r(\alpha)})_{\alpha\in A}$ is a net in T. We show that $t_{r(\alpha)}\longrightarrow s$. In fact, if U is a neighbourhood of s there exists $\alpha_0\in A$ such that $U=K_{\alpha_0}$ and for all $\alpha\geqslant\alpha_0$ we have $t_{r(\alpha)}\in K_\alpha\subset K_{\alpha_0}=U$ and the assertion follows. Now for $n\in\mathbb{N}$, $h_n(t_r)\longrightarrow L_n$. Hence $h_n(\mathscr{E}_0)\longrightarrow L_n$ and $h_n(\mathscr{E})\longrightarrow L_n$. On the other hand, $\mathscr{E}\longrightarrow s$ and $h_n(\mathscr{E})\longrightarrow h_n(s)$ by continuity. Hence $L_n=h_n(s)$. Let $\varepsilon>0$ and $r_0\in\mathbb{N}$. Then there exists α_0 with $E_{r_0}=K_{\alpha_0}$. Hence by (c) there exist $\alpha_1,\ldots,\alpha_k\geqslant\alpha_0$ such that for all $n\in\mathbb{N}$ there is $j\in\{1,\ldots,k\}$ with $|h_n(t_{r(\alpha_j)})-h_n(s)|<\varepsilon$. For $j\in\{1,\ldots,k\}$ we define $r'_j=r(\alpha_j)$ and hence

$$t_{r'_j}=t_{r(\alpha_j)}\in K_{\alpha_j}\subset K_{\alpha_0}=E_{r_0}.$$

Hence $r'_j\geqslant r_0, j=1,\ldots,k$. Thus we have shown that there exist $r'_1,\ldots,r'_k\geqslant r_0$ such that for all $n\in\mathbb{N}$ there exists $j\in\{1,\ldots,k\}$ with $|h_n(t'_{r_j})-h_n(s)|<\varepsilon$, which is (d).

(d)$\Rightarrow$(e): Any subsequence of (h_n) contains a further subsequence (again denoted by (h_n)) such that $\lim_{n\to\infty} h_n(s)=:L$ exists. Let $\varepsilon>0$ and $r_0>0$. Then by (d) there exist $r_1,\ldots,r_m\geqslant r_0$ such that for every $n\in\mathbb{N}$ there is $j\in\{1,\ldots,m\}$ with $|h_n(t_{r_j})-h_n(s)|<\varepsilon/3$. Furthermore there exists $j_0\in\mathbb{N}$ such that for all $n\geqslant j_0$ and for all $j\in\{1,\ldots,m\}$ we have $|h_n(t_{r_j})-h(t_{r_j})|<\varepsilon/3$. Also there exists $i_0\in\mathbb{N}$ such that $|h_n(s)-L|<\varepsilon/3$ for all $n\geqslant i_0$. Hence there exists an $r_j\geqslant r_0$ for some $j\in\{1,\ldots,m\}$ with $|h(t_{r_j})-L|<\varepsilon$. It follows that there exists a subsequence (t_r') of (t_r) such that $h(t_r') \longrightarrow L$. This implies that

$$\lim_{r\to\infty} h(t_r) = L = \lim_{n\to\infty} h_n(s).$$

But then this is true for the original sequence (h_n) also. This shows (e) to be true.

(e)$\Rightarrow$(f): Let $(t_r) \subset T$. There exists a subsequence (t_r') such that $\lim_{r\to\infty} h_n(t_r') = h_n(s)$ exists for $n\in\mathbb{N}$ for some $s\in S$ (see the proof of (c)$\Rightarrow$(d)). By (e) we have

$$\lim_{r\to\infty} h(t_r') = \lim_{n\to\infty} h_n(s) = \lim_{n\to\infty}\lim_{r\to\infty} h_n(t_r').$$

This is (f).

(f)$\Rightarrow$(e): Consider $(t_r) \subset T, s\in S$ such that

$$\lim_{r\to\infty} h_n(t_r) = h_n(s) \quad \text{for } n\in\mathbb{N}.$$

Let (t_r') be any subsequence of (t_r). By (f) there exists a further subsequence (t_r'') such that

$$\lim_{r\to\infty} h(t_r'') = \lim_{n\to\infty}\lim_{r\to\infty} h_n(t_r'') = \lim_{n\to\infty} h_n(s).$$

This shows that $\lim_{r\to\infty} h(t_r)$ exists and is equal to $\lim_{n\to\infty} h_n(s)$.

