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Weakly prime sets for function spaces

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Abstract. We define and study weakly prime sets for a function space and show that it coincides with the known concept of weakly prime sets for function algebras and spaces of affine functions.

Keywords. Weakly prime set; function space; function algebra; space of affine functions.

1. Introduction

A *function space* A on a compact Hausdorff space X is a closed subspace of the space $C(X)$ of all continuous, complex-valued functions on X separating points and containing constants. If A is an algebra, it is called a function algebra. The Bishop and Šilov decompositions play an important role in characterizing function algebras. Later on these decompositions were studied for function spaces [6]. Ellis [3] defined and studied these decompositions for the spaces of affine functions on a compact convex set.

For a function algebra, certain decompositions finer than the Bishop and Šilov decompositions have been defined and studied [6]. One such decomposition of weakly prime sets, was defined and discussed by Ellis [4] for function algebras as well as for spaces of affine functions. Here we generalize this concept for a function space, study its properties and show that it coincides with the corresponding definitions of Ellis.

We also give examples of function spaces whose family of maximal weakly prime sets differ from the corresponding families of its induced algebras.

2. Function space

Let X be a compact Hausdorff space. Throughout this paper we assume that A is a function space on X. For a closed subset E of X, we define

$$N(A|_E) = \{f \in C(E): \ fg \in A|_E \quad \text{for all } g \in A|_E\}.$$

For the concepts like peak set, p-set, etc. related to a function space and for the various properties of a decomposition for a function space, we refer to [2], [5] and [7].

DEFINITION 2.1

A closed subset E of X is called a *weakly prime set* for A if $E = G \cup H$, with G and H generalized peak sets for $N(A_{|E})$, then either $G = E$ or $H = E$.

The function space A is called *weakly prime* if X is a weakly prime set for A.

Remarks 2.2. (i) If A is an algebra, then $N(A_{|E}) = A_{|E}$ and hence Definition 2.1 coincides with the definition for a function algebra given by Ellis [4].

(ii) It can be shown that each weakly prime set is contained in a maximal weakly prime set for A.

The collection of all maximal weakly prime sets for A is denoted by $\mathscr{P}(A)$.

(iii) It is easy to check that $\mathscr{P}(A)$ is finer than the Bishop decompositions for A and hence $\mathscr{P}(C(X)) = \{\{x\}: x\in X\}$.

(iv) It can be easily verified that A is weakly prime if and only if $N(A)$ is weakly prime. Further, for a closed subset E of X, $N(A)_{|E} \subset N(A_{|E})$ and so, $\mathscr{P}(A)$ is weaker than $\mathscr{P}(N(A))$. But, in general, $\mathscr{P}(A) \neq \mathscr{P}(N(A))$ (see Example 2.6(i)).

As in case of a function algebra, we shall show that here also every member of $\mathscr{P}(A)$ is a p-set and $\mathscr{P}(A)$ has the (GA)-property [5] for A.

We shall need the following lemma.

Lemma 2.3. *If E is a p-set for A and $F \subset E$ is a generalized peak set for $N(A_{|E})$, then F is a p-set for A.*

Proof. Let $\mu\in A^{\perp}$ and $\varepsilon > 0$ be given. Then there is an open set U in X such that $|\mu|(U-F) < \varepsilon$. Clearly, $E\cap U$ is open in E and $F \subset E\cap U$. Since F is a generalized peak set for $N(A_{|E})$, there is a peak set T for $N(A_{|E})$ such that $F\subset T\subset E\cap U$. Let $f\in N(A_{|E})$ be a peaking function for T. Define h on E by $h=1$ on T and $h=0$ on $E\setminus T$. Then f^n converges pointwise and boundedly to h on E.

Now let $g\in A$. Then

$$\int_T g\,\mathrm{d}\mu = \int_E gh\,\mathrm{d}\mu = \lim \int_E g f^n \mathrm{d}\mu.$$

But $\int_E g f^n \mathrm{d}\mu = 0$, as $f^n g_{|E}\in A_{|E}$ and E is a p-set for A. Thus $\int_T g\,\mathrm{d}\mu = 0$.

