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Oscillation of higher order delay differential equations

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Abstract. A sufficient condition was obtained for oscillation of all solutions of the *odd-order* delay differential equation

$$x^{(n)}(t)+\sum_{i=1}^{m} p_i(t)x(t-\sigma_i)=0, \tag{$*$}$$

where $p_i(t)$ are non-negative real valued continuous function in $[T,\infty]$ for some $T\geqslant 0$ and $\sigma_i\in(0,\infty)$ $(i=1,2,\dots,m)$. In particular, for $p_i(t)=p_i\in(0,\infty)$ and $n>1$ the result reduces to

$$\frac{1}{m}\left(\sum_{i=1}^{m}(p_i\sigma_i^n)^{1/2}\right)^2>(n-2)!\frac{(n)^n}{e},$$

implies that every solution of ($*$) oscillates. This result supplements for $n>1$ to a similar result proved by Ladas *et al* [*J. Diff. Equn.*, **42** (1982) 134–152] which was proved for the case $n=1$.

Keywords. Odd order; delay equation; oscillation of all solutions.

1. Introduction

This paper was motivated by certain results of the paper [7] and [8] due to Ladas *et al.* In [7] authors proved that all solutions of the *odd-order* delay differential equation

$$x^{(n)}(t)+\sum_{i=1}^{m} p_i x(t-\sigma_i)=0, \tag{1}$$

oscillates (i.e., every solution $x(t)$ has zeros for arbitrarily large t) if and only if the associated characteristic equation

$$\lambda^n+\sum_{i=1}^{m} p_i e^{-\lambda\sigma_i}=0 \tag{2}$$

has no real roots, where p_i and $\sigma_i\in(0,\infty)$ for $i=1,2,\dots,m$. Further, it was proved that (2) has no real roots if and only if

$$(p_1)^{1/n}\sigma_1>\frac{n}{e}.$$

In the literature, it was observed that the *odd-order* differential equations of the form

$$x^{(n)}(t)+\sum_{i=1}^{m} p_i(t)x(t-\sigma_i)=0, \tag{3}$$

where $p_i\in C([T,\infty),(0,\infty))$, $T\geqslant 0$ and $\sigma_i\in(0,\infty)$, is least studied. In this connection, we may refer, in particular, to [4], [5], [9] and the references therein. For $n=1$, (3) is almost well-studied. In this case there are several results associated with its characteristic equation (see [3] and [7]) as well as conditions on coefficients and deviating arguments which ensures that every solution of (3) oscillates. In [8], authors proved that if $p_i \in C([T,\infty),(0,\infty))$, $\sigma_i \in (0,\infty)$ $(i = 1,2,\ldots,m)$ and $n = 1$ then

$$\liminf_{t\to\infty} \int_{t-\sigma_i/2}^{t} p_i(s)\mathrm{d}s > 0 \quad (i = 1,2,\ldots,m) \tag{4}$$

and

$$\frac{1}{m}\sum_{i=1}^{m}\left(\liminf_{t\to\infty}\int_{t-\sigma_i}^{t} p_i(s)\mathrm{d}s\right) + \frac{2}{m}\sum_{\substack{i<j\\ i,j=1}}^{m}\left[\left(\liminf_{t\to\infty}\int_{t-\sigma_j}^{t} p_i(s)\mathrm{d}s\right)\left(\liminf_{t\to\infty}\int_{t-\sigma_i}^{t} p_j(s)\mathrm{d}s\right)\right] > \frac{1}{e} \tag{5}$$

then every solution of (3) oscillates. If $p_i(t) = p_i \in (0,\infty)(i = 1,2,\ldots,m)$ then the above result becomes

$$\frac{1}{m}\left(\sum_{i=1}^{m}(p_i\sigma_i)^{1/2}\right)^2 > \frac{1}{e}$$

implies that every solution of (3) oscillates. In this paper an attempt has been made to obtain a similar result which shows that every solution of (3) oscillates. Our result fails to hold when $n = 1$. Indeed, when $p_i(t) = p_i \in (0,\infty)$, the main result of this paper shows that if

$$\frac{1}{m}\left(\sum_{i=1}^{m}(p_i\sigma_i^n)^{1/2}\right)^2 > (n^n(n-2)!)\frac{1}{e}$$

then every solution of (3) oscillates. Although our result does not generalize the result of Ladas *et al* [8], but certainly supplements for higher order equations.

2. Main results

In the beginning of this section we prove a lemma for its use in the sequel.