(e)$\Rightarrow$(a): First let S be metrizable. Let $s\in S$ and choose $(t_r) \subset T$ such that $t_r \longrightarrow s$ (such sequences exist by denseness of T). Then $h_n(t_r) \to h_n(s), n\in\mathbb{N}$ and the conclusion of (e) implies (a). The general case is reduced to the first case by considering the compact metric quotient $\tilde{S} = S/\sim$ with respect to the equivalence relation $s \sim s'$ iff $h_n(s) = h_n(s')$ for $n\in\mathbb{N}$. The canonical surjection $\sigma: S \to \tilde{S}$ is continuous. The set $\tilde{T} = \sigma T$ is dense in $\tilde{S}$, the sequence $(\tilde{h}_n) \subset C(\tilde{S})$ given by $\tilde{h}_n(\sigma s) = h_n(s)$ converges pointwise on $\tilde{T}$ to the limit $\tilde{h}: \tilde{T} \to \mathbb{C}$ given by $\tilde{h}(\sigma s) = h(s)$. $(\tilde{h}_n)$ satisfies (e) on $\tilde{S}$ and (a) follows for $(\tilde{h}_n)$ which implies that (a) holds for (h_n) on S.

Before applying Lemma 2.1 we note the following result which is in the spirit of Arzelà's Theorem.

Lemma 2.2 Let $(f_n) \subset \mathrm{AP}(\mathbb{R})$ be a bounded sequence converging pointwise on $\mathbb{R}$ to a limit function $f: \mathbb{R} \longrightarrow \mathbb{C}$. Then the convergence is quasiuniform and $f\in\mathrm{AP}(\mathbb{R})$ iff the following condition holds:

$$\text{(A)} \begin{cases} \forall \varepsilon > 0 \exists l > 0 \,\forall a\in\mathbb{R} \,\exists \tau\in[a, a+l] \,\forall n_0\in\mathbb{N} \,\exists n_1, \ldots, n_k \geqslant n_0 \,\forall x\in\mathbb{R} \\ \exists i,j\in\{1,2,\ldots,k\} \text{ with } |f_{n_i}(x) - f(x)| < \varepsilon \text{ and } |f_{n_i}(x) - f_{n_j}(x+\tau)| < \varepsilon. \end{cases}$$

Proof: If $f\in\mathrm{AP}(\mathbb{R})$, then $\forall\varepsilon > 0 \;\exists l > 0 \;\forall a\in\mathbb{R} \,\exists\tau\in[a, a+l]$: $\|f_\tau - f\| < \varepsilon/3$. If the convergence is quasiuniform, then $\forall n_0\in\mathbb{N} \;\exists n_1, \ldots, n_k \geqslant n_0 \;\forall x\in\mathbb{R} \;\exists i,j\in\{1,\ldots,k\}$ with

$$|f_{n_i}(x) - f(x)| < \frac{\varepsilon}{3} \quad \text{and} \quad |f_{n_j}(x+\tau) - f(x+\tau)| < \frac{\varepsilon}{3}.$$

This implies $|f_{n_i}(x) - f_{n_j}(x+\tau)| < \varepsilon$ and proves (A). Conversely, (A) implies that the convergence is quasiuniform and that $f\in C_b(\mathbb{R})$. To prove that $f\in\mathrm{AP}(\mathbb{R})$, let $x\in\mathbb{R}$. By (A), for all $p \in \mathbb{N}$ there exist $n_{i_p}, n_{j_p} \geqslant p$ such that

$$|f_{n_{i_p}}(x) - f(x)| < \varepsilon, \quad |f_{n_{i_p}}(x) - f_{n_{j_p}}(x+\tau)| < \varepsilon,$$

which gives $|f_{n_{j_p}}(x+\tau) - f(x)| < 2\varepsilon$. We may assume $n_{j_1} < n_{j_2} < \cdots < n_{j_p} \uparrow \infty$ as $p \to \infty$. Since $f_{n_{j_p}}(x+\tau) \to f(x+\tau)$ as $p \to \infty$ it follows that

$$|f(x+\tau) - f(x)| \leqslant 2\varepsilon \quad \text{for } x \in \mathbb{R}.$$

This proves that $f \in \mathrm{AP}(\mathbb{R})$.

