Proc. Indian Acad. Sci. (Math. Sci.), Vol. 105, No. 4, November 1995, pp. 425–444.

Nontrivial solution of a quasilinear elliptic equation with critical growth in $\mathbb{R}^n$

RATIKANTA PANDA

T.I.F.R. Centre, P.O. Box No. 1234, I.I.Sc. Campus, Bangalore 560012, India

MS received 10 February 1995; revised 16 June 1995

Abstract. Suppose $\Delta_n u = \mathrm{div}(|\nabla u|^{n-2}\nabla u)$ denotes the n-Laplacian. We prove the existence of a nontrivial solution for the problem

$$\begin{cases} -\Delta_n u + |u|^{n-2}u = f(x,u)u^{n-2} & \text{in } \mathbb{R}^n \\ u \in W^{1,n}(\mathbb{R}^n) \end{cases}$$

where $f(x,t) = o(t)$ as $t \to 0$ and $|f(x,t)| \leqslant C\exp(\alpha_n |t|^{n/(n-1)})$ for some constant $C > 0$ and for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ with $\alpha_n = n\omega_n^{1/(n-1)}$, ω_n = surface measure of S^{n-1}.

Keywords. Elliptic equation; critical growth; Palais–Smale condition; concentration compactness; mountain pass lemma.

1. Introduction

Suppose $\Delta_n u = \mathrm{div}(|\nabla u|^{n-2}\nabla u)$ denotes the n-Laplacian. We look for a solution of the problem

$$\begin{cases} -\Delta_n u + |u|^{n-2}u = f(x,u)u^{n-2} & \text{in } \mathbb{R}^n \\ u \in W^{1,n}(\mathbb{R}^n) \end{cases} \tag{1.1}$$

where $f(x,t) = o(t)$ as $t \to 0$ and $|f(x,t)| \leqslant C\exp(\alpha_n |t|^{n/(n-1)})$ for some constant $C > 0$ and for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ with $\alpha_n = n\omega_n^{1/(n-1)}$, ω_n = surface measure of S^{n-1}.

In the case where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, and $f(x,t) = h(x,t)\exp(\alpha_n |t|^{n/(n-1)})$ with $h(x,t)$ a lower order term in t, the problem (1.1) with Dirichlet boundary condition has been considered by Adimurthi [1] and with Neumann boundary condition by the author [9]. In case of $n = 2$, D M Cao [5] has shown the existence of a nontrivial solution for the problem (1.1). In this paper, applying the concentration-compactness principle of P L Lions [6, 7], we show that the functional associated with (1.1) satisfies $(\text{Palais–Smale})_c$ (in short $(\text{PS})_c$) condition for all $c \in (0, J)$ for some $J > 0$ (for definition of J see §3). Then we show the existence of a nontrivial solution for (1.1) by using Mountain Pass lemma as given in [4] and constructing a critical point of the functional with critical value in $(0, J)$. The main difficulty here is to show that whenever a Palais–Smale sequence $u_m \underset{m}{\rightharpoonup} u$ weakly in $W^{1,n}(\mathbb{R}^n)$,

$$|\nabla u_m|^{n-2}\nabla u_m \underset{m}{\rightharpoonup} |\nabla u|^{n-2}\nabla u \quad \text{weakly in } (L^{n/(n-1)}(\mathbb{R}^n))^n$$

We need the following assumptions on the nonlinearity $f(x,t) \in C(\mathbb{R}^n \times \mathbb{R})$:

(f_1) $|f(x,t)| \leqslant C\exp(\alpha_n |t| n/(n-1))$ for $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, where $C > 0$ is some constant.

(f_2) $f(x,t)t^{n-1} = f(x,-t)(-t)^{n-1}$ for $x \in \mathbb{R}^n$, $t \in \mathbb{R}$; $\dfrac{f(x,t)}{t}$ is nondecreasing with respect to t, for $t>0$;

$$\lim_{t\to 0}\frac{f(x,t)}{t}=0 \quad \text{uniformly with respect to } x\in\mathbb{R}^n;$$

$$\lim_{t\to\infty}\frac{f(x,t)}{t}=\infty \quad \text{uniformly with respect to } x\in\mathbb{R}^n.$$

(f_3) There exists $\theta\in\left(0,\frac{1}{n}\right)$ such that

$$F(x,t)\leqslant \theta t^{n-1}f(x,t) \quad \text{for} \quad x\in\mathbb{R}^n, t\in\mathbb{R},$$

where $F(x,t)=\int_0^t f(x,s)s^{n-2}\mathrm{d}s$

(f_4) $\exists \bar{f}(t)$ such that $\lim_{|x|\to\infty}f(x,t)=\bar{f}(t)$ uniformly for t bounded, more precisely,

$$|f(x,t)-\bar{f}(t)|\leqslant \varepsilon(R)|t|^{n-1} \quad \text{for} \quad |x|\geqslant R,$$

where $\varepsilon(R)\to 0$ as $R\to\infty$.

(f_5) $\exists p>n$ such that $f(x,t)\geqslant \bar{f}(t)\geqslant C_p t^{p-n+1}>(p/n)S_p^p(1-n\theta)^{1-(p/n)}t^{p-n+1}$ for $x\in\mathbb{R}^n$, $t\in\mathbb{R}^+$, where

$$S_p=\inf_{\substack{u\in W^{1,n}(R^n)\\ u\neq 0}}\frac{[\int_{\mathbb{R}_n}(|\nabla u|^n+|u|^n)\mathrm{d}x]^{1/n}}{(\int_{\mathbb{R}_n}|u|^p\mathrm{d}x)^{1/p}}.$$

For $u\in W^{1,n}(\mathbb{R}^n)$ let

$$I^\infty(u)=\frac{1}{n}\int_{\mathbb{R}^n}(|\nabla u|^n+|u|^n)\mathrm{d}x-\int_{\mathbb{R}^n}\bar{F}(u)\mathrm{d}x, \tag{1.2}$$

where $\bar{F}(t)=\int_0^t\bar{f}(s)s^{n-2}\mathrm{d}s$. The main results in this paper are as follows.

Theorem 1.1. *Suppose $f(x,t)\equiv\bar{f}(t)$ does not depend on x and satisfies (f_1)–(f_3) and (f_5). Then* (1.1) *has a nontrivial solution u_0. Moreover $I^\infty(u_0)<(1/n)-\theta$.*

Theorem 1.2. *Suppose $f(x,t)$ satisfies (f_1)–(f_5) and $f(x,t)\not\equiv\bar{f}(t)$ for fixed t with respect to $x\in\mathbb{R}^n$. Then* (1.1) *has a nontrivial solution.*

We remark that the (PS) condition is not needed for the proof of Theorem (1.1).

2. Preliminaries and notations

We shall denote $\int\cdots$ to mean $\int_{\mathbb{R}^n}\cdots\mathrm{d}x$. Define

$$\|u\|=\left(\int(|\nabla u|^n+|u|^n)\right)^{1/n} \quad \text{for} \quad u\in W^{1,n}(\mathbb{R}^n) \tag{2.1}$$

$$|u|_q=\left(\int|u|^q\right)^{1/q} \quad \text{for} \quad u\in L^q(\mathbb{R}^n). \tag{2.2}$$

The variational functional associated with (1.1) is

$$I(u)=\frac{1}{n}\|u\|^n-\int F(x,u) \tag{2.3}$$

where $F(x,t)=\int_0^t f(x,s)s^{n-2}\mathrm{d}s$

Let $I^\infty(u)$ be as in (1.2),

$$M^\infty = \left\{ u \in W^{1,n}(\mathbb{R}^n)\backslash\{0\} \,\middle|\, \|u\|^n = \int u^{n-1}\bar{f}(u) \right\} \tag{2.4}$$

$$C^\infty = \begin{cases} \inf\{I^\infty(u) | u \in M^\infty\}, & \text{if } M^\infty \neq \phi \\ \infty, & \text{if } M^\infty = \phi, \end{cases} \tag{2.5}$$

$$I_0^\infty = \inf\left\{ \int |\nabla u|^n \,\middle|\, u \in W^{1,n}(\mathbb{R})^n\backslash\{0\}, \int \bar{F}(u) = \frac{1}{n}\int |u|^n \right\}.$$

Remark. If $\bar{f}(t)$ satisfies (f_1) and (f_2) then $I_0^\infty > 0$.

