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Uniqueness of the uniform norm and adjoining identity in Banach algebras

S J BHATT and H V DEDANIA*

Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar 388 120, India
*Department of Mathematics, University of Leeds, Leeds LS2 9JT, UK

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Abstract. Let A_e be the algebra obtained by adjoining identity to a non-unital Banach algebra $(A, \|\cdot\|)$. Unlike the case for a C^*-norm on a Banach $*$-algebra, A_e admits exactly one uniform norm (not necessarily complete) if so does A. This is used to show that the spectral extension property carries over from A to A_e. Norms on A_e that extend the given complete norm $\|\cdot\|$ on A are investigated. The operator seminorm $\|\cdot\|_{op}$ on A_e defined by $\|\cdot\|$ is a norm (resp. a complete norm) iff A has trivial left annihilator (resp. $\|\cdot\|_{op}$ restricted to A is equivalent to $\|\cdot\|$).

Keywords. Adjoining identity to a Banach algebra; unique uniform norm property; spectral extension property; regular norm; weakly regular Banach algebra.

1. Introduction

Let $A_e = A + \mathbb{C}1$ be the algebra obtained by adjoining identity to a non-unital Banach algebra $(A, \|\cdot\|)$ [8]. There are two natural problems associated with this elementary unitification construction: (1) which are (all) algebra norms $|\cdot|$ on A_e that are closely related with (e.g. extending) $\|\cdot\|$ on A? (2) Which properties of the Banach algebra $(A, \|\cdot\|)$ are shared by the normed algebra $(A_e, |\cdot|)$? In the present paper, it is shown that A has unique uniform norm (not necessarily complete) (resp. spectral extension property [9]) iff A_e has the same. This is interesting in view of the fact that for a Banach $*$-algebra $(A, \|\cdot\|)$ with a unique C^*-norm, A_e can admit more than one C^*-norm [1, Example 4.4, p. 850]. This holds in spite of apparent similarity between the defining properties $\|x^2\| = \|x\|^2$ and $\|x^*x\| = \|x\|^2$ of uniform norms and C^*-norms respectively. This main result, together with a couple of corollaries, is formulated and proved in § 3. Their proofs require some properties of norms on A that are regular [5]. There are two standard constructs of norms on A_e, viz. the *l^1-norm* $\|x + \lambda 1\|_1 = \|x\| + |\lambda|$ and the *operator norm* $\|x + \lambda 1\|_{op} = \sup\{\|xy + \lambda y\| : \|y\| \leq 1,\ y \in A\}$. In general, $\|\cdot\|_{op}$ need neither be a norm nor be complete [6, Example 4.2]. Also, in general, $\|\cdot\|_{op|A} \neq \|\cdot\|$. It is easy to see that if p is any algebra seminorm on A_e such that $p_{|A} = \|\cdot\|$, then $\|a + \lambda 1\|_{op} \leq p(a + \lambda 1) \leq p(1)\|a + \lambda 1\|_1$. The norm $\|\cdot\|$ on A is *regular* (resp. *weakly regular*) if the restriction of $\|\cdot\|_{op}$ on A $\|\cdot\|_{op|A} = \|\cdot\|$ (resp. $\|\cdot\|_{op|A}$ is equivalent to $\|\cdot\|$). These are essentially non-unital phenomena, for if A is unital (resp. having a bai (e_i)), then any norm $|\cdot|$ on A with $|1| \leq 1$ (or $|e_i| \leq 1$) is regular [5]. It is shown in § 2 that $\|\cdot\|_{op}$ is a norm on A_e iff the left annihilator $\operatorname{lan}(A) = \{0\}$; and in this case, $\|\cdot\|_{op}$ is complete iff $\|\cdot\|$ is weakly regular iff $\|\cdot\|_1$ is equivalent to $\|\cdot\|_{op}$ on A_e.

Throughout, A is a non-unital algebra. By a *norm* on A, we mean an algebra norm; i.e. a norm satisfying $\|xy\| \leq \|x\|\,\|y\|$ for all x, y. A *uniform norm* on A (resp. a *C^*-norm* on a $*$-algebra) is a norm satisfying the square property $\|x^2\| = \|x\|^2$ (resp. the C^*-property $\|x^*x\| = \|x\|^2$) for all x.

2. Weakly regular norms

Let $(A, \|\cdot\|)$ be a normed algebra. The following shows that if $\|\cdot\|_{op}$ is a norm on A_e, then $|\cdot|_{op}$ is also a norm on A_e for all norms $|\cdot|$ on A. The *left annihilator* of A is $\mathrm{lan}(A) = \{x \in A : xA = \{0\}\}$.