The nets $(x_\alpha) \subset \mathbb{R}$ which converge to some $\xi \in \mathbb{R}^a$ in $\mathbb{R}^a$, will be called Bohr nets. Obviously, a net $(x_\alpha) \subset \mathbb{R}$ is a Bohr net iff for all $f \in \mathrm{AP}(\mathbb{R})$, $(f(x_\alpha))$ converges, or for all $\lambda \in \mathbb{R}$, $(e^{i\lambda x_\alpha})$ converges in $\mathbb{C}$, which means

$$\forall \varepsilon > 0 \exists \delta > 0 \forall \lambda \in \mathbb{R} \quad \exists \alpha_0 = \alpha_0(\varepsilon, \lambda) \quad \forall \alpha, \beta \geqslant \alpha_0:$$

$$|\lambda(x_\alpha - x_\beta)| < \delta \,(\mathrm{mod}\ 2\pi).$$

For all $\xi \in \mathbb{R}^a$ there is a Bohr net (x_α) converging to $\xi \in \mathbb{R}^a$ in $\mathbb{R}^a$. Furthermore, one sees that a net $(x_\alpha) \subset \mathbb{R}$ is a Bohr net iff (x_α) is a Cauchy net in the precompact uniform space $(\mathbb{R}, \mathscr{U})$, where the translation invariant uniformity $\mathscr{U}$ is generated by the sets $\{(s,t) \in \mathbb{R} \times \mathbb{R} : \| f_s - f_t \| < \varepsilon\}$ for $f \in \mathrm{AP}(\mathbb{R})$ and $\varepsilon > 0$.

The proof of the following theorem which characterizes weak convergence in AP($\mathbb{R}$) is clear from the preceding two lemmas. The condition (g) is similar to the one given in [7, Theorem IV. 6.31, p. 281].

Theorem 2.3. *A bounded sequence* $(f_n) \subset \mathrm{AP}(\mathbb{R})$ *is weakly convergent iff* (f_n) *converges pointwise on a dense subset D of* $\mathbb{R}$ *to a limit* $f: D \to \mathbb{C}$ *and one of the following conditions is satisfied*:

(a) f *has an extension in* AP($\mathbb{R}$) *and* $\hat{f}_n \to \hat{f}$ *pointwise on* $\mathbb{R}^a$.
(b) f *has an extension in* AP($\mathbb{R}$) *and for all Bohr nets* (x_α) *the following double limits exist and are equal*:

$$\lim_n \lim_\alpha f_n(x_\alpha) = \lim_\alpha \lim_n f_n(x_\alpha).$$

(c) *For every Bohr net* $x_\alpha \to \xi$ *for some* $\xi \in \mathbb{R}^a$ *one has*

$$e_{x_\alpha} \to e_\xi \text{ quasiuniformly on } \{\hat{f}_n : n \in \mathbb{N}\}.$$

(d) *For all sequences* $\{x_r\} \subset D$ *such that* $\lim_{r\to\infty} f_n(x_r) =: L_n$ *exists for* $n \in \mathbb{N}$ *one has*

$$e_{x_r} \to e_\xi \text{ quasiuniformly on } \{\hat{f}_n : n \in \mathbb{N}\}$$

for all $\xi \in \mathbb{R}^a$ *such that* $L_n = \hat{f}_n(\xi)$, $n \in \mathbb{N}$ *(and such* ξ *exist).*

(e) *For all sequences* $(x_r) \subset \mathbb{R}$ *and all* $\xi \in \mathbb{R}^a$ *such that* $\lim_{r\to\infty} f_n(x_r) = \hat{f}_n(\xi)$, $n \in \mathbb{N}$, *the limit* $\lim_{r\to\infty} f(x_r)$ *exists and* $\lim_{r\to\infty} f(x_r) = \lim_{n\to\infty} \hat{f}_n(\xi)$.
(f) *Every sequence* $\{x_r\} \subset D$ *contains a subsequence* $\{x'_r\}$ *such that*

$$\lim_{r\to\infty} f(x'_r) = \lim_{n\to\infty} \lim_{r\to\infty} f_n(x'_r).$$

(g) $f_n \to f$ *quasiuniformly on D together with all subsequences.*
(h) *Every subsequence of* (f_n) *satisfies condition* (A) *of Lemma* 2.2.

3. Weak compactness

We can characterize the relatively weakly compact subsets in $AP(\mathbb{R})$ as follows.

