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An axisymmetric steady-state thermoelastic problem of an external circular crack in an isotropic thick plate

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Abstract. A steady state thermoelastic mixed boundary value problem for an isotropic thick plate is considered in this paper. The faces of an external circular crack situated in the mid-plane of the plate are opened up by the application of temperature while the bounding surface of the plate are maintained at a constant zero temperature. Solution valid for large values of the ratio of the plate thickness to the diameter of the crack has been obtained. Expressions for various quantities of physical interest are derived by finding iterative solutions of the equations and the results are shown graphically.

Keywords. Axisymmetric; steady-state; external circular crack; stress-intensity factor.

1. Introduction

The strength of a material with cracks is an interesting problem in fracture as well as structural mechanics and the knowledge of the elastic stress field is potentially useful for strength estimation based upon brittle fracture theory.

Several papers have appeared which treat distributions of stress in an infinite solid due to the application of temperature or normal pressure on the faces of a flat internal circular crack (Das and Ghosh [2], Lowengrub [5], Bandyopadhyay and Das [1]). The problem of an infinite body containing an external circular crack covering the outside of a circle, due to the application of normal pressure has been considered by Uflyand [12] using toroidal coordinates and by Lowengrub and Sneddon [6] from the dual integral equation point of view. Lowengrub [7] has also solved the two-dimensional plane strain problem for an external crack $y = 0, |x| > 1$ opened up by normal pressure, using dual trigonometric equations. Distribution of stress in a thick plate containing an external circular crack opened up by the application of pressure has been considered by Dhawan [4].

This paper determines the thermoelastic stress distribution in the vicinity of an external circular crack situated in the mid-plane of an isotropic elastic plate of finite thickness and infinite radius. The temperature, the shear component of stress tensor and the normal component of displacement vector vanish over the plane boundaries while the crack is opened up by the application of a prescribed axially symmetric temperature to its faces. The method of solution is to seek suitable representations of the potential of thermoelastic displacements and the Love function and then to reduce the problem to the solution of two pairs of dual integral equations. Finally, these dual integral equations have been further reduced to Fredholm integral equations of the second kind which are solved in terms of power series. The results are illustrated by a number of diagrams (figures 2–7).

2. Basic equations of thermoelasticity

We consider the temperature and displacement fields in an isotropic elastic solid which is conducting heat. If we assume that there is symmetry about an axis, which we take to be the z-axis, then the position of a typical point of the solid may conveniently be expressed by the cylindrical polar coordinates (r, θ, z) and the displacement vector will have the components $(u_r, 0, u_z)$. The non-vanishing components of the stress tensor will be $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz}$.

In the absence of body forces or heat sources within the solid, the steady-state equations of thermoelasticity with symmetry about z-axis are (Sneddon and Berry [10], p. 125)

$$\left.\begin{aligned} 2(1-\nu)\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right)+(1-2\nu)\frac{\partial^2 u_r}{\partial z^2}+\frac{\partial^2 u_z}{\partial r\,\partial z}&=2(1+\nu)\alpha\frac{\partial T}{\partial r}\\ (1-2\nu)\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right)+2(1-\nu)\frac{\partial^2 u_z}{\partial z^2}+\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)&=2(1+\nu)\alpha\frac{\partial T}{\partial z} \end{aligned}\right] \tag{1}$$

and

$$\nabla^2 T = 0, \tag{2}$$

where $T = T(r, z)$ is the deviation of the absolute temperature of the solid from that in a state of zero stress and strain, α is the co-efficient of linear thermal expansion of the solid, ν is its Poisson ratio and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}. \tag{3}$$

3. Boundary conditions

With a suitable choice of our unit of length we can assume that the faces of the crack are described by the relations $z = 0\pm, r \geqslant 1$. The thickness of the plate is assumed to be δ-times the diameter of the crack. We suppose that there is no external force acting on the crack-faces and that the face $z = 0+, r \geqslant 1$ is heated (or cooled) exactly in the same way as the face $z = 0-$, $r \geqslant 1$. Then following Sneddon [9] we reduce the crack problem for the thick plate $r \geqslant 0, |z| \leqslant \delta$ to the mixed boundary value problem for the layer $r \geqslant 0$, $0 \leqslant z \leqslant \delta$ for which the thermal and elastic conditions are:

on $z = 0$:

$$\frac{\partial T}{\partial z}(r,0)=0, \qquad 0 \leqslant r < 1 \tag{4}$$

$$T(r,0)=f(r), \qquad 1<r<\infty \tag{5}$$

$$\sigma_{rz}(r,0)=0, \qquad 0 \leqslant r < \infty \tag{6}$$

$$u_z(r,0)=0, \qquad 0 \leqslant r < 1 \tag{7}$$

$$\sigma_{zz}(r,0)=0, \qquad 1<r<\infty \tag{8}$$

on $z = \delta$:

$$T(r,\delta)=0, \qquad 0 \leqslant r < \infty \tag{9}$$

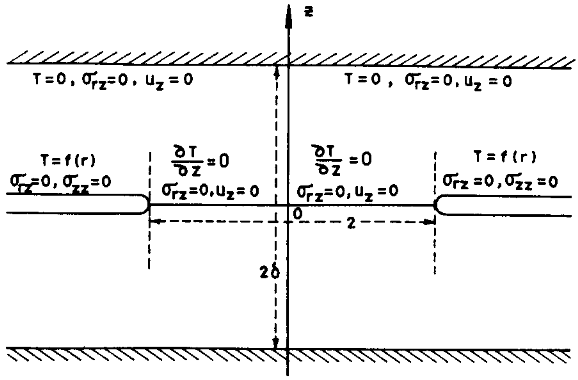

Figure 1.

$$\sigma_{rz}(r,\delta)=0, \qquad 0\leqslant r<\infty \tag{10}$$

$$u_z(r,\delta)=0, \qquad 0\leqslant r<\infty, \tag{11}$$

where $f(r)$ is prescribed.

