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Equivariant cobordism of Grassmann and flag manifolds

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Abstract. We consider certain natural $(\mathbb{Z}_2)^n$ actions on real Grassmann and flag manifolds and S^1 actions on complex Grassmann manifolds with finite stationary point sets and determine completely which of them bound equivariantly.

Keywords. Equivariant cobordism; Grassmann manifold; flag manifold; tangential representation.

1. Introduction

Let G be a compact Lie group. A smooth G-manifold M with a given action of G will be denoted by (M, ϕ), where $\phi: G \times M \to M$ denotes the action map. An element $x \in M$ is called a stationary point if $\phi(g, x) = x$ for all $g \in G$. We shall be concerned with actions of the groups $(\mathbb{Z}_2)^n$ and S^1 with finite stationary point sets. A smooth closed n-dimensional G-manifold (M^n, ϕ) with finite stationary point set, is said to bound equivariantly if and only if there is an action (W^{n+1}, Φ) on a compact $(n+1)$-manifold for which the induced action $(\partial W^{n+1}, \Phi/\partial W^{n+1})$ is equivariantly diffeomorphic to (M^n, ϕ). Two smooth closed G-manifolds (M_1^n, ϕ_1) and (M_2^n, ϕ_2), having finite stationary point sets, are said to be unoriented G-cobordant if and only if the disjoint union $(M_1^n \cup M_2^n, \phi_1 \cup \phi_2)$ bounds equivariantly. This is an equivalence relation and the resulting set of equivalence classes is denoted by $Z_n(G)$. The equivalence class of (M^n, ϕ) is denoted by $[M^n, \phi]_2$. By disjoint union this becomes an abelian group. The cartesian product of G-manifolds with diagonal action makes the direct sum $Z_*(G) = \sum_{n \geq 0} Z_n(G)$ a graded commutative algebra. For a smooth closed oriented n-dimensional G-manifold (M^n, ϕ) (so that for every $g \in G, \phi_g: M \to M, x \mapsto gx$ is orientation preserving) having finite stationary point set, we say (M^n, ϕ) is an oriented equivariant boundary if and only if there is an action (W^{n+1}, Φ) on a compact oriented $(n+1)$-manifold as a group of orientation preserving diffeomorphism for which the induced action $(\partial W^{n+1}, \Phi/\partial W^{n+1})$ is equivariantly diffeomorphic to (M^n, ϕ) by orientation preserving diffeomorphism. We take $-(M^n, \phi)$ to be $(-M^n, \phi)$, by just reversing the orientation of M^n. Two smooth closed oriented n-dimensional G-manifolds (M_1^n, ϕ_1) and (M_2^n, ϕ_2), having finite stationary point sets, are said to be oriented G-cobordant if and only if the disjoint union $(M_1^n \cup -M_2^n, \phi_1 \cup \phi_2)$ is an oriented equivariant boundary. This is again an equivalence relation and the resulting set of equivalence classes is denoted by $\mathscr{F}_n(G)$, which is an abelian group by disjoint union. The equivalence class of (M^n, ϕ) is denoted by $[M^n, \phi]$. As in the unoriented case we have oriented G-cobordism algebra $\mathscr{F}_*(G)$. Let MO_* and MSO_* denote respectively the unoriented and oriented cobordism algebra, notations are as in [3]. We have forgetful homomorphisms $\varepsilon: Z_*(G) \to MO_*$ and $\tilde{\varepsilon}: \mathscr{F}_*(G) \to MSO_*$ given by $[M, \phi]_2 \mapsto [M]_2$ and $[M, \phi] \mapsto [M]$ respectively. It may be noted here that in [9] (Theorem 4.1), it was proved that any element $\alpha \in MSO_*$ admits a representative M on which there exists an action of S^1 with finitely many stationary points. Thus in the case $G = S^1$, the map $\tilde{\varepsilon}$ is surjective.