Now

$$\left|\int_F g\,\mathrm{d}\mu\right| = \left|\int_F g\mathrm{d}\mu - \int_T g\mathrm{d}\mu\right| \leqslant \|g\|(|\mu|(T-F))$$
$$\leqslant \|g\|(|\mu|(U-F))$$
$$< \varepsilon\|g\|.$$

Since ε is arbitrary, $|\int_F g\mathrm{d}\mu| = 0$ or F is a p-set for A.

PROPOSITION 2.4

A maximal weakly prime set for A is a p-set for A.

Proof. Let E be a maximal weakly prime set for A and let F denote the smallest p-set for A which contains E. We shall show that F is a weakly prime set for A.

Let F_1 and F_2 be generalized peak sets for $N(A_{|F})$ with $F_1\cup F_2 = F$. Then $E=(F_1\cap E)\cup(F_2\cap E)$ and since $N(A_{|F})_{|E}\subset N(A_{|E})$, $F_1\cap E$ and $F_2\cap E$ are generalized peak sets for $N(A_{|E})$. Since E is a weakly prime set for A, either $F_1\cap E=E$ or $F_2\cap E=E$, i.e., either $E\subset F_1$ or $E\subset F_2$. If $E\subset F_1$, then $E\subset F_1\subset F$ where F is a p-set for A and F_1 is a generalized peak set for $N(A_{|F})$. So, by Lemma 2.3, F_1 is a p-set for A and hence $F_1=F$. Similarly, if $E\subset F_2$, then $F_2=F$. Thus F is a weakly prime set for A and by the maximality of E, we have $E=F$.

Next, we show that the family $\mathscr{P}(A)$ characterizes a function space A in the sense that it has the (D)-property for A [5.7], i.e. if $f\in C(X)$ and $f_{|E}\in(A_{|E})^-$ for every $E\in\mathscr{P}(A)$, then

$f \in A$. Actually the Bishop's theorem can be restated as "The Bishop decomposition has the (D)-property for A". In fact the Bishop decomposition has a stronger property than the (D)-property, namely the (GA)-property.

By (GA)-property for a family $\mathscr{F}$ of closed subsets of X for A [7] we mean that for each $\mu \in b(A^{\perp})^e$, $\operatorname{supp} \mu \subset F$ for some $F \in \mathscr{F}$, where $b(A^{\perp})^e$ denotes the set of extreme points of the unit ball of $A^{\perp}$.

We shall show that $\mathscr{P}(A)$ has the (GA)-property for A.

Theorem 2.5. *$\mathscr{P}(A)$ has the (GA)-property for A.*

Proof. Let $\mu \in b(A^{\perp})^e$, the set of extreme points of the unit ball in $A^{\perp}$ and let $S = \operatorname{supp} \mu$. It is enough to show that S is a weakly prime set for A.

Let G and H be generalized peak sets for $N(A_{|S})$ with $S = G \cup H$. Let $\mu_1 = \mu_{|G'}$, $\mu_2 = \mu - \mu_1$ and $g \in A$. Since $\mu = \mu_S \in A^{\perp}$ and G is a generalized peak set for $N(A_{|S})$, by Lemma 2.3, we get $\int_G g \, d\mu = 0$. Thus $\mu_1 \in A^{\perp}$ and hence $\mu_2 \in A^{\perp}$. Also, $\|\mu\| = 1 = \|\mu_1\| + \|\mu_2\|$. Hence $\mu_1 = \mu$ or $\mu_2 = \mu$, as $\mu \in b(A^{\perp})^e$, i.e., $G = S$ or $H = S$. Thus S is a weakly prime set for A.

Examples 2.6. (i) Let X be the union of a line segment F and a sequence of disjoint solid rectangles $\{F_n; n = 1, 2, \ldots\}$ converging to F. Let A be the set of all f in $C(X)$ such that $f_{|F_n}$ is a polynomial of degree atmost n. Then A is a function space on X and as in [7] it can be checked that $\mathscr{P}(A) = \{F_n | n \in \mathbb{N}\} \cup \{\{x\}; x \in X\}$. Note that, here $N(A) = \{f \in C(X): f_{|F_n}$ is constant, for each $n \in \mathbb{N}\}$ and hence $\mathscr{P}(N(A)) = \{F_n: n \in \mathbb{N}\} \cup \{F\}$. Therefore, $\mathscr{P}(A) \neq \mathscr{P}(N(A))$.