We further assume that the disturbance is localized i.e. the temperature and the components of stress and displacement all vanish as $\sqrt{(r^2+z^2)}\to\infty$. Position of the crack and the boundary conditions for the plate are indicated in figure 1.

4. The heat conduction problem

A suitable Hankel integral representation of the temperature field satisfying the Laplace's equations (2) and (9) and vanishing at infinity is taken in the form

$$T(r,z)=\int_0^\infty \xi^{-1}B(\xi)\sinh\xi(\delta-z)\operatorname{sech}(\xi\delta)J_0(\xi r)\mathrm{d}\xi, \tag{12}$$

where $B(\xi)$ is an unknown function to be determined from the boundary conditions.

Conditions (4) and (5) are fulfilled if the function $B(\xi)$ is a solution of the set of dual integral equations

$$\int_0^\infty B(\xi)J_0(\xi r)\mathrm{d}\xi=0, \quad 0\leqslant r<1 \tag{13}$$

$$\int_0^\infty \xi^{-1}B(\xi)[1-H_1(\xi\delta)]J_0(\xi r)\mathrm{d}\xi=f(r), \quad 1<r<\infty, \tag{14}$$

where

$$H_1(\xi\delta)=1-\tanh(\xi\delta). \tag{15}$$

To reduce the above equations to a single integral equation, we apply Sneddon's method [11] and put

$$B(\xi)=\xi\int_1^\infty \psi_1(t)\sin(\xi t)\mathrm{d}t, \tag{16}$$

where, for the convergence of the integral, we assume that

$$\lim_{t\to\infty}\psi_1(t)=0. \tag{17}$$

Integrating by parts and making use of (17) we rewrite (16) in the form

$$B(\xi) = \psi_1(1)\cos\xi + \int_1^\infty \psi_1'(t)\cos(\xi t)\mathrm{d}t, \tag{18}$$

where the prime (′) denotes differentiation.

Substituting from (18) and making use of the result ([13] p. 405)

$$\int_0^\infty J_0(\xi r)\cos(\xi t)\mathrm{d}\xi = \begin{cases} 0, & r<t \\ \dfrac{1}{\sqrt{(r^2-t^2)}}, & r>t \end{cases} \tag{19}$$

we can show that

$$\int_0^\infty B(\xi)J_0(\xi r)\mathrm{d}\xi = \begin{cases} 0, & 0\leqslant r<1 \\ \dfrac{\psi_1(1)}{\sqrt{(r^2-1)}} + \displaystyle\int_1^r \dfrac{\psi_1'(t)}{\sqrt{(r^2-t^2)}}\mathrm{d}t, & 1<r<\infty. \end{cases} \tag{20}$$

It is clear from (20) that the form (16) satisfies (13). Now, from (14) we have

$$\int_1^\infty \psi_1(t)\mathrm{d}t \int_0^\infty \sin(\xi t)J_0(\xi r)\mathrm{d}\xi - \int_1^\infty \psi_1(t)\mathrm{d}t \int_0^\infty H_1(\xi\delta)J_0(\xi r)\sin(\xi t)\mathrm{d}\xi$$
$$= f(r), \quad 1<r<\infty. \tag{21}$$

Making use of the result [13], p. 405

$$\int_0^\infty J_0(\xi r)\sin(\xi t)\mathrm{d}\xi = \begin{cases} \dfrac{1}{\sqrt{(t^2-r^2)}}, & r<t \\ 0, & t<r \end{cases} \tag{22}$$

we find from (21)

$$\int_r^\infty \frac{\psi_1(t)\mathrm{d}t}{\sqrt{(t^2-r^2)}} - \int_1^\infty \psi_1(t)\mathrm{d}t \int_0^\infty H_1(\xi\delta)J_0(\xi r)\sin(\xi t)\mathrm{d}\xi = f(r), \quad 1<r<\infty. \tag{23}$$

If we replace $J_0(\xi r)$ by its integral representation,

$$\frac{2}{\pi}\int_r^\infty \frac{\sin(\xi u)\mathrm{d}u}{\sqrt{(u^2-r^2)}}$$

we find that the second term on LHS is equal to

$$\frac{2}{\pi}\int_1^\infty \psi_1(t)\mathrm{d}t \int_0^\infty H_1(\xi\delta)\sin(\xi t)\mathrm{d}\xi \int_r^\infty \frac{\sin(\xi u)\mathrm{d}u}{\sqrt{(u^2-r^2)}}.$$

Simplifying and interchanging the order of integrations the second term on LHS becomes

$$\frac{1}{\pi\delta}\int_r^\infty \frac{\mathrm{d}t}{\sqrt{(t^2-r^2)}}\int_1^\infty \psi_1(u)\{H_1^*(t-u)-H_1^*(t+u)\}\mathrm{d}u$$

where

$$H_1^*(\omega) = \int_0^\infty H_1(u)\cos\left(\frac{u\omega}{\delta}\right)du. \tag{24}$$

