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Local behaviour of the first derivative of a deficient cubic spline interpolator

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Abstract. Considering a given function $f \in C^4$ and its unique deficient cubic spline interpolant, which match the given function and its derivative at mid point between the successive mesh point, we have obtained in the present paper asymptotically precise estimate for $s' - f'$.

Keywords. Local behaviour; deficient cubic spline; mid point interpolation; precise estimate.

1. Introduction

Let $P: 0 = x_0 < x_1, < \cdots < x_n = 1$ denote a partition of $[0, 1]$ with equidistant mesh points so that $h = x_i - x_{i-1} = 1/n$. Let Π_m be the set of all real algebraic polynomials of degree not greater than m. For a function s defined over $[0, 1]$ we denote the restriction of s over $[x_{i-1}, x_i]$ by s_i. The class of periodic deficient cubic splines over $[0, 1]$ with mesh P is defined by

$$S(3, P) = \{s : s_i \in \Pi_3, s \in C^1[0, 1], s^{(j)}(0) = s^{(j)}(1), \quad j = 0, 1\}.$$

Considering a nondecreasing function g on $[0, 1]$ such that $g(x + h) - g(x) = H(\text{const.}) = \int_0^h \mathrm{d}g$, $x \in [0, 1 - h]$, Rana and Purohit [4] have proved the following for deficient cubic splines:

Theorem 1. *Let $f \in C^1[0, 1]$. Then there exists a unique 1-periodic spline $s \in S(3, P)$ which satisfies the following interpolatory conditions,*

$$\int_{x_{i-1}}^{x_i} (f(x) - s(x))\mathrm{d}g = 0, \quad i = 1, 2, \ldots, n, \tag{1.1}$$

$$s'(\theta_i) = f'(\theta_i), \quad \theta_i = (x_i + x_{i-1})/2, \quad i = 1, 2, \ldots, n. \tag{1.2}$$

It is interesting to observe that condition (1.1) reduces to different interpolatory conditions by suitable choice of $g(x)$. Thus, if g is a step function with a single jump of one at $h/2$ then condition (1.1) reduces to the interpolatory condition,

$$s(\theta_i) = f(\theta_i), \quad i = 1, 2, \ldots, n. \tag{1.3}$$

Considering a function $f \in C^4$ and its unique spline interpolant s matching at the mesh points, Rosenblatt [5] has obtained asymptotically precise estimate for $s' - f'$. For further results concerning asymptotically precise estimate for cubic spline interpolant reference may be made to Dikshit and Rana [3]. Similar to the result of Rosenblatt [5], we obtain in the present paper a precise estimate for $s' - f'$ concerning the deficient cubic spline interpolating the given function and its derivative at mid points between the successive mesh points. It may be worthwhile to mention that Boneva, Kendall and

Stefanov [2] have shown the use of derivative of a cubic spline interpolator for smoothing of histograms.

Without any loss of generality, we consider for the rest of this paper that the deficient cubic spline s under consideration satisfies the condition $s'(0) = 0$. Thus, we have from the proof of Theorem 1 that the system of equations for determining the first derivative $m_i = s'(x_i)$ of the deficient cubic spline interpolant s is written as,

$$-(m_{i+1} - 10m_i + m_{i-1})/2 = F_i, \quad i = 1, 2, \ldots, n-1 \tag{1.4}$$

where $F_i = 12\{f(\theta_{i+1}) - f(\theta_i)\}/h - 4\{f'(\theta_{i+1}) + f'(\theta_i)\}$.

2. Estimation of the inverse of the coefficient matrix

Ahlberg, Nilson and Walsh [1] have estimated precisely the inverse of the coefficient matrix appearing in the studies concerning cubic spline interpolant matching at the mesh points. Following Ahlberg *et al* we propose to obtain here a precise estimate for the inverse of the coefficient matrix in (1.4). It may be mentioned that this method permits the immediate application to the spline to standard problem of numerical analysis (see [1], p. 34). For this we introduce the following square matrix of order n.