(ii) Let T denote the unit circle in $\mathbb{C}$ and $A(T)$ denote the disc algebra on T. Let $\Phi \in A(T)$ be such that $\Phi \neq 0$ on T. Define $A = \{\Phi^{-1} f: f \in A(T)\}$. Then A is a function space on T and $N(A) = A(T)$ [8]. It is clear that $N(A)$ is weakly prime and hence by Remark 2.2 (iv), A is also weakly prime, i.e., $\mathscr{P}(A) = \{T\}$. Since $A(T)$ is a maximal function algebra on T and $A(T) \subsetneq A$, the algebra generated by A will be $C(T)$. But $\mathscr{P}(C(T)) = \{\{x\}: x \in T\}$ by Remark 2.2 (iii) while $\mathscr{P}(A) = \{T\}$.

3. Space of affine functions

Let K be a compact convex subset of a locally convex Hausdorff space and let $A(K)$ denote the Banach space of all real-valued continuous affine functions on K with the supremum norm. The set of extreme points of K will be denoted by ∂K.

Ellis [4] has defined weakly prime sets for $A(K)$ with the help of concepts of convexity. Now $A(K)$ can also be looked upon as a function space on K. So we can discuss $\mathscr{P}(A(K))$ for $A(K)$. But, since the functions in $A(K)$ are determined by their values on ∂K, we shall consider the space $A(K)_{|\partial K}$. In fact, weakly prime sets defined by Ellis, are also subsets of ∂K. In this section, we shall prove that $\mathscr{P}(A(K)_{|\partial K})$ coincides with the family of maximal weakly prime sets as defined by Ellis.

For the definitions and results regarding compact convex sets and space of affine functions, we refer to [1] and [2].

Let us recall the definition due to Ellis [4].

DEFINITION 3.1

A subset E of ∂K is called a *weakly prime set for* $A(K)$ if $E = \partial G$ for some closed face G of

K and if every proper facially closed subset of G has empty interior in the facial topology of G.

Equivalently, for a closed face G of K, ∂G is weakly prime if whenever $G = \mathrm{Co}(H_1 \cup H_2)$ for some closed split faces H_1 and H_2 of G, then either $H_1 = G$ or $H_2 = G$.

If ∂K is a weakly prime set, then $A(K)$ is called weakly prime.

We shall denote the family of maximal weakly prime sets for $A(K)$ according to Definition 3.1 by $\mathscr{P}_E(A(K))$.

The following proposition can be easily proved.

PROPOSITION 3.2

$$\mathrm{Ce}(A(K))_{|\partial k} = N(A(K)_{|\partial K}),$$

where $\mathrm{Ce}(A(K)) = \{f \in A(K): fg_{|\partial K} \in A(K)_{|\partial K}$ *for every* $g \in A(K)\}$, *the centre of* $A(K)$.

Since $\mathrm{Ce}(A(K))_{|\partial k}$ is the set of facially continuous functions on ∂K [2, Theorem 1.4, p. 105], we immediately get the following result.

COROLLARY 3.3

A subset E of ∂K *is a facially closed subset of* ∂K *if and only if E is a generalized peak set for* $N(A(K)_{|\partial K})$.

PROPOSITION 3.4

If $E \in \mathscr{P}(A(K)_{|\partial K})$, *then* $\overline{\mathrm{Co}}E$, *the closed convex hull of E, is a closed split face of K.*

Proof. Let F be the smallest closed split face of K containing $\overline{\mathrm{Co}}E$. Then $E \subset \overline{\mathrm{Co}}E \cap \partial K \subset F \cap \partial K = \partial F$, as F is a face. It is enough to show that ∂F is a weakly prime set for $A(K)_{|\partial K}$.