Then from (23) we have

$$\int_r^\infty \frac{\psi_1(t)dt}{\sqrt{(t^2-r^2)}} - \frac{1}{\pi\delta}\int_r^\infty \frac{dt}{\sqrt{(t^2-r^2)}}\int_1^\infty \psi_1(u)\{H_1^*(t-u) - H_1^*(t+u)\}du$$
$$= f(r), \quad 1 < r < \infty$$

or,

$$\int_r^\infty \frac{dt}{\sqrt{(t^2-r^2)}}\left[\psi_1(t) - \frac{1}{\pi\delta}\int_1^\infty \psi_1(u)\{H_1^*(t-u) - H_1^*(t+u)\}du\right]$$
$$= f(r), \quad 1 < r < \infty,$$

which on inversion gives

$$\psi_1(t) - \frac{1}{\pi\delta}\int_1^\infty \psi_1(u)\{H_1^*(t-u) - H_1^*(t+u)\}du$$
$$= -\frac{2}{\pi}\frac{d}{dt}\int_t^\infty \frac{rf(r)}{\sqrt{(r^2-t^2)}}dr,$$

or,

$$\psi_1(t) - \frac{1}{\pi\delta}\int_1^\infty \psi_1(u)K_1(u,t)du = -\frac{2}{\pi}\frac{d}{dt}\int_t^\infty \frac{rf(r)}{\sqrt{(r^2-t^2)}}dr, \quad 1 < t < \infty, \tag{25}$$

where

$$K_1(u,t) = H_1^*(t-u) - H_1^*(t+u). \tag{26}$$

5. The thermoelastic problem

The potential Φ of thermoelastic displacement satisfying the Poisson equation (Nowacki [8], p. 12) $\nabla^2\Phi = mT$, where $m = (1+\nu)\alpha/(1-\nu)$, is

$$\Phi(r,z) = -\frac{1}{2}m\int_0^\infty \xi^{-3}B(\xi)\operatorname{sech}(\xi\delta)[\sinh\xi(\delta-z)$$
$$+ z\xi\cosh\xi(\delta-z)]J_0(\xi r)d\xi. \tag{27}$$

The Love function Ψ satisfying the biharmonic equation (Nowacki [8], p. 17) $\nabla^4\Psi = 0$, is sought in the form of the Hankel integral

$$\Psi(r,z) = -\int_0^\infty \xi^{-2}C(\xi)\operatorname{cosech}(\xi\delta)[2\nu\sinh\xi(\delta-z)$$
$$+ z\xi\cosh\xi(\delta-z) - \delta\xi\sinh(\xi z)\operatorname{cosech}(\xi\delta)]J_0(\xi r)d\xi \tag{28}$$

which vanishes at infinity.

Using basic equations, we have

$$u_z = \frac{1}{2}mz\int_0^\infty \xi^{-1}B(\xi)\operatorname{sech}(\xi\delta)\sinh\xi(\delta-z)J_0(\xi r)d\xi$$

$$+\int_0^\infty C(\xi)\operatorname{cosech}(\xi\delta)[2(1-\nu)\sinh\xi(\delta-z)+z\xi\cosh\xi(\delta-z)$$
$$-\delta\xi\sinh(\xi z)\operatorname{cosech}(\xi\delta)]J_0(\xi r)\mathrm{d}\xi \tag{29}$$

$$\sigma_{zz}=-m\mu\int_0^\infty \xi^{-1}B(\xi)\operatorname{sech}(\xi\delta)[\sinh\xi(\delta-z)+z\xi\cosh\xi(\delta-z)]J_0(\xi r)\mathrm{d}\xi$$
$$-2\mu\int_0^\infty \xi C(\xi)\operatorname{cosech}(\xi\delta)[\cosh\xi(\delta-z)+z\xi\sinh\xi(\delta-z)$$
$$+\delta\xi\cosh(\xi z)\operatorname{cosech}(\xi\delta)]J_0(\xi r)\mathrm{d}\xi \tag{30}$$

$$\sigma_{rz}=-m\mu z\int_0^\infty B(\xi)\operatorname{sech}(\xi\delta)\sinh\xi(\delta-z)J_1(\xi r)\mathrm{d}\xi$$
$$-2\mu\int_0^\infty \xi C(\xi)\operatorname{cosech}(\xi\delta)[z\xi\cosh\xi(\delta-z)$$
$$-\delta\xi\sinh(\xi z)\operatorname{cosech}(\xi\delta)]J_1(\xi r)\mathrm{d}\xi. \tag{31}$$

Equations (6), (10) and (11) are automatically satisfied. Using boundary conditions (7) and (8) we get,

$$\int_0^\infty C(\xi)J_0(\xi r)\mathrm{d}\xi=0,\quad 0\leqslant r<1 \tag{32}$$

$$\int_0^\infty \xi C(\xi)[1+H_2(\xi\delta)]J_0(\xi r)\mathrm{d}\xi=-\frac{m}{2}f(r),\quad 1<r<\infty \tag{33}$$

where

$$1+H_2(\xi\delta)=\frac{\cosh(\xi\delta)\sinh(\xi\delta)+\xi\delta}{\sinh^2(\xi\delta)}. \tag{34}$$

Following Lowengrub and Sneddon [6] we put

$$C(\xi)=\int_1^\infty \psi_2(t)\cos(\xi t)\mathrm{d}t, \tag{35}$$

where we assume that

$$\lim_{t\to\infty}\psi_2(t)=0. \tag{36}$$

Integrating by parts and making use of (36) we rewrite (35) in the form

$$C(\xi)=-\frac{\psi_2(1)\sin\xi}{\xi}-\int_1^\infty \psi_2'(t)\frac{\sin(\xi t)}{\xi}\mathrm{d}t, \tag{37}$$

where the prime (′) denotes differentiation. Substituting (35) and making use of the result (19) we have

$$\int_0^\infty C(\xi)J_0(\xi r)\mathrm{d}\xi=\begin{cases}0, & 0\leqslant r<1\\ \displaystyle\int_1^r\frac{\psi_2(t)\mathrm{d}t}{\sqrt{(r^2-t^2)}}, & 1<r<\infty.\end{cases} \tag{38}$$