PROPOSITION 2.1

The seminorm $\|\cdot\|_{op}$ is a norm on A_e iff $\mathrm{lan}(A) = \{0\}$.

Proof. Let $\|\cdot\|_{op}$ be a norm on A_e. Let $a \in \mathrm{lan}(A)$. Then $ax = 0$ $(x \in A)$, hence $\|a\|_{op} = \sup\{\|ax\| : \|x\| \leqslant 1, x \in A\} = 0$, so that $a = 0$. Hence $\mathrm{lan}(A) = \{0\}$. Conversely, assume that $\mathrm{lan}(A) = \{0\}$. Let $\|a + \lambda 1\|_{op} = 0$. Then $ax + \lambda x = 0$ for all $x \in A$. Suppose $\lambda \neq 0$. Then $-\lambda^{-1} ax = x$ $(x \in A)$. Define $L_e(x) = ex$ $(x \in A)$, where $e = -\lambda^{-1} a$. Then L_e is an identity operator on A. Then, for $x \in A$, $L_x L_e = L_e L_x$, i.e. $xey = L_x L_e(y) = L_e L_x(y) = exy$ $(y \in A)$, i.e. $(xe - ex)y = 0$ $(y \in A)$. Hence, $xe = ex = x$. Thus A has an identity which is a contradiction. Thus $\lambda = 0$. This implies $ax = 0$ for all $x \in A$, hence $a = 0$. This completes the proof.

PROPOSITION 2.2

(*a*) *Let $|\cdot|$ be a uniform norm on A. Then $|\cdot|$ is regular and $|\cdot|_{op}$ is a uniform norm on A_e.*
(*b*) *Let A be a $*$-algebra. Let $|\cdot|$ be a C^*-norm on A. Then $|\cdot|$ is regular and $|\cdot|_{op}$ is a C^*-norm on A_e.*

Note that if a Banach algebra admits a uniform norm, then it is commutative and semisimple. In the above, the proof of (a) is similar to that of (b) in [4, Lemma 19, p. 67]. In the following, the proof of (1) implies (2) is along the lines of [7, Theorem 1]; whereas that of the remaining part is simple.

PROPOSITION 2.3

Let $(A, \|\cdot\|)$ be a Banach algebra. Then the following are equivalent.

(1) $\|\cdot\|$ *is weakly regular (so that* $\|a\|_{op} \leqslant \|a\| \leqslant m\|a\|_{op}$ $(a \in A)$, *for some* $m > 0$).
(2) $\|a + \lambda 1\|_{op} \leqslant \|a + \lambda 1\|_1 \leqslant 2(2 + m)(\exp 1)\|a + \lambda 1\|_{op}$ $(a + \lambda 1 \in A_e)$
(3) $\|\cdot\|_{op}$ *is a complete norm on* A_e.

If $\|\cdot\|$ is regular, then $m = 1$ so that $\|a + \lambda 1\|_{op} \leqslant \|a + \lambda 1\|_1 \leqslant 6(\exp 1)\|a + \lambda 1\|_{op}$ *for all* $a + \lambda 1 \in A_e$ [7, *Theorem* 1].

3. Uniqueness of uniform norm and unitification

A Banach algebra $(A, \|\cdot\|)$ has *unique uniform norm property* (UUNP) if A admits exactly one (not necessarily complete) uniform norm. The uniform algebra $C(X)$ has UUNP, whereas the disc algebra does not have. In [2] and [3], Banach algebras with UUNP have been investigated. Such an A is necessarily commutative, semisimple and the spectral radius $r(= r_A(\cdot))$ is the unique uniform norm. We denote the Hausdorff completion of (A, r) by $U(A)$. The spectral radius on $U(A)$ is the complete uniform norm on $U(A)$. A norm $|\cdot|$ on A is *functionally continuous* (FC) if every multiplicative linear functional on A is $|\cdot|$-continuous. A subset F of the Gelfand space of A is a set of uniqueness for A if $|x|_F = \sup\{|f(x)| : f \in F\}$ defines a norm on A.

Theorem 3.1. *A Banach algebra $(A, \|\cdot\|)$ has UUNP iff A_e has UUNP.*

We shall need the following. The proofs are straightforward. For details we refer to [3].