$$D_n(a,b) = \begin{bmatrix} 2b & a & 0 & \cdots & 0 & 0 & 0 \\ a & 2b & a & \cdots & 0 & 0 & 0 \\ 0 & a & 2b & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a & 2b & a \\ 0 & 0 & 0 & \cdots & 0 & a & 2b \end{bmatrix}$$

where a and b are given real numbers such that $b^2 \geqslant a^2$. By using the induction hypothesis it may be seen easily that $|D_n|$ satisfies the following difference equation,

$$|D_n(a,b)| = 2b|D_{n-1}(a,b)| - a^2|D_{n-2}(a,b)| \tag{2.1}$$

with $|D_{-1}(a,b)| = 0, |D_0(a,b)| = 1$ and $|D_1(a,b)| = 2b$ and for $\alpha = (b^2 - a^2)^{1/2}$,

$$2\alpha|D_n(a,b)| = (b+\alpha)^{n+1} - (b-\alpha)^{n+1}, \quad b^2 > a^2$$

$$|D_n(a,b)| = (n+1)b^n, \quad \text{otherwise}. \tag{2.2}$$

Further, it may be observed that the system of eq. (1.4) may be written as

$$AM = F \tag{2.3}$$

where the coefficient matrix A is a square matrix of order $n-1$, M and F are the transposes of the matrices $[m_1, m_2, \ldots, m_{n-1}]$ and $[F_1, F_2, \ldots, F_{n-1}]$ respectively. In order to determine the inverse of the coefficient matrix A we first observe that for $a = -1/2$,

$$\beta^{-n}(2b + r)|D_n(a,b)| = 2b(1 - r^{2n}) + r(1 - r^{2n-2})/2 \tag{2.4}$$

where $-r = (2\beta)^{-1} = 2[b - (b^2 - 1/4)^{1/2}]$.

Taking $2b = 5$ and $a = -1/2$ in $|D_n(a,b)|$, we observe that the coefficient matrix A satisfies the following difference equation,

$$4|A| = 20|D_{n-2}(-1/2, 5/2)| - |D_{n-3}(-1/2, 5/2)|. \tag{2.5}$$

Thus, using (2.4) in (2.5) we have

$$(5+r)\beta^{2-n}|A| = (5+r/2)^2 - r^{2n-6}(5r+1/2)^2. \tag{2.6}$$

We get the elements $a_{i,j}$ of A^{-1} from the cofactors of the transpose matrix. Thus, for $0 < i \leqslant j \leqslant n-2$ or $i = j = 0$ (cf. [1, pp. 35–38])

$$|A|a_{i,j} = (\beta \cdot r)^{j-i} D_i(-1/2, 5/2) D_{n-j-2}(-1/2, 5/2) \tag{2.7}$$

and

$$|A|a_{0,j} = (\beta \cdot r)^j D_{n-j-2}(-1/2, 5/2) \quad \text{for } 0 < j \leqslant n-2. \tag{2.8}$$

Thus, in view of (2.4) and (2.5), we have for $0 < i \leqslant j < n-2$

$$(5+r)(1-r^{2n})a_{i,j} = r^{j-i}(1-r^{2i+2})(1-r^{2n-2j-2}),$$

$$(5+r/2)(1-r^{2n})a_{i,n-2} = r^{n-2-i}(1-r^{2i+2}), \quad \text{for } 0 < i \leqslant n-2,$$

$$(5+r/2)(1-r^{2n})a_{0,j} = r^j(1-r^{2n-2-2j}), \quad \text{for } 0 < j < n-2,$$

and

$$(5+r/2)^2(1-r^{2n})a_{0,n-2} = r^{n-2}(5+r).$$

From the above expression, we observe that A^{-1} is symmetric. Now considering a fixed value x such that $0 < x < 1$, we see that for fixed $\varepsilon > 0$ and $\varepsilon < i/n$, $j/n < 1-\varepsilon$ the elements $a_{i,j}$ of A^{-1} may be approximated asymptotically by $r^{|j-i|}/(5+r)$.

We thus complete the proof of the following:

Theorem 2. *The coefficient matrix A of (2.3) is invertible and if $A^{-1} = (a_{i,j})$, then $a_{i,j}$ can just be approximated asymptotically by $r^{|j-i|}/(5+r)$ and the row max norm of its inverse; that is,*

$$\|A^{-1}\| \leqslant \frac{(1+r)}{(1-r)(5+r)}, \tag{2.9}$$

where $r = 2\sqrt{6} - 5$.

Remark 1. It is worthwhile to mention that the estimate (2.9) is sharper than that obtained in terms of the infimum of the excess of the positive value of the leading diagonal element over the sum of the positive values of other elements in each row. For adopting the latter approach, we observe from (2.3) that $\|A^{-1}\| \leqslant 0.25$ whereas (2.9) shows that the $\|A^{-1}\|$ does not exceed 1/6.

Since A is invertible, it follows from the proof of Theorem 1 or more precisely (2.3), that there exists a unique spline $s \in S(3, P)$ satisfying the interpolatory conditions (1.2) and (1.3).