It is clear from (38) that the form (35) satisfies (32). From (33) we have

$$\int_0^\infty \xi J_0(\xi r)\mathrm{d}\xi \int_1^\infty \psi_2(t)\cos(\xi t)\mathrm{d}t + \int_0^\infty \xi H_2(\xi\delta)J_0(\xi r)\mathrm{d}\xi$$
$$\times \int_1^\infty \psi_2(t)\cos(\xi t)\mathrm{d}t = -\frac{m}{2}f(r), \quad 1<r<\infty. \tag{39}$$

The first term on the LHS of the above integral equation (39) becomes

$$-\int_r^\infty \frac{\psi_2'(t)\mathrm{d}t}{\sqrt{(t^2-r^2)}}.$$

Replacing $J_0(\xi r)$ by its integral representation

$$\frac{2}{\pi}\int_r^\infty \frac{\sin(\xi t)\mathrm{d}t}{\sqrt{(t^2-r^2)}}$$

the second term on the LHS of the above integral equation (39) becomes

$$\frac{2}{\pi}\int_0^\infty \xi H_2(\xi\delta)\mathrm{d}\xi \int_r^\infty \frac{\sin(\xi t)\mathrm{d}t}{\sqrt{(t^2-r^2)}}\int_1^\infty \psi_2(u)\cos(\xi u)\mathrm{d}u.$$

Interchanging the order of integration and simplifying the above term becomes

$$\frac{1}{\pi\delta}\int_r^\infty \frac{\mathrm{d}t}{\sqrt{(t^2-r^2)}}\int_1^\infty \left[H_2^*(t+u)+H_2^*(t-u)\right]\psi_2(u)\mathrm{d}u,$$

where

$$H_2^*(\omega) = \int_0^\infty \frac{\xi}{\delta}H_2(\xi)\sin\left(\frac{\xi\omega}{\delta}\right)\mathrm{d}\xi. \tag{40}$$

Thus (39) becomes

$$-\int_r^\infty \frac{\psi_2'(t)\mathrm{d}t}{\sqrt{(t^2-r^2)}} + \frac{1}{\pi\delta}\int_r^\infty \frac{\mathrm{d}t}{\sqrt{(t^2-r^2)}}\int_1^\infty [H_2^*(t+u)+H_2^*(t-u)]\psi_2(u)\mathrm{d}u$$
$$= -\frac{m}{2}f(r), \quad 1<r<\infty.$$

or,

$$\int_r^\infty \frac{\mathrm{d}t}{\sqrt{(t^2-r^2)}}\left[-\psi_2'(t)+\frac{1}{\pi\delta}\int_1^\infty [H_2^*(t+u)+H_2^*(t-u)]\psi_2(u)\mathrm{d}u\right]$$
$$= -\frac{m}{2}f(r), \quad 1<r<\infty,$$

which on inversion gives

$$-\psi_2'(t)+\frac{1}{\pi\delta}\int_1^\infty [H_2^*(t+u)+H_2^*(t-u)]\psi_2(u)\mathrm{d}u = \frac{m}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{rf(r)\mathrm{d}r}{\sqrt{(r^2-t^2)}},$$
$$1<t<\infty. \tag{41}$$

Assuming that $f(r)$ is continuous differentiable in $(1,\infty)$, we integrate (41) between the limits t to ∞ and on making use of (36), we obtain the following Fredholm integral equation of the second kind

$$\psi_2(t)+\frac{1}{\pi\delta}\int_1^\infty K_2(u,t)\psi_2(u)\mathrm{d}u=-\frac{m}{\pi}\int_t^\infty \frac{rf(r)\mathrm{d}r}{\sqrt{(r^2-t^2)}} \tag{42}$$

where

$$K_2(u,t)=\int_t^\infty [H_2^*(t_1+u)+H_2^*(t_1-u)]\mathrm{d}t_1. \tag{43}$$

6. Method of solution

Assuming that $\delta \gg 1$, we can write (15) and (34) as

$$H_1(\xi\delta)=2\sum_1^\infty(-1)^{n-1}\mathrm{e}^{-2n\xi\delta} \tag{44}$$

and

$$H_2(\xi\delta)=2\sum_1^\infty(1+2n\xi\delta)\mathrm{e}^{-2n\xi\delta}. \tag{45}$$

Using (24) and (44) we have from (26)

$$K_1(u,t)=2\left[\frac{ut}{\delta^2}H_{12}-\frac{ut^3+u^3t}{6\delta^4}H_{14}+\frac{3ut^5+10u^3t^3+3u^5t}{360\delta^6}H_{16}+\cdots\right] \tag{46}$$

where

$$H_{1n}=\int_0^\infty H_1(\omega)\omega^n\mathrm{d}\omega. \tag{47}$$

To solve the Fredholm integral equation (25) we assume a series solution in the form

$$\psi_1(t)=\psi_{10}(t)+\frac{1}{\delta}\psi_{11}(t)+\frac{1}{\delta^2}\psi_{12}(t)+\frac{1}{\delta^3}\psi_{13}(t)+\cdots \tag{48}$$

Then from the Fredholm integral equation (25), we have

$$\left.\begin{aligned}
\psi_{10}(t)&=-\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty\frac{rf(r)\mathrm{d}r}{\sqrt{(r^2-t^2)}}\\
\psi_{11}(t)&=0\\
\psi_{12}(t)&=0\\
\psi_{13}(t)&=\frac{2}{\pi}H_{12}\int_1^\infty ut\,\psi_{10}(u)\mathrm{d}u\\
\psi_{14}(t)&=0\\
\psi_{15}(t)&=-\frac{1}{3\pi}H_{14}\int_1^\infty(ut^3+u^3t)\psi_{10}(u)\mathrm{d}u\\
\psi_{16}(t)&=\frac{2}{\pi}H_{12}\int_1^\infty ut\,\psi_{13}(u)\mathrm{d}u
\end{aligned}\right\} \tag{49}$$

etc.