Lemma A. *Let $|\cdot|$ be an FC norm on any commutative algebra A. Let B be the completion of $(A, |\cdot|)$. Then the Gelfand space $\Delta(A)$ (resp. Silove boundary ∂A) is homeomorphic to $\Delta(B)$ (resp. ∂B).*

Lemma B. *Let A be a semisimple commutative Banach algebra. Then the following are equivalent.*

(1) *A has UUNP.*

(2) *$U(A)$ has UUNP; and any closed set F in $\Delta(U(A))$ which is a set of uniqueness for A, is also a set of uniqueness for $U(A)$.*

(3) *$U(A)$ has UUNP; and for a non-zero closed ideal I of $U(A)$ with $I = k(h(I))$ (kernel of hull of I), $I \cap A$ is non-zero.*

Lemma C. *Let A be a Banach algebra with UUNP, and I be a closed ideal such that $I = k(h(I))$. Then I has UUNP.*

Proof of Theorem 3.1. Assume that A has UUNP.

Case 1. Let $\|\cdot\|$ have the square property. By Proposition 2.2 (a) and Proposition 2.3, $(A_e, \|\cdot\|_{op})$ is a Banach algebra, $\|\cdot\|_{op}$ has square property and $\|\cdot\|_{op}$ is equivalent to $\|\cdot\|_1$. Let $|\cdot|$ be any uniform norm on A_e, then $|1\cdot 1|_A$ is a uniform norm on A. Since A has UUNP, $|1\cdot 1|_A = \|\cdot\|$. Hence $\|\cdot\|_{op} \leqslant |\cdot| \leqslant \|\cdot\|_1 \leqslant 6(\exp 1)\|\cdot\|_{op}$ on A_e. Thus $\|\cdot\|_{op}$ and $|\cdot|$ are equivalent uniform norms on A_e. Since equivalent uniform norms are equal, $\|\cdot\|_{op} = |\cdot|$ on A_e. Thus A_e has UUNP.

Case 2. In the general case, note that $U(A)$ is an ideal of $U(A_e)$ and, by Lemma A, the Gelfand space $\Delta(U(A_e))$ is homeomorphic to the one point compactifications of each of $\Delta(A)$ and $\Delta(U(A))$. Define $K = \{x \in U(A_e): xU(A) = \{0\}\}$. We prove that $K = \{0\}$. Let $x \in K$. Then its Gelfand transform $\hat{x}: \Delta(U(A_e)) \to \mathbb{C}$ is continuous. Since $x \in K$, $xy = 0$ $(y \in U(A))$. We prove that $\hat{x}$ is zero on $\Delta(U(A_e))$. Since $\Delta(U(A))$ is dense in $\Delta(U(A_e))$, it is enough to prove that $\hat{x}$ is zero on $\Delta(U(A))$. Suppose there exists $\phi \in \Delta(U(A))$ such that $\phi(x) \neq 0$. Since ϕ is non-zero, there exists y in $U(A)$ such that $\phi(y)$ is non-zero. This implies $\phi(xy) \neq 0$, hence $xy \neq 0$ which is a contradiction. Thus $K = \{0\}$. By Lemma B, it is enough to prove that $U(A_e)$ has UUNP; and for every non-zero closed ideal I of $U(A_e)$ with $I = k(h(I))$, $A_e \cap I$ is non-zero. Let I be a non-zero closed ideal of $U(A_e)$ such that $I = k(h(I))$. We prove that $I \cap A_e \neq \{0\}$. Let $J = I \cap U(A)$. Then, first, we prove that $J = k(h(J))$ in $U(A)$. Clearly $J \subseteq k(h(J))$. Let $x \in U(A)$ such that $x \notin J$. Then $x \notin I$, hence there exists $\phi \in h(I) \subseteq \Delta(U(A_e))$ such that $\phi(x) \neq 0$. Then $\psi = \phi|_{U(A)}$ is zero on J and $\psi(x) \neq 0$. Thus $x \notin k(h(J))$, and so $J = k(h(J))$. From $K = \{0\}$, $I \neq \{0\}$ and $IU(A) \subseteq J$, it follows that $J \neq \{0\}$. Since A has UUNP and J is a non-zero closed ideal of $U(A)$ such that $J = k(h(J))$, $A \cap I = A \cap J \neq \{0\}$ by Lemma B. Hence $I \cap A_e \neq \{0\}$. Finally, we show that $U(A_e)$ has UUNP. Note that, by Proposition 2.2 (a) and Proposition 2.3, the operator norm on $U(A)_e$ is a complete uniform norm; and is the spectral radius $r_{U(A)_e}$ itself. Further, $U(A)_e$, is clearly isometrically isomorphic to $U(A_e)$ via the map $T: U(A)_e \to U(A_e)$, $T(a + \lambda 1) = a + \lambda e$, where e is the identity of $U(A_e)$. By Lemma C,

$U(A)$ has UUNP, hence by the isomorphism T and by Case 1, $U(A_e)$ has UUNP. Conversely, if A_e have UUNP, then, A being a closed ideal of A_e satisfying $A = k(h(A))$ in A_e, A has UUNP by Lemma C. This completes the proof.