Proof. Suppose, on the contrary, $I_0^\infty = 0$. Then there exists a sequence $\{u_m\}$ in $W^{1,n}(\mathbb{R}^n)$ such that

$$\int \bar{F}(u_m) = \frac{1}{n}\int |u_m|^n$$

$$\int |\nabla u_m|^n \underset{m}{\longrightarrow} 0.$$

Then by (f_1), (f_2) and Lemma 2.3 (to be proved)

$$\left|\int \bar{F}(u_m)\right| \leqslant \frac{1}{2n}\int |u_m|^n + C_1 \int |u_m|^{n-1}(\exp(\alpha_n |u_m|^{n/(n-1)}) - 1 - \alpha_n |u_m|^{n/(n-1)})$$

$$\leqslant \frac{1}{2n}|u_m|_n^n + C_2 |\nabla u_m|_n |u_m|_n^n,$$

where C_1, $C_2 > 0$ are some constants. Thus

$$\frac{1}{2n}|u_m|_n^n \leqslant C_2 |\nabla u_m|_n |u_m|_n^n,$$

and therefore $|\nabla u_m|_n \geqslant 1/2nC_2$, a contradiction which proves the remark. □

Remark. If $\bar{f}(t)$ satisfies (f_1)–(f_3) then $C^\infty > 0$.

Proof. Suppose $M^\infty \neq \phi$. For $u \in M^\infty$, using (f_3) we get

$$I^\infty(u) = \frac{1}{n}\int (|\nabla u|^n + |u|^n) - \int \bar{F}(u)$$

$$= \frac{1}{n}\int (|\nabla u|^n + |u|^n) - \int \theta \bar{f}(u) u^{n-1}$$

$$= \left(\frac{1}{n} - \theta\right) \|u\|^n.$$

Since $\bar{f}(t)$ satisfies (f_1) and (f_2) we have

$$\bar{f}(t)t^{n-1} \leqslant \frac{1}{n}|t|^n + C_1 |t|^{n-1}(\exp(\alpha_n |t|^{n/(n-1)}) - 1 - \alpha_n |t|^{n/(n-1)})$$

and therefore as in the above Remark we obtain

$$\inf\left\{\int |\nabla u|^n : u\in M^\infty\right\} \geqslant C_2$$

for some positive constant C_2. Therefore by above estimate $C^\infty \geqslant ((1/n)-\theta)C_2$. This proves the Remark. □

Similar to the imbedding of Moser [8] we have

Lemma 2.1. *Suppose $u\in W^{1,n}(\mathbb{R}^n)$, $|\nabla u|_n^n \leqslant r < 1$, $|u|_n \leqslant M < \infty$. Then*

$$\int\left[\exp(\alpha_n |u|^{n/(n-1)}) - \sum_{m=0}^{n-2} \frac{\alpha_n^m |u|^{nm/(n-1)}}{m!}\right] \leqslant C(M,r), \tag{2.6}$$

where $C(M,r) > 0$ is a constant independent of u.

Proof. As in Moser [8] we use the method of symmetrization. Let u^* be the symmetrization of u. Then u^* is a radial, nonnegative and nonincreasing function. Further,

$$\int |u^*|^p = \int |u|^p, \quad 1 < p < \infty \tag{2.7}$$

$$\int G(u) = \int G(u^*), \tag{2.8}$$

$$\int |\nabla u^*|^n \leqslant \int |\nabla u|^n, \tag{2.9}$$

where $G(u)$ is the integrand on the l.h.s. of (2.6). We have

$$\int G(u) = \int G(u^*) = \int_{|x|<s} G(u^*) + \int_{|x|\geqslant s} G(u^*) \tag{2.10}$$

where $s > 0$ is a number to be determined.

First we estimate the second integral in (2.10). By the radial Lemma A. IV in [3] we have

$$|u^*(x)| \leqslant \left(\frac{n}{\omega_n}\right)^{1/n} |u^*|_n |x|^{-1} \quad \text{for} \quad x \neq 0. \tag{2.11}$$

Thus

$$\begin{aligned}\int_{|x|\geqslant s} G(u) &= \frac{\alpha_n^{n-1}|u^*|_n^n}{(n-1)!} + \int_{|x|\geqslant s}\left(\sum_{m=n}^{\infty} \frac{\alpha_n^m |u^*|_{n-1}^{nm}}{m!}\right)\\ &\leqslant \frac{\alpha_n^{n-1}|u^*|_n^n}{(n-1)!} + \sum_{m=n}^{\infty}\frac{1}{m!}\alpha_n^m\left(\frac{n}{\omega_n}\right)^{m/(n-1)} |u^*|_n^{nm/(n-1)} \int_{|x|\geqslant s} \frac{1}{|x|^{nm/(n-1)}}\mathrm{d}x\\ &\leqslant \frac{\alpha_n^{n-1}|u^*|_n^n}{(n-1)!} + \frac{\omega_n}{s^{-n}}\left(\frac{n-1}{n}\right)\sum_{m=n}^{\infty}\frac{1}{m!}\left(\frac{n|u^*|_n}{s}\right)^{nm/(n-1)}\\ &\leqslant C(M) \quad \text{if} \quad s > n|u^*|_n.\end{aligned} \tag{2.12}$$

To estimate the first integral in (2.10), let us put $|x|^n = s^n e^{-t}$, $v(t) = n^{(n-1)/n} u^*(x)$. Then

$$\int_0^\infty \dot{v}^n(t)\mathrm{d}t = \int_{|x|<s} |\nabla u^*|^n, \tag{2.13}$$

$$\int_0^\infty \exp(|v(t)|^{n/(n-1)} - t)\mathrm{d}t = \frac{n}{\omega_n r^n}\int_{|x|<s} \exp(\alpha_n |u^*|^{n/(n-1)})\mathrm{d}x, \tag{2.14}$$

where $\dot{v} = \mathrm{d}v/\mathrm{d}t$. By Hölder inequality we have

$$\begin{aligned} v(t) &= v(0) + \int_0^t \dot{v}(s)\,\mathrm{d}s \\ &\leqslant v(0) + \left(\int_0^t |\dot{v}(s)|^n\right)^{1/n} t^{(n-1)/n} \\ &\leqslant n^{(n-1)/n}\omega_n^{1/n} u^*(sx_0) + |\nabla u^*|_n t^{(n-1)/n} \end{aligned} \tag{2.15}$$

where x_0 is some unit vector in $\mathbb{R}^n$. Now

$$\begin{aligned} \int_{|x|<s} G(u^*) &< \int_{|x|<s} \exp(\alpha_n |u^*|^{n/(n-1)}) \\ &= \frac{\omega_n s^n}{n}\int_0^\infty \exp(|v(t)|^{n/(n-1)} - t)\mathrm{d}t \\ &\leqslant \frac{\omega_n s^n}{n} 2^{n/(n-1)} \exp(\alpha_n |u^*(sx_0)|^{n/(n-1)}) \int_0^\infty \exp(|\nabla u^*|_n^{n/(n-1)} t - t)\mathrm{d}t \\ &\leqslant \frac{\omega_n s^n}{n} 2^{n/(n-1)} \exp(\alpha_n |u^*(sx_0)|^{n/(n-1)}). \end{aligned} \tag{2.16}$$

Combining (2.11), (2.12) and (2.16) we have (2.6). □

Lemma 2.2. *There exists $\beta = \beta(n) > 0$ such that for all $u \in W^{1,n}(\mathbb{R}^n)$ with $|\nabla u|_n^n < 1/\alpha_n \beta e$, we have*

$$\int |u|^{n-1}(\exp(\alpha_n |u|^{n/(n-1)}) - 1 - \alpha_n |u|^{n/(n-1)}) \leqslant C|u|_n^n |\nabla u|_n$$

where $C > 0$ is a constant independent of $\dot{u}$.