3. Error bounds

Considering a 1-periodic function $f \in C^4$ in this section of the paper we shall estimate the precise bounds of the function $e' = s' - f'$ where s is the deficient cubic spline interpolant of a 1-periodic function f which satisfies the interpolatory conditions (1.2), (1.3). Considering the interval $[x_{i-1}, x_i]$, we see that, since s' is quadratic, hence in the interval $[x_{i-1}, x_i]$, we may write

$$h^2 s'(x) = h(x - x_{i-1})m_i + h(x_i - x)m_{i-1} + (x - x_{i-1})(x_i - x)c_i \tag{3.1}$$

where the constant c_i is to be determined. Using the interpolatory condition (1.2), we notice that,

$$4f'(\theta_i) = 2(m_i + m_{i-1}) + c_i. \tag{3.2}$$

Now applying (3.2) in (3.1), we get

$$h^2 s'(x) = (x - x_{i-1})[h - 2(x_i - x)]m_i + (x_i - x)[h - 2(x - x_{i-1})]m_{i-1} + 4(x - x_{i-1})(x_i - x)f'(\theta_i). \tag{3.3}$$

Thus, replacing now m_i by $e'(x_i)$ in (3.3), we have

$$h^2 s'(x) = (x - x_{i-1})[h - 2(x_i - x)]e'(x_i) + (x_i - x)[h - 2(x - x_{i-1})]e'(x_{i-1}) + R_i(f) \tag{3.4}$$

where $R_i(f) = (x - x_{i-1})[h - 2(x_i - x)]f'(x_i) + (x_i - x)[h - 2(x - x_{i-1})]f'(x_{i-1}) + 4(x - x_{i-1})(x_i - x)f'(\theta_i)$.

Now using the fact that $f \in C^4$, we see by Taylor's theorem that $R_i(f)$ may be expressed as a linear combination of the values of the fourth derivative $f^{(4)}$ of f. Thus,

$$R_i(f) = h^2 f'(x) + f^{(4)}(x)(x - x_{i-1})(x_i - x)(2x - x_i - x_{i-1})h^2/12 + 0(h^5) \tag{3.5}$$

where x is an appropriate point in (x_{i-1}, x_i) which is not necessarily the same at each occurrence. Rewriting (2.3) as,

$$A(e'(x_i)) = (F_i) - A(f'(x_i)) = (H_i), \tag{3.6}$$

say, we first estimate (H_i). Thus, applying Taylor's theorem again to the right hand side of (3.6), we get

$$(H_i) = -h^3 f^{(4)}(x)/6 + 0(h^3) \tag{3.7}$$

Recalling eq. (3.6) and noticing that $A^{-1} = (a_{i,j})$, we have

$$(e'(x_i)) = \left(\sum_{|k-i| \geq m} + \sum_{|k-i| < m} (a_{i,k} H_k) \right) = (R_1) + (R_2)$$

say, where m is a sufficiently large but fixed positive integer. We shall estimate R_1 and R_2 separately. Suppose that x is a fixed point in (0, 1) and let $x_i = [nx]/n$ where $[nx]$ denotes the largest integer less than or equal to nx. Then it is clear that as $n \to \infty$, $i \cong nx$ and $n - i \cong n(1 - x)$. Now assuming that $f^{(4)}$ is monotonic, we get from Theorem 2

$$|(R_1)| \leqslant d_1 (1/6)^m h^3 \tag{3.8}$$

where d_1 is some positive constant.

Next, we see that the points x_k for the values of k occurring in R_2 satisfy

$$|x_k - x| = 0(h). \tag{3.9}$$

Thus, using the continuity of $f^{(4)}$ and applying the result of Theorem 2 alongwith (3.7), we have

$$\left|(R_2) - \sum_{|k-i|<m} \frac{r^{|k-i|}}{(5+r)}(-h^3 f^{(4)}(x)/6)\right| = 0(h^3). \tag{3.10}$$

Combining the estimates of (R_1) and (R_2) and noticing that m is arbitrary, we prove the following:

Theorem 3. *Let $s \in S(3, P)$ be the deficient cubic spline interpolant of a 1-periodic function f satisfying the interpolatory conditions* (1.2) *and* (1.3). *Let $f^{(4)}$ exist and be a nonnegative monotonic continuous function. Then for any fixed point x such that* $0 < x < 1$,

$$s'(x) - f'(x) = f^{(4)}(x)[3(x - x_{i-1})(x_i - x)(2x - x_i - x_{i-1}) + 4h(x_i - x)(x - x_{i-1}) - h^3]/36 + 0(h^3) \tag{3.11}$$

as $n \to \infty$.

Remark 2. It may be interesting to investigate the similar precise estimate for deficient cubic spline in the case of nonuniform mesh.

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