Theorem 3.1. *For a bounded subset $F \subset AP(\mathbb{R})$ the following conditions are equivalent:*

(i) *F is relatively compact in the weak topology of $AP(\mathbb{R})$.*
(ii) *For all Bohr nets (x_α) with $x_\alpha \to \xi$ one has $e_{x_\alpha} \to e_\xi$ quasiuniformly on $\hat{F}$.*
(iii) *For all countable subsets $F_0 \subset F$, all sequences $(x_r) \subset \mathbb{R}$ such that $\lim_{r\to\infty} f(x_r) =: L_f$ exists for $f \in F_0$, one has $e_{x_r} \to e_\xi$ quasiuniformly on $\hat{F}_0$ for all $\xi \in \mathbb{R}^a$ such that $L_f = \hat{f}(\xi)$ for $f \in F_0$ (and such ξ exist).*
(iv) *Every sequence in F which converges pointwise on $\mathbb{R}$ (equivalently, on a dense subset of $\mathbb{R}$) is weakly convergent in $AP(\mathbb{R})$.*

Proof. To prove (iv)$\Rightarrow$(i) we remark that each sequence $(f_n) \subset F$ contains a subsequence which converges pointwise on the set $\mathbb{Q}$ of rationals and is weakly convergent by (iv). The other implications follow from arguments already given in the proof of Lemma 2.1.

4. Further results and examples

This section is devoted to miscellaneous results and examples which throw some light on the nature of weak convergence in $AP(\mathbb{R})$.

Example 4.1. For a sequence $(a_n) \subset \mathbb{R}$, let $f_n(x) = e^{ia_n x}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. If $a_n \to a$ for some $a \in \mathbb{R}, f(x) = e^{iax}$ is the pointwise limit of f_n. By Theorem 2.3, $f_n \to f$ weakly in $AP(\mathbb{R})$ iff for all Bohr nets (x_α)

$$\lim_n \lim_\alpha e^{ia_n x_\alpha} = \lim_\alpha e^{iax_\alpha},$$

equivalently, $\chi(a_n) \to \chi(a)$ for every character χ of $\mathbb{R}$. According to a general device given by *v.* Neumann [13, Example IV, 15.4] a discontinuous character χ on $\mathbb{R}$ can be constructed as follows. Let B be a basis for $\mathbb{R}$ over the set $\mathbb{Q}$ of rationals. B contains exactly one rational which we may take as 1. Define

$$\chi(t) = \exp(i(u_1 + \cdots + u_r)),$$

if $t = u_1 t_{\alpha_1} + \cdots + u_r t_{\alpha_r}$ with $t_{\alpha_i} \in B, u_i \in \mathbb{Q}, i = 1, \ldots, r$ is the unique representation of $t \in \mathbb{R}$. There exists $a \in B \backslash \{2n\pi + \frac{1}{2}, n \in \mathbb{Z}\}$ and a sequence (a_n) in $\mathbb{Q}$ with $a_n \to \frac{a}{2}$, and we have

$$\chi(a_n) = e^{ia_n} \to e^{ia} \neq e^{1/2} = \chi(a/2).$$

For this sequence (a_n), $(e^{ia_n(\cdot)})$ does not converge weakly in $AP(\mathbb{R})$ to $e^{ia(\cdot)/2}$.

The following example shows that even for a sequence of periodic functions the concept of quasiuniform convergence and quasiuniform convergence together with all subsequences are two distinct concepts. It also illustrates the condition (h) of Theorem 2.3. For $a \in \mathbb{R}$ and $\lambda > 0$, $\phi_{a,\lambda}$ will denote the function in $C_b(\mathbb{R})$ vanishing outside $[a - \lambda, a + \lambda]$ with $\phi_{a,\lambda}(a) = 1$ and linear in between.

Example 4.2. Consider the sets

$$M_1 = \{\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \dots\}$$

$$M_3 = \{\pm 3, \pm 9, \pm 15, \pm 21, \pm 27, \dots\}$$

$$M_5 = \{\pm 9, \pm 27, \pm \dots\}$$

$$\vdots$$

$$M_2 = \{\pm 2, \pm 6, \pm 10, \pm 14, \pm 18, \dots\}$$

$$M_4 = \{\pm 6, \pm 18, \pm 30, \pm \dots\}$$

$$M_6 = \{\pm 18, \pm 54, \pm \dots\}$$

$$\vdots$$

and define the sequence (f_n) as follows:

$$f_{2n+1} = \sum_{j \in M_{2n+1}} \phi_{j,1/2},$$

$$f_{2n} = \sum_{j \in M_{2n}} \phi_{j,1/2}, \quad n \in \mathbb{N}.$$

This sequence has the following properties:

(1) $f_n \to 0$ pointwise on $\mathbb{R}$.
(2) For all $m, n \in \mathbb{N}$, $\operatorname{supp} f_{2n} \cap \operatorname{supp} f_{2m+1} = \emptyset$, hence $\min(f_{2n}, f_{2m+1}) = 0$ and $f_n \to 0$ quasiuniformly on $\mathbb{R}$.
(3) If $m, n \in \mathbb{N}$ are both even or both odd, then for $n \leqslant m$, $\operatorname{supp} f_n \supset \operatorname{supp} f_m$ and $f_n = f_m$ on $\operatorname{supp} f_m$. Then $f_{2n+1} \not\to 0$ quasiuniformly on $\mathbb{R}$. In fact, for arbitrary odd numbers $n_1 < n_2 \cdots < n_k$ and for $x \in \mathbb{R}$ with $f_{n_k}(x) = 1$, one also has $f_{n_i}(x) = 1, i = 1, \dots, k$. Similarly, $f_{2n} \not\to 0$ quasiuniformly on $\mathbb{R}$.
(4) $f_n \not\to 0$ weakly in AP($\mathbb{R}$). This follows from Theorem 2.3: One can use (g) or disprove (h) for the sequence with odd indices, $\varepsilon = 1$ and $a = 1$: For all $l > 0$ there exists an odd n_0 such that for $\tau \in [1, 1 + l]$ and all odd $n \geqslant n_0$ one has $\operatorname{supp} f_n \cap \operatorname{supp} f_{n,\tau} = \emptyset$. Together with (3) it follows that for all odd $m > n \geqslant n_0$ there exists $x \in \mathbb{R}$ with $|f_m(x) - f_n(x + \tau)| = 1$.
(5) The sequence of periods of the f_n tends to infinity.

Let $(f_n) \subset$ AP($\mathbb{R}$) be a bounded sequence which is pointwise convergent to $f: \mathbb{R} \to \mathbb{C}$. Such a sequence is strongly convergent iff it is equicontinuous and equi-almost periodic, which means

$$\bigcap_{n \in N} \{\tau \in \mathbb{R}: \|f_{n,\tau} - f_n\| < \varepsilon\}$$

is relatively dense in $\mathbb{R}$ [7, p. 345]. The equicontinuity implies the uniform convergence on every compact subinterval and then the equi-almost periodicity gives uniform convergence on $\mathbb{R}$. If one weakens equicontinuity to quasi-equicontinuity and replaces equi-almost periodicity by the following condition (B) one gets weak convergence of (f_n) in AP($\mathbb{R}$) according to the following proposition.

PROPOSITION 4.3

Let $(f_n) \subset \mathrm{AP}(\mathbb{R})$ be bounded and converge pointwise to $f:\mathbb{R}\to\mathbb{C}$. Let (f_n) be quasi-equicontinuous on every compact interval and satisfy the following condition

$$\text{(B)} \qquad \begin{cases} \forall \varepsilon > 0 \, \exists L = L(\varepsilon) > 0 \, \forall \pi \subset \mathbb{N} \text{ finite}, \quad \forall x \in \mathbb{R} \, \exists y_\pi(x) \in [0, L] \, \forall n \in \pi \\ |(f_n - f)(y_\pi(x)) - (f_n - f)(x)| < \varepsilon. \end{cases}$$

Then $f_n \to f$ weakly in $\mathrm{AP}(\mathbb{R})$.

Proof. Because the assumptions of the proposition are satisfied by any subsequence it is sufficient to show that $f_n \to f$ quasiuniformly on $\mathbb{R}$. Let $S \subset \mathbb{R}$ be any compact interval. Because (f_n) is quasi-equicontinuous on S, f is continuous on S by Lemma 2.1 (with $T = S$) and hence on $\mathbb{R}$. In particular $f_n \to f$ quasiuniformly on $[0, L]$:

$$\forall \varepsilon > 0 \, \forall n_0 \in \mathbb{N} \, \exists n_1, \ldots, n_k \geqslant n_0 \, \forall y \in [0, L] \, \exists i \in \{1, \ldots, k\} \text{ with } |f_{n_i}(y) - f(y)| < \varepsilon.$$

Hence by (B) we have

$$\forall \varepsilon > 0 \, \forall n_0 \in \mathbb{N} \, \exists n_1, \ldots, n_k \geqslant n_0 \forall x \in \mathbb{R} \exists i \in \{1, \ldots, k\} \text{ with } |(f_{n_i} - f)(x)| < 2\varepsilon.$$

This means $f_n \to f$ quasiuniformly on $\mathbb{R}$. By Theorem 2.3, $f_n \to f$ weakly in $\mathrm{AP}(\mathbb{R})$.

