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On the partial sums, Cesáro and de la Valleé Poussin means of convex and starlike functions of order 1/2

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Abstract. In this paper we study certain properties of partial sums, cesáro and de la valleé Poussin means of convex and starlike functions.

Keywords. Partial sums; Cesáro; de la Valleé Poussin means.

1. Introduction

Let S denote the class of functions $f(z) = z + a_2 z^2 + \cdots$ which are regular and univalent in the unit disc $E = \{z/|z| < 1\}$. Denote by S_t and K the usual subclasses of S consisting of functions which map E onto starlike (with respect to origin) and convex domains, respectively. Let $S_t(1/2) \subset S_t$ be the class of functions which are starlike of order 1/2. It is known that $K \subset S_t(1/2)$.

For a given function $f(z) = z + \sum_{n=1}^{\infty} a_n z^n$ and $n \in N$, let $s_n(z,f) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n$,

$$v_n(z,f) = \frac{n}{n+1} z + \frac{n(n-1)}{(n+1)(n+2)} a_2 z^2 + \cdots + \frac{n(n-1)(n-2)\ldots 3\cdot 2\cdot 1}{(n+1)(n+2)\ldots(2n)} a_n z^n,$$

$$\sigma_n^{(1)}(z,f) = z + \frac{(n-1)}{n} a_2 z^2 + \frac{(n-2)}{n} a_3 z^3 + \cdots + \frac{1}{n} a_n z^n,$$

and

$$\sigma_n^{(2)}(z,f) = z + \frac{n(n-1)}{n(n+1)} a_2 z^2 + \frac{(n-1)(n-2)}{n(n+1)} a_3 z^3 + \cdots + \frac{2\cdot 1}{n(n+1)} a_n z^n$$

denote, respectively, the nth partial sum, the nth de la Valleé Poussin mean, the nth Cesáro mean of first order and the nth Cesáro mean of second order of f.

A function f is said to be subordinate to a function F (in symbols $f(z) \prec F(z)$) in $|z| < r$ if F is univalent in $|z| < r$, $f(0) = F(0)$ and $f(|z| < r) \subset F(|z| < r)$.

For every $f \in K$ the following results are well-known:

(i) $z/2 = s_1(z,f)/2 = \sigma_1^{(1)}(z,f)/2 = \sigma_1^{(2)}(z,f)/2 \prec f(z)$ in E [2];
(ii) $(4/9)\, s_2(z,f) \prec f(z)$ in E [10];
(iii) $(2/3)\, \sigma_2^{(1)}(z,f) \prec f(z)$ in E [10];
(iv) $v_n(z,f) \prec f(z)$ in E.

The fascinating result (iv) is due to Pólya and Schoenberg [6] (see also Robertson [7]).

In the present paper, we establish the analogue of the Pólya–Schoenberg theorem for a certain transformation of the nth partial sum, $s_n(z,f)$, and the nth Cesáro mean of first order, $\sigma_n^{(1)}(z,f)$, of $f \in K$. We also prove that for every $f \in S_t(1/2)$ and for every positive integer n, $\mathrm{Re}(v_n(z,f)/\sigma_n^{(2)}(z,f)) > 0$, $z \in E$. An alternative and simple proof of a well-known result of Basgöze, Frank and Keogh [1] pertaining to subordination of the partial sums of convex functions is also given.

2. Preliminaries

We shall need the following definitions and results.

DEFINITION 2.1

A sequence $\{b_n\}_1^\infty$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z) = z + \sum_{n=2}^\infty a_n z^n$ is regular, univalent and convex in E, we have

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z), \quad (a_1 = 1)$$

in E.

DEFINITION 2.2

A sequence $\{c_n\}_0^\infty$ of non-negative numbers is said to be a convex null sequence if $c_n \to 0$ as $n \to \infty$ and

$$c_0 - c_1 \geqslant c_1 - c_2 \geqslant \cdots \geqslant c_n - c_{n+1} \geqslant \cdots \geqslant 0.$$

Lemma 2.1. (*Wilf* [11]). *A sequence* $\{b_n\}_1^\infty$ *of complex numbers is a subordinating factor sequence if and only if* $\mathrm{Re}[1 + 2\sum_{n=1}^\infty b_n z^n] > 0$, $z \in E$.