Let H_1 and H_2 be generalized peak sets for $N((A(K)_{|\partial K})_{|\partial F})$ with $\partial F = H_1 \cup H_2$. Then $E = E \cap \partial F = (H_1 \cap E) \cup (H_2 \cap E)$ and $H_1 \cap E$, $H_2 \cap E$ are generalized peak sets for $N((A(K)_{|\partial K})_{|E})$. Since E is a weakly prime set for $A(K)_{|\partial k}$, either $H_1 \cap E = E$ or $H_2 \cap E = E$. Thus, either $E \subset H_1$ or $E \subset H_2$.

Now, since F is a closed split face of K, $(A(K)_{|\partial K})_{|\partial F} = A(K)_{|\partial F} = A(F)_{|\partial F}$. So, H_1 and H_2 are generalized peak sets for $N(A(K)_{|\partial K})$ and hence by Corollary 3.3, H_1 and H_2 are facially closed subsets of ∂F, i.e., $H_1 = \partial G_1$ and $H_2 = \partial G_2$ for some closed split faces G_1 and G_2 of F. Since F is a closed split face of K, G_1 and G_2 are closed split faces of K. Now $E \subset H_1 \Rightarrow \overline{\mathrm{Co}}E \subset \overline{\mathrm{Co}}H_1 = G_1$. Thus we get $\overline{\mathrm{Co}}E \subset G_1 \subset F$ and hence $G_1 = F$, as F is the smallest closed split face containing $\overline{\mathrm{Co}}E$, i.e., $H_1 = \partial F$. Similarly, if $E \subset H_2$, then we get $H_2 = \partial F$. So ∂F is a weakly prime set for $A(K)_{|\partial K}$.

If $\overline{\mathrm{Co}}H$ is a closed face of K for $H \subset \partial K$, then $\partial(\overline{\mathrm{Co}}H) = H$ and hence we get the following result.

COROLLARY 3.5

If $E \in \mathscr{P}(A(K)_{|\partial K})$, *then E is facially closed.*

Now we prove the main result.

Theorem 3.6. $\mathscr{P}(A(K)_{|\partial K}) = \mathscr{P}_E(A(K))$.

Proof. Let $F \in \mathscr{P}_E(A(K))$. We want to show that F is a weakly prime set for $A(K)_{|\partial K}$.

Let H_1 and H_2 be generalized peak sets for $N((A(K)_{|\partial K})_{|F})$ with $H_1 \cup H_2 = F$. Since $F \in \mathscr{P}_E(A(K))$, F is facially closed [4], i.e., $F = \partial G$ for some closed split face G of K. Hence $A(K)_{|G} = A(G)$ and so $(A(K)_{|\partial K})_{|F} = A(G)_{|\partial G}$. Thus H_1 and H_2 are generalized peak sets for $N(A(G)_{|\partial G})$. So by Corollary 3.3, H_1 and H_2 are facially closed subsets of G. Also, by definition, $F = \partial G$, where G is a closed face of K and $F = H_1 \cup H_2$. Since F is a weakly prime set for $A(K)$, either $H_1 = F$ or $H_2 = F$. Hence F is a weakly prime set for $A(K)_{|\partial K}$.

Conversely, let $F \in \mathscr{P}(A(K)_{|\partial K})$. Then by Proposition 3.4, $\overline{\text{Co}}F$ is a closed split face of K. Let $\overline{\text{Co}}F = G$. Then $F = \partial G$ and $A(G)_{|\partial G} = (A(K)_{|\partial K})_{|F}$. Suppose $F = H_1 \cup H_2$, where H_1 and H_2 are facially closed in G. Then by Corollary 3.3, H_1 and H_2 are generalized peak sets for $N((A(K)_{|\partial K})_{|F})$. Since F is a weakly prime set for $A(K)_{|\partial K}$, either $H_1 = F$ or $H_2 = F$. Hence F is a weakly prime set for $A(K)$. Consequently, $\mathscr{P}(A(K)_{|\partial K}) = \mathscr{P}_E(A(K))$.

COROLLARY 3.7

$A(K)$ is weakly prime if and only if $A(K)_{|\partial K}$ is weakly prime.

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