Proof. By the result of Talenti [10] (or of Aubin [2]) we know that if $t, s > 1$, $t < n$ and $1/s = 1/t - 1/n$, then all $\varphi \in W^{1,t}(\mathbb{R}^n)$ satisfy

$$|\varphi|_s \leqslant K(n,t)|\nabla\varphi|_t, \tag{2.17}$$

with

$$K(n,t) = \frac{t-1}{n-t}\left[\frac{n-t}{n(t-1)}\right]^{1/t}\left[\frac{\Gamma(n+1)}{\Gamma(n/t)\Gamma(n+1-n/t)\omega_n}\right]^{1/n}.$$

Let us set $\varphi = |u|^v$, where $v = ((n-1)^2 + nm)/(nm - n + 1)$, $m \geqslant 2$. Then $|\nabla\varphi| = v|u|^{v-1}|\nabla u|$. Taking $s = (n/(n-1))m - 1$, $t = n - (n^2(n-1))/((n-1)^2 + nm)$ and using

Hölder inequality we get

$$\int |u|^{nm/(n-1)+n-1} \leqslant (K\nu)^{nm/(n-1)-1}\left(\int |u|^{t(\nu-1)}|\nabla u|^t\right)^{s/t}$$
$$\leqslant (K\nu)^{nm/(n-1)-1}|u|_n^n|\nabla u|_n^{nm/(n-1)-1}. \tag{2.18}$$

Now

$$K(n,t)=\left(\frac{1}{n}\right)^{1/t}(t-1)^{(t-1)/t}\left[\frac{\Gamma(n+1)}{\Gamma(n/t)\Gamma(n+1-(n/t))\omega_n}\right]^{1/n}(n-t)^{(1-t)/t}$$
$$\leqslant C(n)(n-t)^{(1-t)/t}$$
$$=C(n)\left[\frac{(n-1)^2+nm}{n^2(n-1)}\right]^{(t-1)/t}$$

where $C(n)$ is a constant dependent on n. Since $t<n$ we get $(t-1)/t<(n-1)/n$. Also for all $m\geqslant 2, n\geqslant 2$ we have $((n-1)^2+nm)/n^2(n-1)<m$ and $\nu\leqslant C_1(n)$, a constant dependent on n. Hence we get $(K\nu)^{nm/(n-1)-1}\leqslant C(\beta m)^m$ for some $\beta=\beta(n)>0$ and $C=C(n)>0$. Therefore by (2.18)

$$\int |u|^{n-1}(\exp(\alpha_n|u|^{n/(n-1)})-1-\alpha_n|u|^{n/(n-1)})$$
$$=\int\left(\sum_{m=2}^{\infty}\frac{1}{m!}\alpha_n^m|u|^{nm/(n-1)+n-1}\right)$$
$$\leqslant C\sum_{m=2}^{\infty}\frac{1}{m!}(m\alpha_n\beta)^m|\nabla u|_n^{nm/(n-1)-1}|u|_n^n.$$

For $m\geqslant 2$ we have $mn/(n-1)-2\geqslant m/(n-1)$. Thus for $|\nabla u|_n^{1/(n-1)}<1/\alpha_n\beta e$ we have

$$\int |u|^{n-1}(\exp(\alpha_n|u|^{n/(n-1)})-1-\alpha_n|u|^{n/(n-1)})$$
$$\leqslant C\sum_{m=2}^{\infty}\frac{1}{m!}(m\alpha_n\beta)^m(|\nabla u|_n^{1/(n-1)})^m|\nabla u|_n|u|_n^n$$
$$\leqslant C|\nabla u|_n|u|_n^n.$$

where we have used the same C to denote various constants. □

Lemma 2.3. *Let $\bar{f}(t)$ satisfy (f_1) and (f_2). Suppose there exists $u_0\in W^{1,n}(\mathbb{R}^n)\setminus\{0\}$ such that $\int\bar{F}(u_0)\geqslant(1/n)\int|u_0|^n$ and $|\nabla u_0|_n^n<1$. Then I_0^∞ is achieved. Moreover $I_0^\infty\leqslant|\nabla u_0|_n^n$.*

Proof. Using (f_2) and the hypothesis that $\int\bar{F}(u_0)\geqslant(1/n)|u_0|_n^n$ it is easy to see that there exists $t_0\in(0,1]$ such that $\int\bar{F}(t_0u_0)=(1/n)\int|t_0u_0|^n$. Thus $I_0^\infty\leqslant\int|\nabla u_0|^n$. Let $\{u_m\}$ be a minimizing sequence for I_0^∞. Without loss of generality we can assume that $\int|\nabla u_m|^n\leqslant r<1$. Denote by u_m^* the symmetrization of u_m. Then u_m^* is a radial, nonincreasing function. Furthermore,

$$\int F(u_m^*)=\int F(u_m)$$

$$\int |u_m^*|^n = \int |u_m|^n$$

$$\int |\nabla u_m^*|^n \leqslant \int |\nabla u_m|^n.$$

Thus $\{u_m^*\}$ is still a minimizing sequence of I_0^∞. We denote it simply by $\{u_m\}$ in what follows. Without loss of generality we can assume that $|u_m|_n = 1$. Thus $\{u_m\}$ is bounded in $W^{1,n}(\mathbb{R}^n)$ and so there exists $u \in W^{1,n}(\mathbb{R}^n)$ such that for a subsequence

$$u_m \to u \quad \text{weakly in } W^{1,n}(\mathbb{R}^n)$$

$$u_m \to u \quad \text{a.e. in } \mathbb{R}^n.$$

We want to use a compactness lemma of Strauss (see Theorem A.I. of [3]). Set $P(t) = \bar{F}(t)$,

$$Q(t) = \exp\left(\frac{2\alpha_n}{1+r}|t|^{n/(n-1)}\right) - \sum_{j=0}^{n-2} \frac{1}{j!}\left(\frac{2\alpha_n}{1+r}\right)^j |t|^{nj/(n-1)} + |t|^n.$$

Then using (f_1) we get $\lim_{|t|\to\infty} P(t)/Q(t) = 0$ and using (f_2), $\lim_{|t|\to 0} P(t)/Q(t) = 0$. Again by radial lemma A.IV of [3] we get (2.11) and so as $|x| \to \infty$, $u_m(x) \to 0$ uniformly in m. Further by Lemma 2.1,

$$\sup_m \int Q(u_m) \leqslant C.$$

Thus all the conditions of Strauss' lemma are satisfied and we get

$$\lim_{m\to\infty} \int \bar{F}(u_m) = \int \bar{F}(u).$$

Since $\int F(u_m) = \frac{1}{2}$ we get $u \not\equiv 0$. Now

$$\frac{1}{n}\int |u|^n \leqslant \frac{1}{n} \liminf_{m\to\infty} \int |u_m|^n = \liminf_{m\to\infty} \int \bar{F}(u_m) = \int \bar{F}(u).$$

If $(1/n)\int |u|^n = \int \bar{F}(u)$, then

$$I_0^\infty \leqslant \int |\nabla u|^n \leqslant \liminf_{m\to\infty} \int |\nabla u_m|^n = I_0^\infty,$$

and so I_0^∞ is achieved by u. If on the other hand $(1/n)\int |\nabla u|^n < \int \bar{F}(u)$, then there exists $t \in (0, 1)$ such that $(1/n)\int |tu|^n = \int \bar{F}(tu)$. Hence

$$I_0^\infty \leqslant t^n \int |\nabla u|^n < \int |\nabla u|^n \leqslant \liminf_{m\to\infty} \int |\nabla u_m|^n = I_0^\infty,$$

a contradiction which proves the lemma □

Lemma 2.4. *Suppose* $\{u_m\} \subset W^{1,n}(\mathbb{R}^n)$ *satisfies* $|\nabla u_m|_n < 1$, $|u_m|_n^n < M$ *and*

$$\lim_{m\to\infty} \sup_{y\in\mathbb{R}^n} \int_{y+B_R} (|\nabla u_m|^n + |u_m|^n)dx = 0 \quad \textit{for some } R > 0,$$

where $B_R = \{x \in \mathbb{R}^n; |x| < R\}$. *Then*

$$\lim_{m\to\infty} \int F(x,u_m) = 0$$

$$\lim_{m\to\infty} \int f(x,u_m)u_m^{n-1} = 0. \tag{2.19}$$

Proof. Let $\xi \in C_0^\infty(\mathbb{R}^n)$ be such that $\xi \equiv 1$ for $|x| < R/2$; $\xi \equiv 0$ for $|x| > R$ and $|\nabla \xi| < 4n/R$. Let $\xi_y = \xi(\cdot - y)$. Then

$$\int_{y+B_R} |\nabla(\xi_y u_m)|^n \leqslant 2^n \int_{y+B_R} [|\nabla \xi_y|^n |u_m|^n + |\xi_y|^n |\nabla u_m|^n] \mathrm{d}x$$

$$\leqslant 2^n \left[1 + \left(\frac{4n}{R}\right)^n\right] \int_{y+B_R} (|\nabla u_m|^n + |u_m|^n) \mathrm{d}x. \tag{2.20}$$