Lemma 2.2. *For all* θ, $0 \leqslant \theta \leqslant \pi$,

$$\frac{1}{2} + \sum_{k=1}^{n} \frac{\cos k\theta}{k+1} \geqslant 0.$$

Lemma 2.2 is due to Rogosinski and Szegö [8].

Lemma 2.3. (*Fejér* [4]). *Let* $\{c_n\}_0^\infty$ *be a convex null sequence. Then the function*

$$q(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n$$

is analytic in E and $\mathrm{Re}\, q(z) > 0$, $z \in E$

Lemma 2.4. *Let*

$$g_n(z) = \frac{(n+1)}{2} + nz + (n-1)z^2 + \cdots + z^n.$$

Then $\mathrm{Re}\, g_n(z) > 0$ *in E.*

Proof. In view of the minimum principle for harmonic functions, we have

$$\min_{z \in E} \mathrm{Re}\, g_n(z) = \min_{0 \leqslant \theta \leqslant 2\pi} \mathrm{Re}\, g_n(e^{i\theta})$$

$$= \min_{0\leqslant\theta\leqslant 2\pi} \operatorname{Re}\left[\frac{n+1}{2} + \sum_{k=1}^{n}(n-k+1)\cos k\theta + i\sum_{k=1}^{n}(n-k+1)\sin k\theta\right]$$

$$= \min_{0\leqslant\theta\leqslant 2\pi} \operatorname{Re}\left[\frac{\sin^2[(n+1)\theta/2]}{2\sin^2(\theta/2)} + i\frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2(\theta/2)}\right], \quad \text{(p3, [5])}$$

$$> 0.$$

Lemma 2.5. Let f and g be starlike of order $1/2$. Then for each function F analytic in E and satisfying

$$\operatorname{Re} F(z) > 0 \quad (z\in E),$$

we have

$$Re\frac{f(z)*F(z)g(z)}{f(z)*g(z)} > 0 \quad (z\in E).$$

Lemma 2.5 is due to Ruscheweyh and Sheil-Small [9].

3. Theorems and their proofs

Theorem 3.1. *Let $f\in K$ and let $s_n(z,f)$, $n\in N$, denote its nth partial sum. Then*

$$S_n(z,f) = \frac{1}{2}\int_0^z s_n(t,f)\,\mathrm{d}t \prec f(z)$$

in E for every $n = 1, 2, 3 \ldots$.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in K. Then

$$S_n(z,f) = \frac{1}{2}z + \frac{a_2}{3}z^2 + \frac{a_3}{4}z^3 + \cdots + \frac{a_n}{n+1}z^n.$$

In view of the Definition 2.1, the desired conclusion will follow if and only if the sequence $\langle 1/2, 1/3, \ldots, 1/(n+1), 0, 0 \ldots \rangle$ is a subordinating factor sequence. By Lemma 2.1, this will be the case if and only if

$$\operatorname{Re}\left(1 + 2\sum_{k=1}^{n}\frac{z^k}{k+1}\right) > 0, \quad z\in E. \tag{3.1}$$

Putting $z = re^{i\theta}$, $0\leqslant r<1$, $-\pi\leqslant\theta\leqslant\pi$ and making use of the minimum principle for harmonic functions along with Lemma 2.2, we have

$$\operatorname{Re}\left(1 + 2\sum_{k=1}^{n}\frac{z^k}{k+1}\right) = 1 + 2\sum_{k=1}^{n}\frac{r^k\cos k\theta}{k+1} > 2\min_{0\leqslant\theta\leqslant\pi}\left(\frac{1}{2} + \sum_{k=1}^{n}\frac{\cos k\theta}{k+1}\right) > 0,$$

showing that the inequality (3.1) holds and, therefore, the proof of our theorem is complete.

Taking $n = 1$, we obtain the following well-known result (also cited in the Introduction).

COROLLARY 3.1

$(1/2)z \prec f(z)$ *in E, for all $f\in K$.*

Theorem 3.2. *For all elements f of K and for all positive integers n, we have*

$$(n/(n+1))\,\sigma_n^{(1)}(z,f) \prec f(z)$$

in E. This result is sharp for every n.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be any element of K. Since

$$\frac{n}{n+1}\sigma_n^{(1)}(z,f) = \frac{n}{n+1}z + \frac{n-1}{n+1}a_2 z^2 + \frac{n-2}{n+1}a_3 z^3 + \cdots + \frac{1}{n+1}a_n z^n,$$