Lemma 1. *Let* $f \in C^{(n)}([T,\infty),(0,\infty))$, $T \geq 0$ *such that* $f^{(n)}(t) \leq 0$, $t \geq T$. *If* n *is odd and* $\sigma \in (0,\infty)$ *then there exists* $T_0 \geq T$ *such that*

$$\frac{f(t-\sigma)}{f^{(n-1)}(t)} \geq \frac{\sigma^{n-1}}{(n-1)!}, \quad t \geq T_0. \tag{6}$$

Proof. Since $f(t) \geq 0$ and $f^{(n)}(t) \leq 0$ for $t \geq T$, there exists $T_1 \geq T$ and $0 \leq k \leq n-1$ such that

$$f^{(j)}(t) > 0 \quad \text{for } j \leq k$$

and

$$f^{(j)}(t)f^{(j+1)}(t) \leq 0 \quad \text{for } k \leq j \leq n-1.$$

Expanding $f(t)$ by Taylor's theorem, there exists $x \in (t-\sigma, t)$ such that

$$\begin{aligned} f(t) &= \sum_{j=0}^{k-1}\frac{\sigma^j}{j!}f^{(j)}(t-\sigma) + \frac{\sigma^k}{k!}f^{(k)}(x) \\ &\geq \frac{\sigma^k}{k!}f^{(k)}(t), \quad t \geq T_1 + \sigma. \end{aligned} \tag{7}$$

Similarly, expanding $f^{(k)}(t)$ by Taylor's theorem we get

$$f^{(k)}(t-\sigma) \geqslant \frac{(-\sigma)^{n-k-1}}{(n-k-1)!} f^{(n-1)}(t), \quad t \geqslant T_1 + \sigma. \tag{8}$$

Replacing t by $t-\sigma$ in the inequality (7) we get

$$f(t-\sigma) \geqslant \frac{\sigma^k}{k!} f^{(k)}(t-\sigma), \quad t \geqslant T_1 + 2\sigma. \tag{9}$$

Further, using (8) in (9) along with the fact that $k!(n-k-1)! \leqslant (n-1)!$ and setting $T_0 = T_1 + 2\sigma$ we have our proposed inequality.

This completes the proof of the lemma.

Theorem 1. *Suppose that* $p_i \in C([T,\infty),(0,\infty))$, $T>0$ *and* $\sigma_i \in (0,\infty)$ $(i=1,2,3,\ldots,m)$. *Further if*

$$\liminf_{t\to\infty} \int_{t-\omega\sigma_i}^{t} p_i(s)\mathrm{d}s > 0 \quad (i=1,2,\ldots,m)$$

and

$$\frac{1}{m}\sum_{i=1}^{m} \sigma_i^{n-1} p_{ii} + \frac{2}{m} \sum_{\substack{i<j\\ i,j=1}}^{m} (p_{ij}p_{ji}(\sigma_i\sigma_j)^{n-1})^{1/2} > (n-1)! \frac{(n)^{n-1}}{e}, \tag{10}$$

where

$$p_{ij} = \int_{t-\omega\sigma_j}^{t} p_i(s)\mathrm{d}s$$

and

$$\omega = \left(\frac{n-1}{n}\right),$$

then every solution of (3) *oscillates.*

Proof. On the contrary, suppose that $x(t)>0$ for $t \geqslant t_0$. Dividing (3) throughout by $x^{(n-1)}(t)$ we get

$$\frac{x^{(n)}(t)}{x^{(n-1)}(t)} + \sum_{i=1}^{m} p_i(t) \frac{x(t-\sigma_i)}{x^{(n-1)}(t)} = 0, \tag{11}$$

that is,

$$\frac{x^{(n)}(t)}{x^{(n-1)}(t)} + \sum_{i=1}^{m} p_i(t) \frac{x(t-\sigma_i)}{x^{(n-1)}(t-\omega\sigma_i)} \frac{x^{(n-1)}(t-\omega\sigma_i)}{x^{(n-1)}(t)} = 0. \tag{12}$$

By Lemma 1, there exists $t_1 \geqslant t_0$ such that

$$\frac{x(t-\sigma_i)}{x^{(n-1)}(t-\omega\sigma_i)} = \frac{x(t-\omega\sigma_i-\sigma_i/n)}{x^{(n-1)}(t-\omega\sigma_i)} \geqslant \frac{(\sigma_i/n)^{n-1}}{(n-1)!}, \quad t \geqslant t_1,$$

and the use of this inequality in (12) results

$$\frac{x^{(n)}(t)}{x^{(n-1)}(t)} + \sum_{i=1}^{m} K_i p_i(t) \frac{x^{(n-1)}(t-\omega\sigma_i)}{x^{(n-1)}(t)} \leqslant 0, \tag{13}$$

where

$$K_i = \frac{1}{(n-1)!}\left(\frac{\sigma_i}{n}\right)^{n-1}.$$

Integrating both sides of (13) from $t-\omega\sigma_k$ to t we get

$$\log\left(\frac{x^{(n-1)}(t-\omega\sigma_k)}{x^{(n-1)}(t)}\right) \geqslant \sum_{i=1}^{m} K_i \int_{t-\omega\sigma_k}^{t} p_i(s)\frac{x^{(n-1)}(s-\omega\sigma_i)}{x^{(n-1)}(s)}\mathrm{d}s.$$