The aim of this paper is to consider certain natural $(\mathbb{Z}_2)^n$ actions on real Grassmann and flag manifolds and S^1 actions on complex Grassmann manifolds with finite stationary point sets and generate elements in the kernel of ε and $\tilde{\varepsilon}$. Group actions with finite stationary point sets are particularly interesting, as in this case, the tangential representations of the group $G = (\mathbb{Z}_2)^n$, at stationary points, completely determine the equivariant cobordism class of manifolds [3]. In case $G = S^1$, although the tangential representations do not determine the equivariant cobordism class of a manifold completely, they carry lot of information about the bordism structure of the manifold. As for example, Atiyah-Singer [1] and Bott [2] have shown that if S^1 acts on an oriented compact manifold M with a finite stationary point set S, then the oriented $\mathbb{R}S^1$-modules $\{T_x M : x \in S\}$ determine the Pontrjagin numbers of M, (also cf. § 2).

For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, Conner and Floyd have described the structure of $Z_*(G)$ completely (cf. [3]). Stong and Kosniowski [4], have also derived this result from a more general consideration. They showed that $Z_*(G)$ is the polynomial algebra over $\mathbb{Z}_2$ generated by the class $[\mathbb{R}P^2, \phi]_2$, where ϕ is given by the generators T_1 and T_2 as follows. $T_1([x, y, z]) = [-x, y, z]$ and $T_2([x, y, z]) = [x, -y, z]$. In particular, the kernel of ε is trivial in this case. No neat description of $Z_*(G)$ for $G = (\mathbb{Z}_2)^n$, $n > 2$, is known. Our results show that in general the kernel of ε is nontrivial. a cobordism class $[M^d, \phi]_2 \in Z_d((\mathbb{Z}_2)^n)$ is equivariantly decomposable if (M^d, ϕ) is equivariantly cobordant to a disjoint union of products of lower dimensional manifolds with $(\mathbb{Z}_2)^n$ action with finite stationary point sets, otherwise it is equivariantly indecomposable. The first step towards understanding the structure of $Z_*(G)$ in general, would be to know the indecomposable elements in $Z_*(G)$, which may be considered as the generators. Unfortunately, there is no indecomposability criterion known in the equivariant case. Clearly if $[M]_2 \in MO_*$ is indecomposable (in the non-equivariant sense) and M admits an action of $(\mathbb{Z}_2)^n$, with finite stationary point set, then $[M, \phi]_2$ is indecomposable. But there exist some elements in the kernel of ε which are indecomposable in $Z_*((\mathbb{Z}_2)^n)$. For example, it is easy to argue that $[\mathbb{R}P^3, \phi]_2$ is indecomposable in $Z_*((\mathbb{Z}_2)^3)$, where ϕ is given by the generators as follows. $T_1([x, y, z, w]) = [-x, y, z, w]$, $T_2([x, y, z, w]) = [x, -y, z, w]$ and $T_3([x, y, z, w]) = [x, y, -z, w]$. By knowing enough elements in the kernel, perhaps it would be possible to get an idea about the indecomposable elements in general. We believe that all the elements in the kernel given by Theorem 1.1 and Theorem 1.2 are indecomposable. This motivates our study of these actions.

To determine which real flag manifolds bound, in [9] the authors gave a partial answer to this question. Real Grassmann and flag manifolds come equipped with certain natural $(\mathbb{Z}_2)^n$ actions having finite stationary point sets, to be made precise later. Although, it seems difficult to determine the unoriented cobordism class of flag manifolds, the determination of $(\mathbb{Z}_2)^n$-cobordism class of flag manifolds is easy. In the present paper, which real flag manifolds and Grassmann manifolds bound equivariantly, is completely determined. More precisely, we prove

Theorem 1.1. **(a)** *$(G_{n,k}, \phi)$ bounds equivariantly if $n = 2k$.*
(b) *$(G_{n,k}, \phi)$ does not bound equivariantly if $n \neq 2k$.*

Theorem 1.2. *$(G(n_1, n_2, \ldots, n_s), \phi)$ bounds equivariantly if and only if $n_i = n_j$ for some i, j, $i \neq j$.*

Precise definitions of the actions ϕ on Grassmann and flag manifolds are given in the subsequent sections. Perhaps, by knowing sufficiently many elements in the kernel of ε it would be possible to determine whether the unoriented $(\mathbb{Z}_2)^n$-cobordism class of flag manifolds lie in the kernel of ε or not, and that might lead to a complete answer to the question, which real flag manifolds bound? We also consider certain natural S^1-actions on complex Grassmann manifolds to produce nontrivial elements in the kernel of $\tilde{\varepsilon}$ (cf. Theorem 3.4). In this case, our result produce an infinitely many nontrivial elements in the kernel of $\tilde{\varepsilon}$. As a consequence, we deduce that for each $d > 1$, $\mathscr{F}_{2d}(S^1)$ is not finitely generated as abelian group.