One can verify easily that the condition (B) is satisfied if $(f_n - f)$ is equi-almost periodic. This is the case if (f_n) is equi-almost periodic, $f_n \to f$ pointwise on $\mathbb{R}$ and f is continuous on $\mathbb{R}$. Hence we have

COROLLARY 4.4

Let $(f_n) \subset \mathrm{AP}(\mathbb{R})$ be bounded and equi-almost periodic and converge pointwise to $f \in C_b(\mathbb{R})$. Then $f_n \to f$ weakly in $\mathrm{AP}(\mathbb{R})$.

The following example shows that (B) is not necessary for weak convergence.

Example 4.5. Consider the sequence (f_n) where

$$f_n = \sum_{i=-\infty}^{\infty} \phi_{(2i+1)2^{n-1}, 1/2}, n \in \mathbb{N}.$$

Then (f_n) has the following properties:

(1) f_n has period 2^n for all $n \in \mathbb{N}$.
(2) For all $L > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have $f_n = 0$ on $[0, L]$.
(3) For all $n, m \in \mathbb{N}$, $n \neq m$, $\inf(f_n, f_m) = 0$, because the sets $\{(2i+1)2^{n-1}, i \in \mathbb{N}\}$, $n \in \mathbb{N}$, are pairwise disjoint.
(4) $f_n \to 0$ quasiuniformly on $\mathbb{R}$ together with all subsequences and hence $f_n \to 0$ weakly in $\mathrm{AP}(\mathbb{R})$.
(5) For all $L > 0$ there exists $n_L \in \mathbb{N}$ with $f_{n_L} = 0$ on $[0, L]$ and $x_L \in \mathbb{R}$ with $f_{n_L}(x_L) = 1$ and hence the condition (B) is not satisfied.

The following two examples show that boundedness, equicontinuity and pointwise convergence to a limit $f \in \mathrm{AP}(\mathbb{R})$ together with the periodicity of the functions having periods converging to a limit in $\mathbb{R}$ or diverging to infinity do not ensure weak convergence in $\mathrm{AP}(\mathbb{R})$.

Example 4.6. Define $\phi := \Sigma_{n\in\mathbb{Z}} \phi_{n+1/2,\,1/2}$. Then there exists a sequence $(\lambda_n) \subset \mathbb{R}$ such that $\lambda_n \downarrow 1$ and the functions f_n, $n\in\mathbb{N}$, defined by $f_n(x) = \phi(x/\lambda_n)$ have the following properties:

(1) (f_n) is equicontinuous.
(2) $f_n \to \phi$ pointwise on $\mathbb{R}$.
(3) $f_n \not\to \phi$ quasiuniformly on $\mathbb{R}$ and hence $f_n \not\to \phi$ weakly in AP($\mathbb{R}$). In fact, let $\lambda_1 = \frac{2}{3}$. There exist two odd positive numbers p_2, q_2 such that if we let $\lambda_2 = p_2/q_2$, then $1 < \lambda_2 < \frac{1+\lambda_1}{2}$. Then by induction we can define two sequences (p_i), (q_i) of odd positive integers such that the sequence (λ_i) defined by $\lambda_i = (p_i/q_i)\lambda_{i-1}$ satisfies $1 < \lambda_i < \frac{1+\lambda_{i-1}}{2}$, $i = 2, 3, \ldots$. It then follows that $\lambda_n \downarrow 1$. Now let $t_k = p_k p_{k-1} \cdots p_2 \lambda_1 = p_k p_{k-1} \cdots p_3 q_2 \lambda_2 = \cdots = q_k q_{k-1} \cdots q_2 \lambda_k$. Clearly $t_k \in \mathbb{N} + \frac{1}{2}$. Hence by periodicity, $\phi(t_k) = \phi(\frac{1}{2}) = 1$. But $t_k/\lambda_n \in \mathbb{N}$ and hence $f_n(t_k) = \phi(t_k/\lambda_n) = 0$ for all $n \in \{1, \ldots, k\}$. Hence the convergence is not quasiuniform.