Following [1], a Banach $*$-algebra B has unique C^*-norm (i.e. B has UC^*NP) if B admits exactly one C^*-norm (not necessarily complete). In spite of the apparent similarity between the square property and the C^*-property of norms the above result differs from the corresponding situation in B, viz. UC^*NP for B need not imply UC^*NP for B_e [1, Example 4.4, p. 850]. In fact, by [1, Theorem 4.1, p. 849], for a non-unital B with UC^*NP, B_e has UC^*NP iff the enveloping C^*-algebra $C^*(B)$ is non-unital. Like $C^*(B)$ for B, the uniform Banach algebra $U(A)$ is universal for A in an appropriate sense. Unlike the case of B, it happens that A is unital iff $U(A)$ is unital. This explains why the above result for A differs from the corresponding result for B.

A Banach algebra $(A, \|\cdot\|)$ has *the spectral extension property* (SEP) [9] (i.e. A is a *permanent Q-algebra* [10]), if for every Banach algebra B such that A is algebraically embedded in B, $r_A(x) = r_B(x)$ for all $x \in A$; equivalently, every norm $|\cdot|$ on A satisfies $r_A(x) \leqslant |x|$ for all $x \in A$ [9, Proposition 1].

COROLLARY 3.2

Let $(A, \|\cdot\|)$ be a semisimple commutative Banach algebra. Then A has SEP iff A_e has SEP.

Proof. Let A have SEP. Then, by [2, Proposition 2.1] and Theorem 3.1, A_e has UUNP. By [2, Proposition 2.6], it is enough to prove that A_e has (P)-property; i.e. every non-zero closed ideal I of A_e has an element $a + \lambda 1$ such that $r_1(a + \lambda 1) > 0$, where $r_1(a + \lambda 1) = \inf\{|a + \lambda 1| : |\cdot| \text{ is a norm on } A_e\}$, called the permanent radius of $a + \lambda 1$ in A_e [9]. Let I be a non-zero closed ideal of A_e. Then $J = I \cap A$ is a non-zero closed ideal of A by [8, Theorem 1.1.6, p. 11]. Since A has SEP, by [2, Proposition 2.6], it has (P)-property, hence there exists $a \in J$ such that the permanent radius, say $r_2(a)$, of a in A is positive. Then clearly $r_1(a) \geqslant r_2(a) > 0$. Thus A_e has (P)-property. Conversely, assume that A_e has SEP. Let $|\cdot|$ be any norm on A. Then, since A is semisimple, Proposition 2.1 implies the operator norm $|\cdot|_{\mathrm{op}}$ is a norm on A_e. Since A_e has SEP, $r_A(a) = r_{A_e}(a) \leqslant |a|_{\mathrm{op}} \leqslant |a|\ (a \in A)$. Thus $r_A(a) \leqslant |a|$ for all a in A and for any norm $|\cdot|$ on A. Hence, A has SEP. This completes the proof.

By [9, Corollary 2], a regular Banach algebra has SEP. In understanding the relation between UUNP and SEP, a weaker notion of regularity has been found useful in [2], viz. a semisimple commutative Banach algebra $(A, \|\cdot\|)$ is *weakly regular* if for any proper closed subset F of the Gelfand space $\Delta(A)$ of A, there exists a non-zero element a in A such that $\hat{a}|F = 0$.

COROLLARY 3.3

Let $(A, \|\cdot\|)$ be a semisimple commutative Banach algebra. Then A is weakly regular iff A_e is weakly regular.

Proof. Let A be weakly regular. Then, by [2, Corollary 2.4(II)], A has UUNP and $\Delta(A) = \partial A$, the Silov boundary of A. By Theorem 3.1, A_e has UUNP. Note that $\Delta(A) = \partial A \subseteq \partial A_e$, $\Delta(A)$ is dense in $\Delta(A_e)$ and ∂A_e is closed. These imply $\partial A_e = \Delta(A_e)$. Hence, again by [2, Corollary 2.4 (II)], A_e is weakly regular. Conversely, assume that A_e

is weakly regular. The proof of Lemma C will work for the following statement; If A is weakly regular and I is a closed ideal of A such that $I = k(h(I))$, then I is also weakly regular. Since A is a closed ideal of A_e with $k(h(A)) = A$, A is weakly regular.

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