Similarly for the Fredholm integral equation (42), we assume a series solution in the form

$$\psi_2(t) = \psi_{20}(t) + \frac{1}{\delta}\psi_{21}(t) + \frac{1}{\delta^2}\psi_{22}(t) + \frac{1}{\delta^3}\psi_{23}(t) + \cdots \tag{50}$$

and we obtain a set of equations of the form (49).

7. Solution for a particular type of temperature distribution: Quantities of physical interest

In this section we solve the integral equations (25) and (42) for large values of δ, by giving a particular value of $f(r)$ which is important from the physical point of view.

Let $f(r)$ be defined as

$$f(r) = -f_0 H(a-r), \quad a > 1 \tag{51}$$

where $H(t)$ is the Heaviside unit function.

Then

$$\psi_{10}(t) = \begin{cases} -\dfrac{2f_0}{\pi}\dfrac{t}{\sqrt{(a^2-t^2)}} & t < a \\ 0 & t > a. \end{cases} \tag{52}$$

Substituting this value in (25) we get

i) For $t > a$:

$$\psi_1(t) - \frac{1}{\pi\delta}\int_1^\infty \psi_1(u) K_1(u,t)\mathrm{d}u = 0. \tag{53}$$

It can be shown that its trivial solution is

$$\psi_1(t) = 0. \tag{54}$$

ii) For $t < a$:

In this case integral equation (25) becomes

$$\psi_1(t) - \frac{1}{\pi\delta}\int_1^\infty \psi_1(u) K_1(u,t)\mathrm{d}u = -\frac{2f_0}{\pi}\frac{t}{\sqrt{(a^2-t^2)}}, \tag{55}$$

which on considering terms up to δ^{-6} gives

$$\psi_{10}(t) = -\frac{2f_0}{\pi}\frac{t}{\sqrt{(a^2-t^2)}}$$

$$\psi_{11}(t) = 0$$

$$\psi_{12}(t) = 0$$

$$\psi_{13}(t) = -\frac{4f_0 H_{12}}{\pi^2}\left(\frac{a^2}{2}\cos^{-1}\frac{1}{a} + \frac{\sqrt{(a^2-1)}}{2}\right)t$$

$$\psi_{14}(t) = 0$$

$$\psi_{15}(t)=\frac{2f_0H_{14}}{3\pi^2}\left[t^3\left(\frac{a^2}{2}\cos^{-1}\frac{1}{a}+\frac{\sqrt{(a^2-1)}}{2}\right)\right.$$
$$\left.+t\left(\frac{3a^4}{8}\cos^{-1}\frac{1}{a}+\frac{(3a^2+2)\sqrt{(a^2-1)}}{8}\right)\right]$$
$$\psi_{16}(t)=-\frac{8f_0H_{12}^2}{3\pi^3}(a^3-1)\left(\frac{a^2}{2}\cos^{-1}\frac{1}{a}+\frac{\sqrt{(a^2-1)}}{2}\right)t.$$

Substituting the above values for $\psi_1(t)$ we have from (12)

$$T(r,0)=-\frac{2f_0}{\pi}\left[\sin^{-1}\sqrt{\left(\frac{a^2-1}{a^2-r^2}\right)}-\frac{H_{11}}{\delta^2}A_{11}+\frac{2H_{12}A_{11}Q(r)}{\pi\delta^3}\right.$$
$$+\frac{H_{13}}{6\delta^4}\left(A_{12}+\frac{3r^2}{2}A_{11}\right)-\frac{1}{\delta^5}\left\{\frac{2H_{12}H_{11}A_{11}(a^3-1)}{3\pi}\right.$$
$$\left.+\frac{H_{14}A_{11}}{3\pi}\left(Q(r)r^2+\frac{R(r)}{3}\right)+\frac{H_{14}A_{12}Q(r)}{3\pi}\right\}$$
$$+\frac{1}{\delta^6}\left\{\frac{4H_{12}^2A_{11}(a^3-1)Q(r)}{3\pi^2}\right.$$
$$\left.\left.-\frac{H_{15}}{120}\left(A_{13}+5r^2A_{12}+\frac{15}{8}r^4A_{11}\right)\right\}\right],\quad 0\leqslant r<1, \tag{56}$$

where

$$\left.\begin{aligned}A_{11}&=\frac{a^2}{2}\cos^{-1}\frac{1}{a}+\frac{\sqrt{(a^2-1)}}{2}\\A_{12}&=\frac{3a^4}{8}\cos^{-1}\frac{1}{a}+\frac{(3a^2+2)\sqrt{(a^2-1)}}{8}\\A_{13}&=\frac{5a^6}{16}\cos^{-1}\frac{1}{a}+\frac{(15a^4+10a^2+8)\sqrt{(a^2-1)}}{48}\end{aligned}\right\} \tag{57}$$

and

$$\left.\begin{aligned}Q(r)&=\sqrt{(a^2-r^2)}-\sqrt{(1-r^2)}\\R(r)&=(a^2-r^2)^{3/2}-(1-r^2)^{3/2}\end{aligned}\right\}. \tag{58}$$

Similarly we get a trivial solution $\psi_2(t)=0$ of the Fredholm integral equation (42), for $t>a$.