In view of (f_1) and (f_2), given $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\int_{y+B_{R/2}} |F(x,u_m)| \mathrm{d}x$$

$$\leqslant C_\varepsilon \int_{y+B_{R/2}} |u_m|^{n-1}(\exp(\alpha_n |u_m|^{n/(n-1)}) - 1 - \alpha_n |u_m|^{n/(n-1)}) \mathrm{d}x$$

$$+ \varepsilon \int_{y+B_{R/2}} |u_m|^n \mathrm{d}x$$

$$\leqslant C_\varepsilon \int_{y+B_R} |u_m|^{n-1}(\exp(\alpha_n |\xi_y u_m|^{n/(n-1)}) - 1 - \alpha_n |\xi_y u_m|^{n/(n-1)}) \mathrm{d}x$$

$$+ \varepsilon \int_{y+B_{R/2}} |u_m|^n \mathrm{d}x.$$

For m large enough $|\nabla(\xi_y u_m)|_n^n < 1/\alpha_n \beta e$ and hence Lemma 2.3 gives

$$\int_{y+B_{R/2}} |F(x,u_m)| \mathrm{d}x \leqslant \tilde{C}_\varepsilon |\nabla(\xi_y u_m)|_n \int_{y+B_R} |u_m|^n \mathrm{d}x + \varepsilon \int_{y+B_R} |u_m|^n \mathrm{d}x.$$

We cover $\mathbb{R}^n$ by balls $B_{R/2}(x_i)$ in such a way that any point of $\mathbb{R}^n$ is contained in at most k balls $B_R(x_i)$ of radius R. For large m we have

$$\int_{\mathbb{R}^n} |F(x,u_m)| \mathrm{d}x \leqslant k\tilde{C}_\varepsilon \sup_{y\in\mathbb{R}^n} \left[\int_{y+B_R} (|\nabla u_m|^n + |u_m|^n) \mathrm{d}x\right]^{1/n} \left[\int_{\mathbb{R}^n} |u_m|^n \mathrm{d}x\right]$$

$$+ k\varepsilon \int_{\mathbb{R}^n} |u_m^{n-1} f(x,u_m)|. \tag{2.21}$$

Making $m \to \infty$ and then $\varepsilon \to 0+$ in (2.21) we obtain

$$\lim_{m\to\infty} \int F(x,u_m) = 0. \tag{2.22}$$

Similarly we have

$$\lim_{m\to\infty}\int f(x,u_m)u_m^{n-1}=0. \tag{2.23}$$

□

3. Proof of the main results

First we prove the following

Lemma 3.1. *Let* C^∞ *be as in* (2.5) *and*

$$J=\min\left(C^\infty,\frac{1}{n}-\theta\right).$$

Suppose $f(x,t)$ *satisfies* (f_1)–(f_4). *Then* $I(u)$ *satisfies* $(\mathrm{PS})_c$ *condition for* $c\in(0,J)$.

Proof. Let $\{u_m\}$ be a $(\mathrm{PS})_c$ sequence in $W^{1,n}(\mathbb{R}^n)$. That is,

$$I(u_m)\underset{m}{\to}C\in(0,J)$$

$$I'(u_m)\underset{m}{\to}0\quad\text{in }(W^{1,n}(\mathbb{R}^n))^*.$$

Then

$$\frac{1}{n}\int(|\nabla u_m|^n+|u_m|^n)-F(x,u_m)=c+o(1) \tag{3.1}$$

$$\int(|\nabla u_m|^{n-2}\nabla u_m\cdot\nabla\varphi+|u_m|^{n-2}u_m\varphi)-\int f(x,u_m)u_m^{n-2}\varphi=\langle\xi_m,\varphi\rangle \tag{3.2}$$

where $o(1)$ denotes the quantities that tend to 0 as $m\to\infty$ and $\xi_m\underset{m}{\to}0$ in $(W^{1,n}(\mathbb{R}^n))^*$. Taking $\varphi=u_m$ in (3.2) we obtain

$$\int(|\nabla u_m|^n+|u_m|^n)-\int f(x,u_m)u_m^{n-1}=\langle\xi_m,u_m\rangle. \tag{3.3}$$

Claim 1: $\|u_m\|^n\leqslant nc/(1-n\theta)+o(1)$

From (3.1) and (3.3) we get

$$\int[f(x,u_m)u_m^{n-1}-F(x,u_m)]\leqslant nc+o(1)+|\langle\xi_m,u_m\rangle|.$$

Thus using (f_3),

$$(1-n\theta)\int f(x,u_m)u_m^{n-1}\leqslant nc+o(1)+|\langle\xi_m,u_m\rangle|$$

and hence in view of (3.3) $\{u_m\}$ is bounded. Further (3.3) gives

$$\|u_m\|^n\leqslant\frac{nc}{1-n\theta}+o(1)\quad\text{as }m\to\infty$$

as desired.

Thus for m large enough

$$|\nabla u_m|_n^n \leqslant r \tag{3.4}$$

where $r \in (0,1)$ is some fixed number, and there exists $u \in W^{1,n}(\mathbb{R}^n)$ such that for a subsequence

$$\begin{cases} u_m \to u \quad \text{weakly in } W^{1,n}(\mathbb{R}^n) \\ u_m \to u \quad \text{a.e. in } \mathbb{R}^n \\ (|\nabla u_m|^n + |u_m|^n)\mathrm{d}x \to \mathrm{d}\mu \quad \text{in measure} \\ |\nabla u_m|^{n-2}\nabla u_m \to T \quad \text{weakly in } (L^{n/(n-1)}(\mathbb{R}^n))^n. \end{cases} \tag{3.5}$$

Without loss of generality we may assume

$$\|u_m\|^n \to l \geqslant 0$$

$$|\nabla u_m|_n^n \leqslant r < 1 \quad \text{for all } m \geqslant 1.$$

Claim 2: $l > 0$.

If not, suppose $l = 0$. Then by (f_1), (f_2) and Lemma 2.3

$$\left|\int f(x, u_m)u_m^{n-1}\right|$$

$$\leqslant C\int |u_m|^{n-1}(\exp(\alpha_n|u_m|^{n/(n-1)}) - 1 - \alpha_n|u_m|^{n/(n-1)}) + C\int |u_m|^n$$

$$\leqslant C\int |u_m|^n \underset{m}{\to} 0.$$

Similarly $|\int F(x, u_m)| \underset{m}{\to} 0$.

So $I(u_m) = (1/n)\|u_m\|^n - \int F(x, u_m) \underset{m}{\to} 0$, which contradicts the fact that $I(u_m) \underset{m}{\to} c \neq 0$. Hence $l > 0$.

We want to apply concentration-compactness principle of P L Lions [6, 7] to the sequence $\{\rho_m\}$ where $\rho_m = |\nabla u_m|^n + |u_m|^n$. Applying Lemma 1.1 of [6] we conclude that for a subsequence one of the three possibilities holds: (a) vanishing, (b) dischotomy (c) compactness. We use contradiction argument to show that only (c) compactness occurs.

Step 1: Vanishing does not occur.
Suppose instead that

$$\lim_{m\to\infty} \sup_{y\in\mathbb{R}^n} \int_{y+B_R} (|\nabla u_m|^n + |u_m|^n) = 0 \quad \text{for all } R > 0.$$

Then Lemma 2.4 yields

$$\int F(x, u_m) \underset{m}{\to} 0, \quad \int f(x, u_m)u_m^{n-1} \underset{m}{\to} 0.$$

This implies, in view of (3.3), that $\|u_m\|^n \underset{m}{\to} 0$, which is not possible since $l > 0$. So vanishing does not occur.

Step 2: Dichotomy does not occur.