in the light of Definition 2.1, the assertion $(n/(n+1))\,\sigma_n^{(1)}(z,f) \prec f(z)$ in E will hold if and only if the sequence $\langle n/(n+1), (n-1)/(n+1), \ldots, 1/(n+1), 0, 0, \ldots \rangle$ is a subordinating factor sequence. By Lemma 2.1, we see that this is equivalent to

$$\operatorname{Re}\left[1 + \frac{2}{n+1}(nz + (n-1)z^2 + (n-2)z^3 + \cdots + z^n)\right] > 0, \quad z \in E,$$

or

$$\operatorname{Re}\left[\frac{n+1}{2} + nz + (n-1)z^2 + (n-2)z^3 + \cdots + z^n\right] > 0, \quad z \in E,$$

which is true in view of Lemma 2.4. To establish the claim regarding sharpness we consider the function $h(z) = z/(1-z)$ which is a member of K. For any positive real number ρ, we have

$$\begin{aligned}\rho\sigma_n^{(1)}(e^{i\theta},h) &= \frac{\rho}{n}\left[-\frac{(n+1)}{2} + \sum_{k=1}^{n}(n-k+1)\cos k\theta + i\sum_{k=1}^{n}(n-k+1)\sin k\theta\right]\\ &= \frac{\rho}{n}\left[-\frac{(n+1)}{2} + \frac{\sin^2[(n+1)\theta/2]}{2\sin^2(\theta/2)} + i\frac{(n+1)\sin\theta - \sin(n+1)\theta}{4\sin^2(\theta/2)}\right].\end{aligned}$$

Now let $\theta = \theta_0 = 2\pi/(n+1)$. Then

$$\operatorname{Re}\rho\sigma_n^{(1)}(e^{i\theta_0},h) = -\frac{\rho}{2}\frac{(n+1)}{n}.$$

Now, if $\rho > n/(n+1)$, then it follows that $\operatorname{Re}\rho\sigma_n^{(1)}(z,h) < -1/2$ and hence (since h maps E onto the right half plane $\operatorname{Re} w > -1/2$) we conclude that $\rho\sigma_n^{(1)}(z,h)$ will not be subordinate to h in E.

Taking $n = 2$, we obtain the following result of Singh and Singh [10].

COROLLARY 3.2

$(2/3)\sigma_2^{(1)}(z,f) \prec f(z)$ *in E, for every* $f \in K$.

In the next theorem we present a simple and interesting proof of a well-known result which was established by Basgöze, Frank and Keogh [1] in 1970.

Theorem 3.3. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$ *and let* $s_n(z,f)$ *denote its nth partial sum. Then*

$$s_n(z/2,f) \prec f(z)$$

in E for every $n = 1, 2, 3, \ldots$. *The constant 1/2 cannot be replaced by a larger one.*

Proof. Since $s_n(z/2,f)=(1/2)z+(1/2^2)a_2z^2+(1/2^3)a_3z^3+\cdots+(1/2^n)a_nz^n$, the conclusion $s_n(z/2,f)\prec f(z)$ in E will follow if and only if the sequence $\langle 1/2, 1/2^2,\ldots,1/2^n, 0,0,\ldots\rangle$ is a subordinating factor sequence. In view of Lemma 2.1, this will be the case if and only if

$$\operatorname{Re}\left[1+2\sum_{k=1}^{n}\frac{z^k}{2^k}\right]>0,\quad z\in E. \tag{3.2}$$

It is readily seen that the sequence $\{c_k\}_0^\infty$ defined by $c_0=1$, $c_k=1/2^k$, $k=1,2,3,\ldots,n$ and $c_k=0$ if $k=n+1,n+2,\ldots$, is a convex null sequence. Thus using Lemma 2.3 we get

$$\operatorname{Re}\left(\frac{1}{2}+\sum_{k=1}^{n}\frac{z^k}{2^k}\right)>0,\quad z\in E,$$

which in turn shows that the inequality (3.2) holds. The function $h(z)=z/(1-z)\in K$, which maps E onto the half plane $\operatorname{Re} w>-1/2$, shows that the constant $1/2$ cannot be replaced by any larger number. This completes the proof of our theorem.