ii)′ For $t<a$:
In this case

$$K_2(u,t)=2\left[\frac{H_{22}(a^2-t^2)}{2\delta^2}-\frac{H_{24}}{6\delta^4}\left\{\frac{a^4-t^4}{4}+\frac{3u^2}{2}(a^2-t^2)\right\}\right.$$
$$\left.+\frac{H_{26}}{120\delta^6}\left\{\frac{a^6-t^6}{6}-\frac{10u^2}{4}(a^4-t^4)+\frac{5u^4}{2}(a^2-t^2)\right\}+\cdots\right] \tag{59}$$

where

$$H_{2n}=\int_0^\infty p^nH_2(p)\mathrm{d}p. \tag{60}$$

Using the method used earlier an iterative solution for $\psi_2(t)$ is obtained in the form

$$\psi_2(t)=\frac{mf_0}{\pi}\left[\sqrt{(a^2-t^2)}-\frac{H_{22}}{\pi\delta^3}B_{11}(a^2-t^2)+\frac{H_{24}}{3\pi\delta^5}\left\{\frac{a^4-t^4}{4}B_{11}+\frac{3(a^2-t^2)}{2}B_{12}\right\}+\frac{H_{22}^2}{3\pi^2\delta^6}B_{11}(a-1)(2a^2-a-1)(a^2-t^2)\right], \tag{61}$$

where

$$B_{11}=\frac{a^2}{2}\cos^{-1}\frac{1}{a}-\frac{\sqrt{(a^2-1)}}{2},\quad B_{12}=\frac{a^4}{8}\cos^{-1}\frac{1}{a}+\frac{(a^2-2)\sqrt{(a^2-1)}}{8}. \tag{62a}$$

Now we derive expressions for quantities of physical interest.

Using (35) in (29) we have on the crack plane $z=0$,

$$u_z(r,0)=2(1-\nu)\int_1^a\psi_2(t)\mathrm{d}t\int_0^\infty J_0(\xi r)\cos(\xi t)\mathrm{d}\xi. \tag{63}$$

Now substituting the value of $\psi_2(t)$ from (61), we can easily find that

$$u_z(r,0)=\begin{cases}\dfrac{2(1-\nu)mf_0}{\pi}\left[a\left\{E\left(\dfrac{r}{a}\right)-E\left(\alpha,\dfrac{r}{a}\right)\right\}-\dfrac{H_{22}B_{11}}{\pi\delta^3}P(r)+\dfrac{H_{24}}{3\pi\delta^5}\left\{\dfrac{B_{11}}{4}S(r)+\dfrac{3B_{12}}{2}P(r)\right\}+\dfrac{H_{22}^2B_{11}(a-1)(2a^2-a-1)}{3\pi^2\delta^6}P(r)\right], & 1<r<a\\[2ex] \dfrac{2(1-\nu)mf_0}{\pi}\left[r\left\{E\left(\dfrac{a}{r}\right)-E\left(\beta,\dfrac{a}{r}\right)\right\}-\dfrac{r^2-a^2}{r}\left\{K\left(\dfrac{a}{r}\right)-F\left(\beta,\dfrac{a}{r}\right)\right\}-\dfrac{H_{22}B_{11}}{\pi\delta^3}M(r)+\dfrac{H_{24}}{3\pi\delta^5}\left\{\dfrac{B_{11}}{4}N(r)+\dfrac{3B_{12}}{2}M(r)\right\}+\dfrac{H_{22}^2B_{11}}{3\pi^2\delta^6}(a-1)(2a^2-a-1)M(r)\right], & r>a,\end{cases} \tag{64}$$

where

$$\alpha=\sin^{-1}\frac{1}{r},\quad \beta=\sin^{-1}\frac{1}{a} \tag{65}$$

$$\left.\begin{aligned}P(r)&=\left(a^2-\frac{r^2}{2}\right)\cos^{-1}\frac{1}{r}-\frac{\sqrt{(r^2-1)}}{2}\\ S(r)&=\left(a^4-\frac{3r^4}{8}\right)\cos^{-1}\frac{1}{r}-\frac{(3r^2+2)\sqrt{(r^2-1)}}{8}\\ M(r)&=\left(a^2-\frac{r^2}{2}\right)\left(\sin^{-1}\frac{a}{r}-\sin^{-1}\frac{1}{r}\right)+\frac{1}{2}(a\sqrt{(r^2-a^2)}-\sqrt{(r^2-1)})\\ N(r)&=\left(a^4-\frac{3r^4}{8}\right)\left(\sin^{-1}\frac{a}{r}-\sin^{-1}\frac{1}{r}\right)+\frac{a(3r^2+2a^2)\sqrt{(r^2-a^2)}}{8}-\frac{(3r^2+2)\sqrt{(r^2-1)}}{8}\end{aligned}\right\} \tag{66}$$

and

$E(a/r)$, $E(\beta, a/r)$, $K(a/r)$, $F(\beta, a/r)$ are elliptic integrals.

The normal component of stress on $z=0$ is given by

$$\sigma_{zz}(r,0) = -m\mu\, T(r,0) + \frac{2m\mu f_0}{\pi\sqrt{(1-r^2)}}\Bigg[\sqrt{(a^2-1)} - \frac{H_{22}B_{11}}{\pi\delta^3}(a^2-1) + \frac{H_{24}}{3\pi\delta^5}\left\{\frac{a^4-1}{4}B_{11} + \frac{3(a^2-1)}{2}B_{12}\right\} + \frac{H_{22}^2 B_{11}}{3\pi^2\delta^6}(a-1)^2(a+1)(2a^2-a-1)\Bigg]$$
$$-\frac{2m\mu f_0}{\pi}\Bigg[\sin^{-1}\sqrt{\left(\frac{a^2-1}{a^2-r^2}\right)} - \frac{2H_{22}B_{11}}{\pi\delta^3}Q(r) + \frac{H_{24}}{3\pi\delta^5}\left\{B_{11}\left(r^2 Q(r) + \frac{1}{3}R(r)\right) + 3B_{12}Q(r)\right\} + \frac{2H_{22}^2 B_{11}(a-1)(2a^2-a-1)}{3\pi^2\delta^6}Q(r)\Bigg] - \frac{2m\mu f_0}{\pi\delta^2}\Bigg[H_{21}B_{11}$$
$$-\frac{H_{23}}{2\delta^2}\left(B_{12} + \frac{r^2}{2}B_{11}\right) + \frac{H_{25}}{24\delta^4}\left(B_{13} + 3r^2 B_{12} + \frac{3r^4}{8}B_{11}\right) - \frac{H_{21}H_{22}B_{11}}{3\pi\delta^3}(a-1)(2a^2-a-1)\Bigg], \quad 0 \leqslant r < 1, \tag{67}$$