Suppose dichotomy occurs. Let $Q_m(t) = \sup_{y\in\mathbb{R}^n}\int_{y+B_t}\rho_m(x)dx$ denote the concentration function of ρ_m. Then $\{Q_m\}$ is a sequence of nondecreasing nonnegative uniformly bounded functions on $\mathbb{R}^+$. As in [6], by extracting a subsequence we can assume that there exists $Q(t)$ such that $Q_m(t) \underset{m}{\to} Q(t)$ and since dichotomy occurs, $\lim_{t\to\infty} Q(t) = \alpha \in (0, l)$. For any $\varepsilon > 0$, $\varepsilon < 1/(2n)^n\alpha_n\beta e$, we can choose $t_0 > 0$ such that $Q(t) \geqslant \alpha - (\varepsilon/4)$ if $t \geqslant t_0$. Then for m large enough $\alpha - (\varepsilon/4) \leqslant Q_m(t) \leqslant \alpha + \varepsilon/4$ if $t \geqslant t_0$. Furthermore, there exists $\{y_m\} \subset \mathbb{R}^n$ such that

$$\int_{y_m+B_t} \rho_m \in \left(\alpha - \frac{\varepsilon}{4}, \alpha + \frac{\varepsilon}{4}\right) \tag{3.6}$$

for $t \geqslant t_0$ and m large enough. Also we can find $t_m \underset{m}{\to} \infty$ such that

$$\int_{y_m+B_{2t_m+2}} \rho_m \in \left(\alpha - \frac{\varepsilon}{2}, \alpha + \frac{\varepsilon}{2}\right). \tag{3.7}$$

Let $\varphi, \psi \in C^\infty(\mathbb{R}^n)$ be bounded cut-off functions such that $0 \leqslant \varphi, \psi \leqslant 1$, $\psi \equiv 0$ if $|x| \geqslant 3$, $\psi \equiv 1$ if $|x| \leqslant 2$; $\varphi \equiv 1$ if $|x| \geqslant 2$; $\varphi \equiv 0$ if $|x| \leqslant 1$. Set $\psi_m = \psi((\cdot - y_m)/t_1)$ and $\varphi_m = \varphi((\cdot - y_m)/t_m)$, where $t_1 > t_0$. Denote $v_m = \psi_m u_m$ and $w_m = \varphi_m u_m$.

Claim 3: $I(u_m) \geqslant I(v_m) + I(w_m) - C\varepsilon$.

By computation we deduce

$$\left|\int(\psi_m^n|\nabla u_m|^n - |\nabla v_m|^n)\right| \leqslant \frac{C}{t_1}.$$

Choosing t_1 large enough we have

$$\left|\int(\psi_m^n|\nabla u_m|^n - |\nabla v_m|^n)\right| < \varepsilon. \tag{3.8}$$

With m large enough so that $t_m > 3t_1$, using (f_1), (f_2) we get

$$\begin{aligned}
&\left|\int[\psi_m^n u_m^{n-1}f(x,u_m) - v_m^{n-1}f(x,v_m)]\right| \\
&= \left|\int_{2t_1<|x-y_m|<3t_1}[\psi_m^n u_m^{n-1}f(x,u_m) - v_m^{n-1}f(x,v_m)]\right| \\
&= C\int_{2t_1<|x-y_m|<3t_1}|u_m|^{n-1}(\exp(\alpha_n|u_m|^{n/(n-1)}) - 1 - \alpha_n|u_m|^{n/(n-1)}) \\
&\quad + C\int_{2t_1<|x-y_m|<3t_1}|u_m|^n.
\end{aligned} \tag{3.9}$$

Let $\eta_m \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leqslant \eta_m \leqslant 1$, $\eta_m \equiv 0$ if $|x - y_m| < t_1$ or $|x - y_m| > 2t_m + 2$; $\eta_m \equiv 1$ if $2t_1 \leqslant |x - y_m| \leqslant 2t_m$ and $|\nabla\eta_m| \leqslant n$. Then using (3.6) and (3.7) we get

$$\begin{aligned}
\int|\nabla(\eta_m u_m)|^n &\leqslant 2^n\int[|\nabla\eta_m|^n|u_m|^n + |\nabla u_m|^n|\eta_m|^n] \\
&\leqslant (2n)^n\int_{t_1\leqslant|x-y_m|\leqslant 2t_m+2}[|u_m|^n + |\nabla u_m|^n] \leqslant \frac{1}{\alpha_n\beta e}.
\end{aligned}$$

Hence applying Lemma 2.3 we get

$$\left|\int[\psi_m^n u_m^{n-1}f(x,u_m)-v_m^{n-1}f(x,v_m)]\right|$$
$$\leqslant C\int_{2t_1\leqslant|x-y_m|}|\eta_m u_m|^{n-1}(\exp(\alpha_n|\eta_m u_m|^{n/(n-1)})-1-\alpha_n|\eta_m u_m|^{n/(n-1)})$$
$$+C\int_{2t_1\leqslant|x-y_m|\leqslant 3t_1}|u_m|^n$$
$$\leqslant C\int_{t_1\leqslant|x-y_m|\leqslant 2t_m+2}|\eta_m u_m|^n+C\int_{2t_1\leqslant|x-y_m|\leqslant 3t_1}|u_m|^n<C\varepsilon. \tag{3.10}$$

Similarly we have for m large,

$$\left|\int[\varphi_m^n|\nabla u_m|^n-|\nabla w_m|^n]\right|<C\varepsilon \tag{3.11}$$

$$\left|\int[\varphi_m^n u_m^{n-1}f(x,u_m)-w_m^{n-1}f(x,w_m)]\right|<C\varepsilon. \tag{3.12}$$

Combining (3.6), (3.8) with (3.11) we get

$$\left|\int[|\nabla u_m|^n-|\nabla v_m|^n-|\nabla w_m|^n]\right|\leqslant C\varepsilon \tag{3.13}$$

$$\left|\int[|u_m|^n-|v_m|^n-|w_m|^n]\right|<C\varepsilon. \tag{3.14}$$

Also using (f_1) and (f_2) we get

$$\left|\int[F(x,u_m)-F(x,v_m)-F(x,w_m)]\right|$$
$$\leqslant C\int_{2t_1\leqslant|x-y_m|\leqslant 2t_m}|u_m|^{n-1}(\exp(\alpha_n|u_m|^{n/(n-1)})-1-\alpha_n|u_m|^{n/(n-1)})$$
$$+C\int_{2t_1\leqslant|x-y_m|\leqslant 2t_m}|u_m|^n,$$

and as in the proof of (3.10) we obtain

$$\left|\int[F(x,u_m)-F(x,v_m)-F(x,w_m)]\right|\leqslant C\varepsilon. \tag{3.15}$$

From (3.13), (3.14) and (3.15) we have

$$I(u_m)\geqslant I(v_m)+I(w_m)-C\varepsilon \tag{3.16}$$

and this proves the claim.

We will now consider two cases, Case 1: $\{y_m\}$ is bounded, and Case 2: $\{y_m\}$ is unbounded.

Case 1: $\{y_m\}$ is bounded.

Claim 4: $I(w_m) \geqslant I^\infty(w_m) - O(\varepsilon) - o(1)$ as $\varepsilon \to 0+$, $m \to \infty$.

We have

$$I(w_m) = I^\infty(w_m) - \int [F(x, w_m) - \bar{F}(w_m)]$$

and for $\delta > 0$,

$$\left| \int [F(x, w_m) - \bar{F}(w_m)] \right|$$
$$= \left| \left(\int_{\substack{|x - y_m| \geqslant t_m \\ |w_m| \leqslant \delta}} + \int_{\substack{|x - y_m| \geqslant t_m \\ \delta \leqslant |w_m| \leqslant 1/\delta}} + \int_{\substack{|x - y_m| \geqslant t_m \\ |w_m| \geqslant 1/\delta}} \right) (F(x, w_m) - \bar{F}(w_m)) \right|.$$

By (f_4),

$$\left| \int_{\substack{|x - y_m| \geqslant t_m \\ |w_m| \leqslant \delta}} (F(x, w_m) - \bar{F}(w_m)) \right| \leqslant \frac{\varepsilon(t_m)}{n} |u_m|_n^n \leqslant C\varepsilon(t_m) \tag{3.17}$$

where $\varepsilon(t_m) \to 0$ as $t_m \to \infty$. Here we have used the assumption that $\{y_m\}$ is bounded. Also when $|x|$ is large enough, for $|w_m| \leqslant 1/\delta$,

$$|F(x, w_m) - \bar{F}(w_m)| \leqslant \varepsilon(R), \quad |x| \geqslant R.$$

Thus

$$\left| \int_{\substack{|x - y_m| \geqslant t_m \\ \delta < |w_m| \leqslant 1/\delta}} (F(x, w_m) - \bar{F}(w_m)) \right| \leqslant \varepsilon(t_m) \operatorname{meas} \left\{ \delta \leqslant |w_m| \leqslant \frac{1}{\delta} \right\}$$
$$\leqslant \frac{1}{\delta^n} \varepsilon(t_m) |w_m|_n^n \leqslant C\delta^{-n} \varepsilon(t_m). \tag{3.18}$$