Example 4.7. Let ϕ be as above, $f_n(x) = \phi(x/2^n)$, $n\in\mathbb{N}$. Then

(1) (f_n) is equicontinuous.
(2) $f_n \to 0$ pointwise on $\mathbb{R}$.
(3) $f_n \not\to 0$ quasiuniformly on $\mathbb{R}$. To prove this, let $t_k = \frac{2}{3}2^k$ for $k \geqslant 0$. Then it is easy to see that $t_k = \pm\frac{2}{3}$ (mod 2) according to k is even or odd. Hence for all $k \geqslant n$ we have $f_n(t_k) = \phi(\frac{2}{3}2^{k-n}) = \phi(t_{k-n}) = \phi(\pm\frac{2}{3}) = \frac{2}{3}$. Now take $\varepsilon = \frac{2}{3}$ and let $n_1, n_2, \ldots, n_k$ be arbitrary in $\mathbb{N}$. Suppose $n = \max(n_1, \ldots, n_k)$ and $k \geqslant n$. Then $f_{n_j}(t_k) = \frac{2}{3}$, $j = 1, \ldots, k$. This shows that the pointwise convergence $f_n \to 0$ is not quasiuniform on $\mathbb{R}$.

Examples 4.6 and 4.7 show that the pointwise convergence of a bounded, equicontinuous sequence of periodic functions to a continuous periodic limit function may not even imply quasiuniform convergence on $\mathbb{R}$. However, the periods of the differences $f_n - f$ in the examples tend to infinity. If these periods are bounded, one at once gets uniform convergence by the following proposition. Here for $f\in$AP($\mathbb{R}$) and $\varepsilon > 0$, $l(f, \varepsilon)$ denotes the infimum of all the ε-periods of f.

PROPOSITION 4.8

Suppose $(f_n) \subset$ AP($\mathbb{R}$) is equicontinuous on compact intervals and $f_n \to f$ pointwise on $\mathbb{R}$ with $f\in$AP($\mathbb{R}$). Suppose further that for every $\varepsilon > 0$, $L(\varepsilon) = \sup_{n\geqslant 1} l(f_n - f, \varepsilon) < \infty$. Then $f_n \to f$ uniformly on $\mathbb{R}$.

Proof. Assume $\| f_n - f \| \not\to 0$. For convenience let $\Delta_n = f_n - f$. Then for $\varepsilon > 0$ there exists $(n_i) \subset \mathbb{N}$ and $(t_i) \subset \mathbb{R}$ such that

$$|\Delta_{n_i}(t_i)| \geqslant \varepsilon \quad \text{for all } i\in\mathbb{N}. \tag{1}$$

Now for every $i\in\mathbb{N}$ there exists $\sigma_i \in [-t_i, -t_i + L(\varepsilon/2)]$ with $\|\Delta_{n_i}(\cdot + \sigma_i) - \Delta_{n_i}\| < \varepsilon/2$, which together with (1) gives

$$|\Delta_{n_i}(t_i + \sigma_i)| > \varepsilon - \varepsilon/2 = \varepsilon \quad \text{for all } i\in\mathbb{N}. \tag{2}$$

Since $0 \leqslant t_i + \sigma_i \leqslant L(\varepsilon/2)$ for all $i \geqslant n$, there exists a $y\in[0, L(\varepsilon/2)]$ and a subsequence $(\sigma_{i_k} + t_{i_k})$ with $\sigma_{i_k} + t_{i_k} \to y$ as $k \to \infty$. Since $\Delta_n \to 0$ pointwise, there exists $k_0 \in\mathbb{N}$ with

$$|\Delta_{n_{i_k}}(y)| < \varepsilon/4 \quad \text{for all } k \geqslant k_0 = k_0(y). \tag{3}$$

Now by assumption (f_n) and therefore (Δ_n) is equicontinuous on $[0, L(\varepsilon/2)]$. Hence for all sufficiently large k, $|\Delta_{n_{i_k}}(\sigma_{i_k}+t_{i_k})-\Delta_{n_{i_k}}(y)|<\varepsilon/4$. Then (3) gives $|\Delta_{n_{i_k}}(\sigma_{i_k}+t_{i_k})|<\frac{\varepsilon}{2}$ for sufficiently large k, which contradicts (2). This completes the proof.