Egerváry [3] has shown that

$$\begin{aligned}&\sigma_n^{(2)}(z,z/(1-z))\\&=\frac{1}{n(n+1)}[(n+1)nz+n(n-1)z^2+(n-1)(n-2)z^3+\cdots+2\cdot 1\cdot z^n]\end{aligned}$$

is a member of $S_t(1/2)$. Using this fact and the well-known result of Ruscheweyh and Sheil-Small (Theorem 3.1, [9]) we conclude that for every $f\in S_t(1/2)$

$$\sigma_n^{(2)}(z,f)=f(z)*\sigma_n^{(2)}(z,z/(1-z))$$

is a member of $S_t(1/2)$.

Theorem 3.4. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be any member of $S_t(1/2)$. Then for every positive integer n, we have*

$$\operatorname{Re}\frac{v_n(z,f)}{\sigma_n^{(2)}(z,f)}>0,\quad z\in E.$$

Proof. Consider the function F_n defined by

$$\begin{aligned}F_n(z)=(1-z)\Bigg[&\frac{n}{n+1}+\frac{n}{n+2}z+\frac{n^2}{(n+2)(n+3)}z^2+\frac{n^2(n-1)}{(n+2)(n+3)(n+4)}z^3\\&+\frac{n^2(n-1)(n-2)}{(n+2)(n+3)(n+4)(n+5)}z^4+\cdots+\frac{n^2(n-1),\ldots,3}{(n+1)(n+2),\ldots,(2n)}z^n\Bigg].\end{aligned} \tag{3.3}$$

Obviously F_n is regular in E (in fact it is an entire function), and we can write it in the form

$$F_n(z)=\frac{n}{n+1}-\frac{n}{n+1}\left(1-\frac{n+1}{n+2}\right)z-\frac{n}{n+2}\left(1-\frac{n}{n+3}\right)z^2$$

$$-\frac{n^2}{(n+2)(n+3)}\left(1-\frac{n-1}{n+4}\right)z^3$$

$$-\frac{n^2(n-1)(n-2),\ldots,4}{(n+2)(n+3),\ldots,(2n-1)}\left(1-\frac{3}{2n}\right)z^{n-1}$$

$$-\frac{n(n-1)(n-2),\ldots,3}{(n+2)(n+3),\ldots,(2n)}z^n.$$

In view of (3.3) and (3.4) it is now easy to see that in E we have

$$\mathrm{Re}\,F_n(z) \geqslant F_n(|z|) > F(1) = 0.$$

In Lemma 2.5 taking $f(z) = \sigma_n^{(2)}(z, f)$, $g(z) = z/(1-z)$ and $F(z) = F_n(z)$ we get

$$\mathrm{Re}\frac{\sigma_n^{(2)}(z,f) * z/(1-z)F(z)}{\sigma_n^{(2)}(z,f) * z/(1-z)} = \mathrm{Re}\frac{v_n(z,f)}{\sigma_n^{(2)}(z,f)} > 0, \quad z \in E.$$

This completes the proof.

References

[1] Basgöze T, Frank J L and Keogh F R, On convex univalent functions, *Can. J. Math.* **22** (1970) 123–127

[2] Peter L Duren, *Univalent functions*, (New York: Springer-Verlag) (1983)

[3] Egerváry E, Abbildungseigenschaften der arithmetischen Mittel der geometrischen Reihe, *Math. Z.* **42** (1987) 221–230

[4] Fejĕr L, Uber die positivitat Von summen, die nach trigonometrischen order Legendreschen funktionen fortschreiten, *Acta Litt. Ac. Sci. Szeged* (1925) 75–86

[5] Goodman A W, *Univalent functions*, (Mariner Publishing Company) Vol. I

[6] Pólya G and Schoenberg I S, Remarks on de la Valleé Poussin means and convex conformal maps of the circle, *Pac. J. Math.* **8** (1958) 295–334

[7] Robertson M S, Applications of the subordination principle to univalent functions, *Pac. J. Math.* **11** (1961) 315–324

[8] Rogosinski W and Szegö G, Uber die Abschimlte Von potenzreihen die inernein Kreise be Schrankt bleiben, *Math. Z.* **28** (1928) 73–94

[9] Ruscheweyh St and Sheil-Small T, Hadamard products of Schlicht functions and the Pólya–Schoenberg conjecture, *Comment. Math. Helv.* **48** (1973) 119–135

[10] Singh S and Singh R, Subordination by univalent functions, *Proc. Am. Math. Soc.* **82** (1981) 39–47

[11] Wilf H S, Subordinating factor sequence for convex maps of the unit circle, *Proc. Am. Math. Soc.* **12** (1961) 689–693