By (f_1) and (f_5), for $t > 0$ we have

$$\lim_{t \to \infty} \frac{F(x, t)}{\exp((2\alpha_n/(1 + r)) |t|^{n/(n-1)}) - 1} = 0 \quad \text{uniformly in } x \text{ and}$$
$$\lim_{t \to \infty} \frac{\bar{F}(t)}{\exp((2\alpha_n/(1 + r)) |t|^{n/(n-1)}) - 1} = 0.$$

Hence by (3.4) and Lemma 2.1

$$\left| \int_{|w_m| > 1/\delta} F(x, w_m) \right| = \left| \int_{|w_m| > 1/\delta} \frac{F(x, w_m)}{\exp((2\alpha_n/(1 + r)) |w_m|^{n/(n-1)}) - 1} \right.$$
$$\left. \times \left(\exp((2\alpha_n/(1 + r)) |w_m|^{n/(n-1)} - 1 \right) \right|$$
$$\leqslant O(\delta) \tag{3.19}$$

as $\delta \to 0$. Similarly

$$\left| \int_{|w_m| > 1/\delta} \bar{F}(w_m) \right| \leqslant O(\delta). \tag{3.20}$$

Thus (3.17)–(3.20) imply

$$\left|\int[F(x,w_m)-\bar{F}(w_m)]\right| \xrightarrow[m]{} 0. \tag{3.21}$$

Therefore we get, as desired

$$I(w_m)\geqslant I^{\infty}(w_m)-O(\varepsilon)-o(1) \tag{3.22}$$

where $O(\varepsilon)\to 0$ as $\varepsilon\to 0$ and $o(1)\to 0$ as $m\to\infty$.

Claim 5: $I(w_m)>C^{\infty}-O(\varepsilon)-o(1)$.

We have

$$\langle I'(w_m),w_m\rangle=\langle I^{\infty\prime}(w_m),w_m\rangle+\int(w_m^{n-1}\bar{f}(w_m)-w_m^{n-1}f(x,w_m)).$$

Arguing as in the proof of (3.21) we can prove that

$$\int(u_m^{n-1}\bar{f}(u_m)-u_m^{n-1}f(x,u_m))=o(1). \tag{3.23}$$

Also by using (3.3), (3.6), (3.11) and (3.12) we can get

$$\begin{aligned}\langle I'(w_m),w_m\rangle&=\langle I'(u_m),\varphi_m^n u_m\rangle+O(\varepsilon)\\&=o(1)+O(\varepsilon).\end{aligned}$$

Hence

$$\langle I^{\infty\prime}(w_m),w_m\rangle=o(1)+O(\varepsilon). \tag{3.24}$$

With $\tilde{w}_m(x)=w_m(\sigma x)$ we have

$$\begin{aligned}&\int(|\nabla\tilde{w}_m|^n+|\tilde{w}_m|^n-\tilde{w}_m^{n-1}\bar{f}(\tilde{w}_m))\\&=\int|\nabla w_m|^n+\sigma^{-n}\int(|w_m|^n-w_m^{n-1}\bar{f}(w_m))\\&=(1-\sigma^{-n})\int|\nabla w_m|^n+\sigma^{-n}\langle I^{\infty\prime}(w_m),w_m\rangle.\end{aligned} \tag{3.25}$$

We want to choose σ_m close to 1 in such a way that $\tilde{w}_m\in M^{\infty}$. First we show that $|\nabla w_m|_n^n$ has a lower bound $A>0$ independent of ε small enough and independent of m. If not, then there is a sequence $\delta_k \xrightarrow[k]{} 0$ such that

$$\lim_{m\to\infty}|\nabla w_m(\delta_k)|_n^n=\bar{\mu}(\delta_k),\ \bar{\mu}(\delta_k)\to 0 \quad \text{as } \delta_k\to 0,$$

where $w_m(\delta_k)$ is a subsequence selected by the above process for each δ_k. Now, by dichotomy we have

$$|\nabla w_m(\delta_k)|_n^n+|w_m(\delta_k)|_n^n\geqslant l-\alpha-\delta_k. \tag{3.26}$$

On the other hand using (3.24), (f_1), (f_2) and Lemma 2.3

$$|\nabla w_m(\delta_k)|_n^n+|w_m(\delta_k)|_n^n=O(\delta_k)+o(1)+\int w_m^{n-1}(\delta_k)\bar{f}(w_m(\delta_k))$$

$$\leqslant O(\delta_k) + o(1) + \tfrac{1}{2}|w_m(\delta_k)|_n^n$$

$$+ C\int w_m^{n-1}(\delta_k)[\exp(\alpha_n|w_m(\delta_k)|^{n/(n-1)})$$

$$- 1 - \alpha_n|w_m(\delta_k)|^{n/(n-1)}]$$

$$\leqslant O(\delta_k) + o(1) + \tfrac{1}{2}|w_m(\delta_k)|_n^n + C|\nabla w_m(\delta_k)|_n|w_m(\delta_k)|_n^n.$$

Thus $|\nabla w_m(\delta_k)|_n^n + |w_m(\delta_k)|_n^n \leqslant O(\delta_k) + o(1) + C|\bar{\mu}(\delta_k)|$, which contradicts (3.26). Therefore

$$\lim_{m\to\infty}\int|\nabla w_m|^n \geqslant A > 0 \quad \text{for } \varepsilon \text{ small enough.}$$

Now choosing $\sigma_m^n = 1 - \langle I'_\infty(w_m), w_m\rangle|\nabla w_m|_n^{-n}$ we see from (3.25) that $\tilde{w}_m \in M^\infty$, which implies that $M^\infty \neq \phi$ if dichotomy occurs. Also from (3.24) it follows that

$$\sigma_m^n = 1 + O(\varepsilon) + o(1).$$

Again, in view of (3.11) we can assume that $|\nabla w_m|_n^n < (1+r)/2$ for ε small enough, and therefore $\int \bar{F}(w_m)$ is bounded. Hence

$$I^\infty(w_m) = I^\infty(\tilde{w}_m) - \frac{1-\sigma_m^n}{2\sigma_m}|w_m|_n^n + \frac{1-\sigma_m^n}{2\sigma_m}\int \bar{F}(w_m)$$

$$\geqslant I^\infty(\tilde{w}_m) - O(\varepsilon) - o(1)$$

$$\geqslant C^\infty - O(\varepsilon) - o(1).$$

Thus, in view of (3.22), we obtain

$$I(w_m) \geqslant C^\infty - O(\varepsilon) - o(1) \tag{3.27}$$

and this proves the claim.

Now as in (3.24) we obtain

$$\langle I'(v_m), v_m\rangle = O(\varepsilon) + o(1). \tag{3.28}$$

So, in view of (f_3) we have

$$I(v_m) = \langle I'(v_m), v_m\rangle + \frac{1}{n}\int(v_m^{n-1}f(x,v_m) - F(x,v_m))$$

$$\geqslant \left(\frac{1}{n} - \theta\right)\int v_m^{n-1}f(x,v_m) + O(\varepsilon) + o(1)$$

$$= \left(\frac{1}{n} - \theta\right)\int(|\nabla v_m|^n + |v_m|^n) + O(\varepsilon) + o(1)$$

$$\geqslant \left(\frac{1}{n} - \theta\right)\alpha + O(\varepsilon) + o(1).$$

Therefore (3.16) and (3.27) imply

$$I(u_m) \geqslant C^\infty + \left(\frac{1}{n} - \theta\right)\alpha - O(\varepsilon) - o(1).$$

Letting $m \to \infty$ and then $\varepsilon \to 0$ we get $c \geqslant C^\infty + ((1/n) - \theta)\alpha$, a contradiction. This completes the case of bounded $\{y_m\}$.

Case 2: $\{y_m\}$ is unbounded.