where

$$B_{13} = \frac{a^6}{16}\cos^{-1}\frac{1}{a} + \frac{(3a^4+2a^2-8)}{48}\sqrt{(a^2-1)}. \tag{62b}$$

The stress intensity factor is given by

$$N = \lim_{r\to 1-} \sqrt{(1-r)}\,\sigma_{zz}(r,0). \tag{68}$$

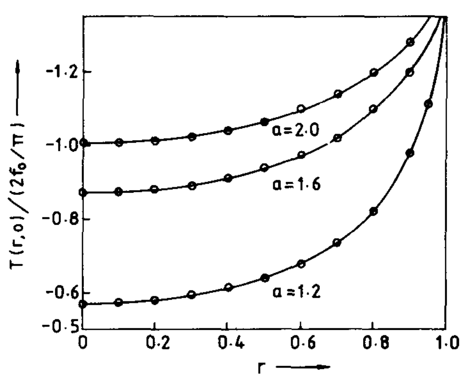

Figure 2. Variation of $T(r,0)/(2f_0/\pi)$ with r for $a = 1{\cdot}2$, $1{\cdot}6$, $2{\cdot}0$ and $\delta = 5$.

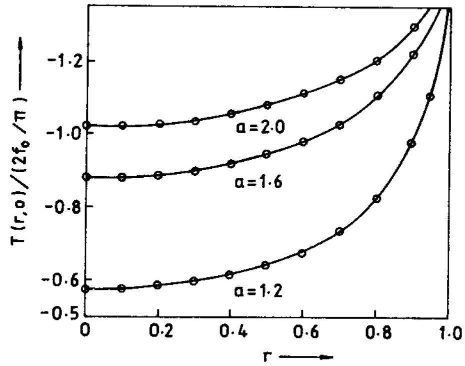

Figure 3. Variation of $T(r,0)/(2f_0/\pi)$ with r for $a = 1.2, 1.6, 2.0$ and $\delta = 7$.

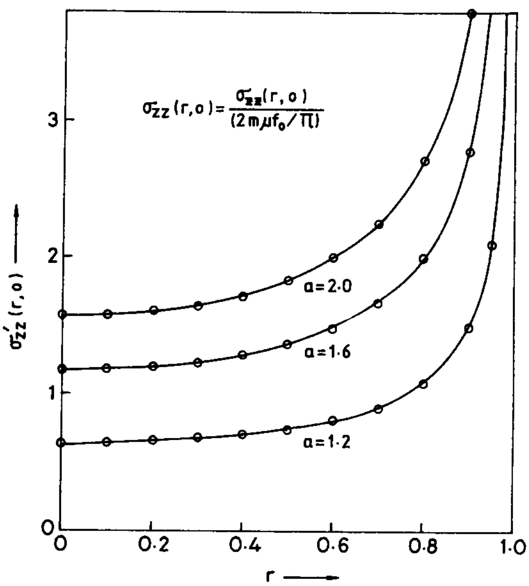

Figure 4. Variation of $\sigma_{zz}(r,0)/(2\,\mathrm{m}\mu f_0/\pi)$ with r for $a = 1.2, 1.6, 2.0$ and $\delta = 5$.

Using (67) we have from (68)

$$N = \frac{\sqrt{2}m\mu f_0}{\pi}\left[\sqrt{(a^2-1)} - \frac{H_{22}B_{11}}{\pi\delta^3}(a^2-1) + \frac{H_{24}}{3\pi\delta^5}\left\{\frac{a^4-1}{4}B_{11} + \frac{3}{2}(a^2-1)B_{12}\right\} + \frac{H_{22}^2B_{11}}{3\pi^2\delta^6}(a-1)^2(a+1)(2a^2-a-1)\right]. \qquad (69)$$

Quantities of physical interest namely, the temperature and the normal components of stress and displacement on the crack plane $z = 0$ have been calculated for $a = 1.2, 1.6, 2.0$ and $\delta = 5, 7$. Variations of $T(r,0)$, $\sigma_{zz}(r,0)$ and $u_z(r,0)$ with r are shown graphically in figures 2–7 respectively.

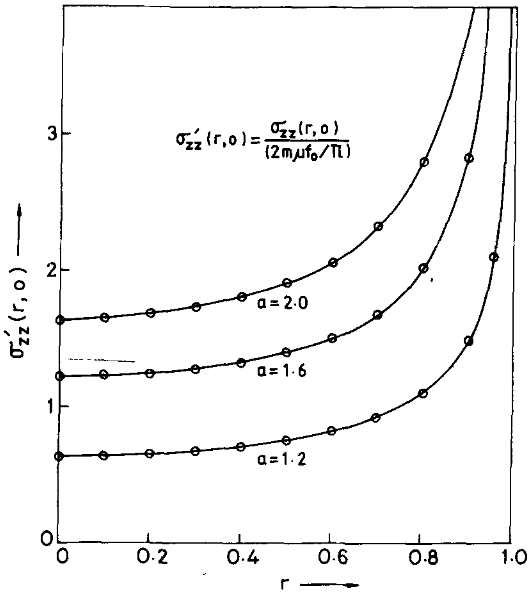

Figure 5. Variation of $\sigma_{zz}(r, 0)/(2\,m\mu f_0/\pi)$ with r for $a = 1.2, 1.6, 2.0$ and $\delta = 7$.