Example 4.9. Let $\phi\in C(\mathbb{R})$ have period 2π. Let $\phi(0)=0$ and assume $\phi(x)\neq 0$ in $(0,2\pi)$. Let (λ_n) be a sequence of positive numbers with $\lambda_n\to\infty$. Define f_n on $\mathbb{R}$ by $f_n(x)=\phi(x/\lambda_n)$, $n\geqslant 1$, $x\in\mathbb{R}$. Assume: $\exists\delta\in(0,1)\,\forall i\in\mathbb{N}\;\exists\tau_i\in\mathbb{R}\;\forall n=1,\dots,i$

(C) $$\left|\frac{\tau_i}{\lambda_n}-\pi\right|<\delta\pi(\text{mod }2\pi).$$

Then $f_n\to 0$ pointwise on $\mathbb{R}$, but not quasiuniformly.

Proof. $f_n\to 0$ pointwise on $\mathbb{R}$ is trivially true. For the second assertion observe that for a fixed $i\in\mathbb{N}$ there exist $j_1,\dots,j_i\in\mathbb{Z}$ with the following property: if we define $\delta_n=\frac{\tau_i}{\lambda_n}-\pi+2\pi j_n$, then $|\delta_n|<\delta\pi$, $n=1,\dots,i$. Hence for $n=1,\dots,i$

$$\begin{aligned}|f_n(\tau_i)|&=|\phi(\tau_i/\lambda_n)|=|\phi(\pi+\delta_n)|\\&\geqslant\inf\{|\phi(\pi+x)|:|x|\leqslant\delta\pi\}=:\varepsilon.\end{aligned}$$

Then $\varepsilon>0$ since $\phi(x)\neq 0$ in $[\pi-\delta\pi,\pi+\delta\pi]$. Thus we have shown that $\exists\varepsilon>0$ $\forall i\in\mathbb{N}\,\exists\tau_i\in\mathbb{R}\;\forall n=1,\dots,i$ we have $|f_n(\tau_i)|\geqslant\varepsilon$, which proves the assertion.

Remark. In the light of Kronecker's theorem [11, Theorem 3.1] the condition (C) is satisfied if $\{\lambda_n:n\in\mathbb{N}\}$ is linearly independent over the rationals. The existence of sequences satisfying (C) is then guaranteed, for example, by the theorem of Lindemann–Weierstrass–Kronecker [2, Theorem 1.4].

We conclude this section with an example which shows that if $F\subset \mathrm{AP}(\mathbb{R})$ is weakly compact, the set of all its translates may not be so. This is in contrast with the corresponding norm case.

Example 4.10 [10]. Let (f_n), $n\geqslant 2$, be the sequence of continuous periodic functions of period 1 defined on $\mathbb{R}$ by

$$\begin{aligned}f_n(x)&=\phi_{1/n,1/n}(x)\quad && 0\leqslant x\leqslant 2/n\\&=0 && 2/n<x\leqslant 1,\end{aligned}$$

and extended by periodicity. Then $f_n\to 0$ weakly in $\mathrm{AP}(\mathbb{R})$ but the set of its translates $\{f_{n,a}:n\geqslant 2, a\in\mathbb{R}\}$ is not relatively weakly compact.

Proof. It is easy to see that $f_n\to 0$ pointwise and quasiuniformly on $\mathbb{R}$ together with all subsequences. Hence $f_n\to 0$ weakly in $\mathrm{AP}(\mathbb{R})$. For the other assertion observe that for all $n\geqslant 2$, we have $(f_n)_{1/n}(-\frac{1}{n})=0$ while $(f_n)_{1/n}(0)=1$ so that $\lim_{n\to\infty}(f_n)_{1/n}$ is not even continuous.

5. Open questions

5.1 Let $(f_n)\subset \mathrm{AP}(\mathbb{R})$ be bounded, $f\in C_b(\mathbb{R})$ and $f_n\to f$ pointwise on $\mathbb{R}$. Give necessary and sufficient conditions for (f_n) such that $f\in \mathrm{AP}(\mathbb{R})$.

5.2 Give a nontrivial illustration of condition (B) of Proposition 4.3.
5.3 Is it true that for all $\lambda_k \to \infty$ and the sequence (f_n) of Example 4.9 one has $f_n \not\to 0$ quasiuniformly on $\mathbb{R}$?
5.4 Give nontrivial examples of equi-almost periodic families in AP($\mathbb{R}$), which are not relatively norm compact.
5.5 Find a Bohr net not converging in $\mathbb{R}$.
5.6 Does there exist a strictly monotone sequence $(a_n) \subset \mathbb{R}$ such that $a_n \to a$ in $\mathbb{R}$ and $e^{ia_n(.)} \to e^{ia(.)}$ weakly in AP($\mathbb{R}$)?

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