Setting

$$\alpha_i = \liminf_{t\to\infty} \frac{x^{(n-1)}(t-\omega\sigma_i)}{x^{(n-1)}(t)}$$

and

$$p_{ik} = \liminf_{t\to\infty} \int_{t-\omega\sigma_k}^{t} p_i(s)\mathrm{d}s \quad i,j=1,2,\ldots,m$$

we see that

$$\log(\alpha_k) \geqslant \sum_{i=1}^{m} K_i\alpha_i p_{ik}.$$

Suppose that $\alpha_k<\infty$ for $k=1,2,3,\ldots,m$. In this case, dividing both sides of the above inequality by α_k and using the fact that

$$\frac{\log(\alpha_k)}{\alpha_k} \leqslant \frac{1}{e} \quad \text{for } \alpha_k \geqslant 1,$$

and $\alpha_k \geqslant 1$ (since $x^{(n-1)}(t)$ is positive decreasing) it follows that

$$\frac{1}{e} \geqslant \sum_{i=1}^{m} K_i\frac{\alpha_i}{\alpha_k}p_{ik}, \quad k=1,2,\ldots,m.$$

Summing the above inequality for $k=1,2,\ldots,m$ we obtain

$$\frac{m}{e} \geqslant \sum_{k=1}^{m}\sum_{i=1}^{m} K_i\frac{\alpha_i}{\alpha_k}p_{ik}.$$

that is,

$$\begin{aligned}\frac{m}{e} \geqslant\ & K_1\frac{\alpha_1}{\alpha_1}p_{11} + K_2\frac{\alpha_2}{\alpha_1}p_{21} + \cdots + K_m\frac{\alpha_m}{\alpha_1}p_{m1}\\ & + K_1\frac{\alpha_1}{\alpha_2}p_{12} + K_2\frac{\alpha_2}{\alpha_2}p_{22} + \cdots + K_m\frac{\alpha_m}{\alpha_2}p_{m2}\\ & + \cdots\cdots\cdots\\ & + \cdots\cdots\cdots\\ & + K_1\frac{\alpha_1}{\alpha_m}p_{1m} + K_2\frac{\alpha_2}{\alpha_m}p_{2m} + \cdots + K_m\frac{\alpha_m}{\alpha_m}p_{mm}.\end{aligned}$$

Rearranging the right hand side elements of the above inequality first along the diagonal then above and below the diagonal respectively, we get

$$\frac{m}{e} \geqslant \sum_{i=1}^{m} K_i\frac{\alpha_i}{\alpha_i}p_{ii} + \sum_{\substack{i>j\\ i,j=1}}^{m} K_i\frac{\alpha_i}{\alpha_j}p_{ij} + \sum_{\substack{i<j\\ i,j=1}}^{m} K_i\frac{\alpha_i}{\alpha_j}p_{ij}. \tag{14}$$

that is,

$$\frac{m}{e} \geqslant \sum_{i=1}^{m} K_i p_{ii} + \sum_{\substack{i<j\\ i,j=1}}^{m}\left(K_i p_{ij}\frac{\alpha_i}{\alpha_j} + K_j\frac{\alpha_j}{\alpha_i}p_{ji}\right). \tag{15}$$

Since the arithmetic mean is greater than the geometric mean

$$K_i p_{ij}\frac{\alpha_i}{\alpha_j} + K_j p_{ji}\frac{\alpha_j}{\alpha_i} \geqslant 2\sqrt{(p_{ij}p_{ji})(K_iK_j)}. \tag{16}$$

In view of (16), (15) reduces to

$$\frac{m}{e} \geqslant \sum_{i=1}^{m} K_i p_{ii} + 2\sum_{\substack{i<j\\ i,j=1}}^{m} ((p_{ij}p_{ji})(K_iK_j))^{1/2}.$$

Putting the value of K_i and K_j in the above inequality we obtain

$$\frac{1}{m}\sum_{i=1}^{m} \sigma_i^{n-1*} p_{ii} + \frac{2}{m}\sum_{\substack{i<j\\ i,j=1}}^{m} ((p_{ij}p_{ji})(\sigma_i\sigma_j)^{n-1})^{1/2} \leqslant (n-1)!\frac{n^{n-1}}{e},$$

which is a contradiction to our assumption.