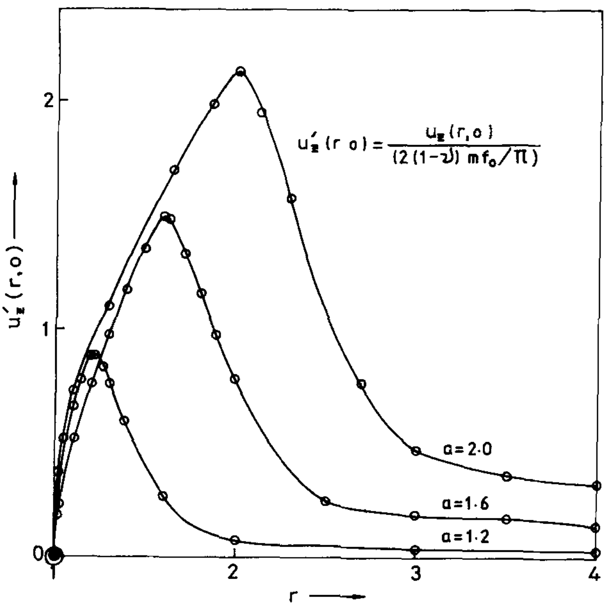

Figure 6. Variation of $u_z(r, 0)/(2(1-\nu)mf_0/\pi)$ with r for $a = 1.2, 1.6, 2.0$ and $\delta = 5$.

8. Conclusions

When $\delta \to \infty$, the problem reduces to that of an infinite medium containing an external circular crack which has been solved by Das [3]. It is found that the limiting values as $\delta \to \infty$ of the temperature, stress intensity factor and the normal components of stress

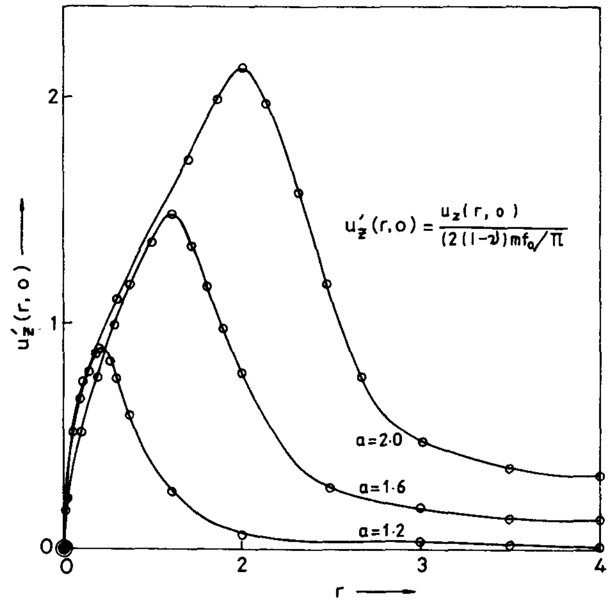

Figure 7. Variation of $u_z(r, 0)/(2(1-\nu)mf_0/\pi)$ with r for $a = 1.2, 1.6, 2.0$ and $\delta = 7$.

and displacement given by (56), (69), (67) and (64) are the same as those obtained by Das.

References

[1] Bandyopadhyay S and Das B R, Stress in the vicinity of a penny-shaped crack in a transversely isotropic thick plate, *Proc. Indian Nat. Sci. Acad.* **A60** (1994) 503–512

[2] Das B R and Ghosh S, Thermoelastic stresses in a thick plate containing a penny shaped crack in the mid-plane, *Geophys. Res. Bull.* **15** (1977) 65–70

[3] Das B R, Some axially symmetric thermal stress distributions in elastic solids containing cracks-I: An external crack in an infinite solid, *Int. J. Engg. Sci.* **9** (1971) 469–478

[4] Dhawan G K, The distribution of stress in the vicinity of an external crack in an infinite elastic thick plate, *Acta Mech.* **16** (1973) 255–270

[5] Lowengrub M, Stress in the vicinity of a crack in a thick elastic plate, *Q. J. Appl. Math.* **19** (1961) 119

[6] Lowengrub M and Sneddon I N, The distribution of stress in the vicinity of an external crack in an infinite elastic solid, *Int. J. Engg. Sci.* **3** (1965) 451

[7] Lowengrub M, A two-dimensional crack problem, *Int. J. Engg. Sci.* **4** (1966) 289–299

[8] Nowacki W, *Thermoelasticity* (London: Pergamon Press) (1962)

[9] Sneddon I N, The distribution of stress in the neighbourhood of a crack in an elastic solid, *Proc. R. Soc. London* **A187** (1946) 229

[10] Sneddon I N and Berry D S, *The classical theory of elasticity*, Handbuch der Physik, Bd. VI. (Springer), (1958)

[11] Sneddon I N, The elementary solution of dual integral equations, *Proc. Glasg. Math. Asso.* **4** (1960) 108

[12] Uflyand Ya S, Elastic equilibrium in an infinite body weakened by an external circular crack, *J. Appl. Math. Mech.* **23** (1959) 134

[13] Watson G N, *A Treatise on Bessel functions* (1st paperback edition) (Cambridge: University Press) (1966)