In this case we change the role of $\{v_m\}$ and $\{w_m\}$ and then we can still get a contradiction as above.

Thus we have ruled out dichotomy and therefore by Lemma 1.1 of [5] there exists $\{y_m\}$ in $\mathbb{R}^n$ such that for any $\varepsilon > 0$, there is $t = t(\varepsilon) > 0$ such that

$$\int_{|x - y_m| > t} (|\nabla u_m|^n + |u_m|^n) < \varepsilon. \tag{3.29}$$

Claim 6: $\{y_m\}$ is bounded.

If not, then without loss of generality suppose $y_m \xrightarrow[m]{} \infty$. Now

$$I(u_m) = I^\infty(u_m) + \int_{|x - y_m| < t+1} [\bar{F}(u_m) - F(x, u_m)] + \int_{|x - y_m| \geqslant t+1} [\bar{F}(u_m) - F(x, u_m)]. \tag{3.30}$$

Let η_m be cut-off functions such that $0 \leqslant \eta_m \leqslant 1$, $\eta_m \equiv 0$ for $|x - y_m| \leqslant t$; $\eta_m \equiv 1$ for $|x - y_m| \geqslant t + 1$, $|\nabla \eta_m| \leqslant 2n$. Then for $\varepsilon < 1/\alpha_n (4n)^n \beta e$ and m large,

$$\int |\nabla(\eta_m u_m)|^n < \frac{1}{\alpha_n \beta e}.$$

Then by Lemma 2.3 and (3.29)

$$\begin{aligned} &\left| \int_{|x - y_m| \geqslant t+1} F(x, u_m) \right| \\ &\leqslant C \int_{|x - y_m| \geqslant t+1} |u_m|^n \\ &\quad + C \int_{|x - y_m| \geqslant t+1} |\eta_m u_n|^{n-1} (\exp(\alpha_n |\eta_m u_m|^{n/(n-1)}) - 1 - \alpha_n |\eta_m u_m|^{n/(n-1)}) \\ &< O(\varepsilon). \end{aligned} \tag{3.31}$$

Similarly

$$\left| \int_{|x_m| \geqslant t+1} \bar{F}(u_m) \mathrm{d}x \right| \leqslant O(\varepsilon). \tag{3.32}$$

Again as in the proof of (3.21), using the assumption $y_m \xrightarrow[m]{} \infty$ we obtain

$$\int_{|x_m| \geqslant t+1} (F(x, u_m) - \bar{F}(u_m)) \leqslant o(1) \quad \text{as } m \to \infty. \tag{3.33}$$

Thus $I(u_m) \geqslant I^\infty(u_m) - O(\varepsilon) - o(1)$. Again as earlier we can choose σ_m such that $\sigma_m = 1 - O(\varepsilon) + o(1)$, $\tilde{u}_m(x) = u_m(\sigma_m x)$ is in M^∞ and

$$I(u_m) \geqslant I^\infty(\tilde{u}_m) - O(\varepsilon) + o(1) \geqslant C^\infty - O(\varepsilon) + o(1).$$

Taking $m \to \infty$ and then $\varepsilon \to 0$ we obtain $c \geqslant C^\infty$, a contradiction which proves the claim.

Therefore, for any $\varepsilon > 0$, there exists $t = t(\varepsilon) > 0$ such that

$$\int_{|x|>t} (|\nabla u_m|^n + |u_m|^n)\mathrm{d}x < \varepsilon. \tag{3.34}$$

To use Strauss' lemma as in [3] we set $P(s) = s^{n-1}f(x,s)$, $Q(s) = \exp((2\alpha_n/(1+r))|s|^{n/(n-1)}) - \sum_{m=0}^{n-2}(1/m!)(2\alpha_n/(1+r))^m|s|^{n/(n-1)} + |s|^n$, so that $\lim_{|s|\to\infty} P(s)/Q(s) = 0$. Also by Lemma 2.1, $\int Q(u_m) \leqslant C$ for some constant $C > 0$. Therefore by Strauss' lemma, for any bounded Borel set Ω

$$\lim_{n\to\infty} \int_\Omega u_m^{n-1}f(x,u_m) = \int_\Omega u^{n-1}f(x,u).$$

In particular

$$\lim_{n\to\infty} \int_{|x|\leqslant t} u_m^{n-1}f(x,u_m) = \int_{|x|\leqslant t} u^{n-1}f(x,u). \tag{3.35}$$

Again, as in the proof of (3.21) we obtain

$$\left|\int_{|x|>t} u_m^{n-1}f(x,u_m)\right| \leqslant O(\varepsilon). \tag{3.36}$$

Thus

$$\lim_{m\to\infty} \int u_m^{n-1}f(x,u_m) = \int u^{n-1}f(x,u). \tag{3.37}$$

Claim 7: $u_m \to u$ in $W^{1,n}(\mathbb{R}^n)$.

Since $u_m \underset{m}{\rightharpoonup} u$ weakly in $W^{1,n}(\mathbb{R}^n)$ we have by Rellich's lemma $u_m \underset{m}{\to} u$ strongly in $L^n(\Omega)$ for any bounded smooth Ω. In particular,

$$\int_{|x|\leqslant t} |u_m|^n \underset{m}{\to} \int_{|x|\leqslant t} |u|^n.$$

Thus using (3.29) we get

$$\int |u_m|^n \underset{m}{\to} \int |u|^n. \tag{3.38}$$

As in (3.35), we have for any $\varphi \in C_b^0(\mathbb{R}^n)$,

$$\int u_m^{n-1}f(x,u_m)\varphi \underset{m}{\to} \int u^{n-1}f(x,u)\varphi. \tag{3.39}$$

Now, for any $\varphi \in C_b^\infty(\mathbb{R}^n)$ we have, by (3.5), (3.38) and (3.39)

$$\begin{aligned} 0 &= \lim_{m\to\infty} \langle I'(u_m), \varphi\rangle \\ &= \int T\cdot\nabla\varphi + \int |u|^{n-2}u\varphi - \int f(x,u)u^{n-2}\varphi, \end{aligned} \tag{3.40}$$

$$0 = \lim_{m\to\infty} \langle I'(u_m), u_m\varphi \rangle$$
$$= \int \varphi \mathrm{d}\mu + \int uT\cdot \nabla\varphi - \int f(x,u)u^{n-1}\varphi, \tag{3.41}$$
$$0 = \lim_{m\to\infty} \langle I'(u_m), u\varphi \rangle$$
$$= \int uT\cdot \nabla\varphi + \int \varphi T\cdot \nabla u + \int |u|^n\varphi - \int f(x,u)u^{n-1}\varphi. \tag{3.42}$$

Thus

$$\int uT\cdot \nabla\varphi = \int f(x,u)u^{n-1}\varphi - \int \varphi T\cdot \nabla u - \int |u|^n \varphi,$$

and substituting in (3.41) we get

$$\int \varphi \mathrm{d}\mu = \int \varphi T\cdot \nabla u + \int |u|^n\varphi.$$

In view of (3.5) we get $\int \varphi |\nabla u_m|^n \underset{m}{\longrightarrow} \int \varphi T\cdot \nabla u$ and hence $\int_{|x|\leq t} |\nabla u_m|^n \longrightarrow \int_{|x|\leq t} T\cdot \nabla u$. This implies, using (3.29), $\int |\nabla u_m|^n \underset{m}{\longrightarrow} \int T\cdot\nabla u$. That is,

$$\lim_{m\to\infty} \int |\nabla u_m|^n = \lim_{m\to\infty} \int |\nabla u_m|^{n-2}\nabla u_m \cdot \nabla u.$$

Then

$$\lim_{m\to\infty} \int (|\nabla u_m|^{n-2}\nabla u_m - |\nabla u|^{n-2}\nabla u)\cdot(\nabla u_m - \nabla u) = 0,$$

which implies

$$\lim_{m\to\infty} \int |\nabla u_m - \nabla u|^n = 0,$$

by using an inequality

$$|a-b|^p \leq 2^{p-1}(|a|^{p-2}a - |b|^{p-2}b)\cdot(a-b)$$

for any $a, b \in \mathbb{R}^n$, $p \geq 2$. Therefore $u_m \underset{m}{\longrightarrow} u$ strongly in $W^{1,n}(\mathbb{R}^n)$ as desired. □

Proof of Theorem 1.1: By the definition of S_p, for any $\varepsilon > 0$ there exists $u_\varepsilon \in W^{1,n}(\mathbb{R}^n)$ such that $(\|u_\varepsilon\|/|u_\varepsilon|_p) < S_p + \varepsilon$. Let $v_\varepsilon = ((1-n\theta)^{1/n}/\|u_\varepsilon\|)|u_\varepsilon|_p$. Then $\|v_\varepsilon\|^n = 1 - n\theta$, $|v_\varepsilon|_p = ((1-n\theta)^{1/n}/\|u_\varepsilon\|)|u_\varepsilon|_p$ and so $\|v_\varepsilon\|/|v_\varepsilon|_p = \|u_\varepsilon\|/|u_\varepsilon|_p$.