Next, assume that $\alpha_i = \infty$ for some $i = 1, 2, \ldots, m$. That is,

$$\liminf_{t\to\infty} \frac{x^{(n-1)}(t-\omega\sigma_i)}{x^{(n-1)}(t)} = \infty, \tag{17}$$

for some $i = 1, 2, \ldots, m$. From (3) it follows that

$$x^{(n)}(t) + p_i(t)x(t-\sigma_i) \leqslant 0,$$

for the value of i for which (17) holds. From the inequality above (13) it follows that

$$\frac{x(t-\sigma_i)}{x^{(n-1)}(t-\omega\sigma_i)} \geqslant \frac{(\sigma_i/n)^{n-1}}{(n-1)!}. \tag{18}$$

From (17) and (18) it follows that

$$x^{(n)}(t) + p_i(t)\frac{(\sigma_i/n)^{n-1}}{(n-1)!}x^{(n-1)}(t-\omega\sigma_i) \leqslant 0. \tag{19}$$

Integrating both sides of (19) from $t-\omega\sigma_i/2$ to t and using the fact that $x^{(n-1)}(t) > 0$ and decreasing we get

$$x^{(n-1)}(t) - x^{(n-1)}(t-\omega\sigma_i/2) + \frac{(\sigma_i/n)^{n-1}}{(n-1)!}\int_{t-\omega\sigma_i/2}^{t} p(s)\mathrm{d}s \leqslant 0. \tag{20}$$

Dividing both sides of (20) first by $x^{(n-1)}(t)$ and then by $x^{(n-1)}(t-\omega\sigma_i/2)$ we have the following inequalities respectively:

$$1 - \frac{x^{(n-1)}(t-\omega\sigma_i/2)}{x^{(n-1)}(t)} + \frac{(\sigma_i/n)^{n-1}}{(n-1)!}\frac{x^{(n-1)}(t-\omega\sigma_i)}{x^{(n-1)}(t)}\int_{t-\omega\sigma_i/2}^{t} p_i(s)\mathrm{d}s < 0 \tag{21}$$

and

$$\frac{x^{(n-1)}(t)}{x^{(n-1)}(t-\omega\sigma_i/2)} - 1 + \frac{(\sigma_i/n)^{n-1}}{(n-1)!}\frac{x^{(n-1)}(t-\omega\sigma_i)}{x^{(n-1)}(t-\omega\sigma_i/2)}\int_{t-\omega\sigma_i/2}^{t} p_i(s)\mathrm{d}s < 0. \tag{22}$$

In view of (10), (17) and (21) we obtain

$$\lim_{t\to\infty} \frac{x^{(n-1)}(t-\omega\sigma_i/2)}{x^{(n-1)}(t)} = \infty. \tag{23}$$

Using (23) in (22) along with (10) we see that

$$\lim_{t\to\infty} \frac{x^{(n-1)}(t-\omega\sigma_i)}{x^{(n-1)}(t-\omega\sigma_i/2)} < \infty,$$

Replacing t by $t+\omega\sigma_i/2$ in the above inequality we get

$$\lim_{t\to\infty} \frac{x^{(n-1)}(t-\omega\sigma_i/2)}{x^{(n-1)}(t)} < \infty,$$

which is a contradiction to (23).

This completes the proof of the theorem.

COROLLARY 1.

If $p_i(t) = p_i \in (0,\infty)$ *and* $\sigma_i \in (0,\infty)$ *then*

$$\frac{1}{m}\left(\sum_{i=1}^{m} (p_i\sigma_i^n)^{1/2}\right)^2 > (n^n(n-2)!)\frac{1}{e}$$

implies that every solution of (3) *oscillates.*

Proof. In this particular case

$$p_{ij} = \left(\frac{n-1}{n}\right)p_i\sigma_j$$

and hence (10) reduces to

$$\frac{1}{m}\left(\frac{n-1}{n}\right)\left\{\sum_{i=1}^{m} \sigma_i^n p_i + 2\sum_{\substack{i<j\\ i,j=1}}^{m} (p_ip_j\sigma_i^n\sigma_j^n)^{1/2}\right\} > (n-1)!\frac{n^{n-1}}{e},$$

that is, (24) holds. Hence the proof follows from Theorem 1.

Example 1. The equation

$$x^{(3)}(t) + \left(4+\frac{1}{t}\right)x(t-1) + \left(16+\frac{1}{t^2}\right)x(t-2) = 0,$$

satisfies the hypotheses of Theorem 1 and hence every solution of it oscillates. But Theorem 5.2 of [8] is not applicable to it.

Example 2. Consider the equation

$$x'(t) + \left(4+\frac{1}{t}\right)x(t-1) + \left(16+\frac{1}{t^2}\right)x(t-2) = 0.$$

By Theorem 5.2 of [8], every solution of it oscillates. But Theorem 1 of this paper is not applicable to this equation. This is due to the fact that Theorem 1 holds only for $n > 1$ and is an odd integer.

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