Claim: $\int \bar{F}(v_\varepsilon) > (1/n)|v_\varepsilon|_n^n$.

Choose ε small enough so that $C_p > (p/n)(S_p + 2\varepsilon)^p(1-n\theta)^{1-(p/n)}$. Now $S_p + \varepsilon > \|v_\varepsilon\|/|v_\varepsilon|_p$ and so

$$\int \bar{F}(v_\varepsilon) \geq \frac{C_p}{p}\int |v_\varepsilon|^p > \frac{C_p}{p}\frac{\|v_\varepsilon\|^p}{(S_p+\varepsilon)^p}$$
$$> \frac{1-n\theta}{n}\left(\frac{S_p+2\varepsilon}{S_p+\varepsilon}\right)^p > \frac{1}{n}\int (|\nabla v_\varepsilon|^n + |v_\varepsilon|^n)$$

and this proves the claim.

Therefore by Lemma 2.2 I_0^∞ is achieved by some u_0 and $I_0^\infty \leqslant 1 - n\theta$. Then

$$-\Delta_n u_0 = \lambda(\bar{f}(u_0)u_0^{n-2} - |u_0|^{n-2}u_0)$$

for some Lagrange multiplier $\lambda \in \mathbb{R}$. By (f_3) we have

$$\int \bar{F}(u_0) = \frac{1}{n}\int |u_0|^n \leqslant \theta \int u_0^{n-1}\bar{f}(u_0).$$

Also we know that $I_0^\infty > 0$. Thus $\lambda > 0$. Let $u(x) = u_0(\lambda^{-1/n}x)$. Then u satisfies (1.1), $\int \bar{F}(u) = (1/n)\int |u|^n$ and $I(u) = (1/n)\int |\nabla u|^n = (1/n)\int |\nabla u_0|^n < (1/n) - \theta$. This proves the theorem. □

Proof of Theorem 1.2: By the assumptions we see that $\bar{f}(t)$ satisfies the conditions of Theorem 1.1. Thus there exists $\bar{u} \in W^{1,n}(\mathbb{R}^n)$ satisfying

$$-\Delta_n \bar{u} + |\bar{u}|^{n-2}\bar{u} = \bar{f}(\bar{u})\bar{u}^{n-2} \quad \text{in } \mathbb{R}^n. \tag{3.43}$$

Moreover, $I^\infty(\bar{u}) < (1/n) - \theta$. Let

$$h(t) = I^\infty(t\bar{u}) = \frac{t^n}{n}\int [|\nabla \bar{u}|^n + |\bar{u}|^n] - \int \bar{F}(t\bar{u}).$$

By (f_2) and (3.43) we have

$$h'(t) \geqslant 0 \quad \text{for } 0 \leqslant t < 1;\ h'(1) = 0$$

$$h'(t) \leqslant 0 \quad \text{for } t > 1.$$

Hence $I^\infty(\bar{u}) = \max_{t \geqslant 0} I^\infty(t\bar{u})$. Further, since $I(t\bar{u}) \to -\infty$ as $t \to \infty$, there exists $t_0 \in (0, \infty)$ such that $I(t_0\bar{u}) = \max_{t \geqslant 0} I(t\bar{u})$. Now, by (f_5) and the hypothesis that $f(x,t) \not\equiv \bar{f}(t)$ we have

$$I(t_0\bar{u}) < I^\infty(t_0\bar{u}) \leqslant \max_{t \geqslant 0} I^\infty(t\bar{u}) = I^\infty(\bar{u}). \tag{3.44}$$

We claim that $C^\infty = I^\infty(\bar{u})$. Clearly $C^\infty \leqslant I_0^\infty = I^\infty(\bar{u})$. Further, given $\varepsilon > 0$, we can find $u \in M^\infty$ such that $I^\infty(u) < C^\infty + \varepsilon$. Using (f_2) we can find $t \in \mathbb{R}^+$ such that $\int F(x, tu) = (1/n)|tu|_n^n$. Again as above we can show that $I^\infty(u) = \max_{t \geqslant 0} I^\infty(tu)$. Thus

$$I^\infty(\bar{u}) = I_0^\infty \leqslant I(tu) \leqslant I^\infty(tu) \leqslant I^\infty(u) < C^\infty + \varepsilon,$$

which gives the other inequality, since $\varepsilon > 0$ was arbitrary.

Therefore from (3.44) we get

$$I(t_0\bar{u}) < I^\infty(\bar{u}) = C^\infty < \frac{1}{n} - \theta. \tag{3.45}$$

It is easy to see, using (f_1), (f_2) and Lemma 2.3, that there exist $\rho, \alpha > 0$ with

$$I(u) > \alpha \quad \text{for all } u \text{ satisfying } \|u\| = \rho.$$

Choose $t_1 > t_0$ sufficiently large so that $I(t\bar{u}) < 0$ for $t > t_1$. Let Γ be the set of all continuous paths connecting 0 and $t_1\bar{u}$. Define

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma} I(u). \tag{3.46}$$

Then $c > \alpha$. Also

$$c \leqslant \max_{0 \leqslant t \leqslant t_1} I(t\bar{u}) < C^{\infty} = \min\left\{C^{\infty}, \frac{1}{n} - \theta\right\}.$$

By Mountain Pass lemma (see [4]), there exists a sequence $\{u_m\}$ in $W^{1,n}(\mathbb{R}^n)$ such that

$$I(u_m) \underset{m}{\rightarrow} c, \quad I'(u_m) \underset{m}{\rightarrow} 0 \quad \text{in } (W^{1,n}(\mathbb{R}^n))^*.$$

By Lemma 3.1, for a subsequence $u_m \underset{m}{\rightarrow} u$ strongly in $W^{1,n}(\mathbb{R}^n)$. Thus $I(u) = c$, $I'(u) = 0$, which implies that $u \not\equiv 0$ and u is a nontrivial solution of (1.1). This completes the proof of the theorem. □

Acknowledgement

The author would like to thank Prof. Adimurthi for his valuable advice and the National Board for Higher Mathematics, India for the financial support. He would also like to thank Prof. Tilak Bhattacharya for his valuable suggestions.

References

[1] Adimurthi, Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian, *Ann. Scu. Norm. Sup. Pisa* **17** 393–413 (1990)
[2] Aubin Th, *Nonlinear analysis on manifolds: Monge–Ampere equations.* (New York: Springer-Verlag) (1982)
[3] Berestycki H and Lions P L, Nonlinear scalar field equations, I. Existence of ground state, *Arch. Ration. Mech. Anal.* **82** (1983) 313–346
[4] Brezis H and Nirenberg L, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent, *Commun. Pure Appl. Math.* **36** (1983) 437–477
[5] Cao D M, Nontrivial solution of semilinear elliptic equations with critical exponent in $\mathbb{R}^2$, *Commun. Partial Differ. Equ.* **17** (1992) 407–435
[6] Lions P L, The concentrated-compactness principle in the calculus of variations. The locally compact case, part I, *Ann. I.H.P. Anal. Nonlin.* **1** (1984) 109–145
[7] Lions P L, The concentrated-compactness principle in the calculus of variations. The locally compact case, part II, *Ann. I.H.P. Anal. Nonlin.* **1** (1984) 223–283
[8] Moser J, A sharp form of an inequality by N Trudinger, *Indiana Univ. Math. J.* **20** (1971) 1077–1092
[9] Panda R, On semilinear Neumann problems with critical growth for the n-Laplacian, *Nonlinear analysis, theory, methods and applications*
[10] Talenti G, Best constants in Sobolev inequality, *Ann. Mat. Pura. Appl.* **110** (1976) 353–372