2. Representation and cobordism

In this section we briefly recall [3] the relation between tangential representations at stationary points and cobordism and a result of Stong.

Let G be a finite group. Let $R_n(G)$ denote the vector space over the field $\mathbb{Z}_2$, with basis the set of representation classes of degree n. The elements in $R_n(G)$ are formal sums of n-dimensional representation classes with coefficients in $\mathbb{Z}_2$. If $R_*(G) = \sum R_n(G)$, then $R_*(G)$ admits a graded commutative algebra structure with unit over $\mathbb{Z}_2$. The product is given as follows. Suppose $(V_1, G), (V_2, G)$ are representations. We take $(V_1 \oplus V_2, G)$ to be $g(v_1, v_2) = (gv_1, gv_2)$. Then the product is $(V_1, G)\cdot(V_2, G) = (V_1 \oplus V_2, G)$. The identity element is the representation class of degree 0. In fact, $R_*(G)$ is the graded polynomial ring over $\mathbb{Z}_2$ generated by the set of isomorphism classes of irreducible finite dimensional real representations of G.

Consider now an action (M^n, ϕ) with finite stationary point set S. For each $x \in S$, we have a real linear representation of G on the tangent space to M^n at x. We denote the resulting representation class by $X(x) \in R_n(G)$. Since x is an isolated stationary point, it is clear that $X(x)$ contains no trivial summand. To (M^n, ϕ) we assign the element $\sum_{x\in S} X(x) \in R_n(G)$. This element is zero in $R_n(G)$ if and only if each tangential representation class which occurs is present at an even number of stationary points. The correspondence $(M^n, \phi) \mapsto \sum_{x\in S} X(x)$ induces an algebra homomorphism $\eta: Z_*(G) \to R_*(G)$ with image $S_*(G)$. Stong [11] showed that for $G = (\mathbb{Z}_2)^n, Z_*(G) \cong S_*(G)$. In other words, (M_1, ϕ_1) and (M_2, ϕ_2) are G-cobordant if and only if $\sum_{x\in S_1} X(x) = \sum_{y\in S_2} X(y)$, where $\sum_{x\in S_1} X(x)$ and $\sum_{y\in S_2} X(y)$ correspond to (M_1, ϕ_1) and (M_2, ϕ_2) respectively. In particular, if $\sum_{x\in S} X(x) = 0$ for (M^d, ϕ), then $[M^d, \phi]_2 = 0$ in $Z_d((\mathbb{Z}_2)^n)$. Thus the unoriented cobordism class $[M]_2$ of a manifold M on which there exists an action of $(\mathbb{Z}_2)^n$ with finite stationary point set S is determined by the tangential $(\mathbb{Z}_2)^n$-modules $\{T_x M : x \in S\}$.

To deal with the oriented case of S^1 action on complex Grassmann manifolds, we need an 'oriented' version of representation ring, which is briefly introduced.

Let G be a compact connected Lie group. For our purpose G will be the circle group S^1. Let V be a (finite dimensional) oriented real representation space. If $\dim_{\mathbb{R}} V > 0$, then denote by $-V$ the same $\mathbb{R}G$-module but with opposite orientation on it. If V and W are oriented $\mathbb{R}G$-modules, then $V \oplus W$ is the oriented $\mathbb{R}G$-module where G acts diagonally and the orientation is the 'direct sum' orientation. We regard the 0-dimensional vector space as having a unique orientation. Then for any two oriented $\mathbb{R}G$-modules V and W, $V \oplus W \cong (-1)^{\dim V \cdot \dim W}(W \oplus V)$ as oriented $\mathbb{R}G$-modules, and if $\dim V$ and $\dim W$ are positive, $(-V) \oplus W \cong V \oplus (-W) \cong -(V \oplus W)$ as oriented $\mathbb{R}G$-modules. Note that if $\dim V$ is odd, then $V \cong -V$ as oriented $\mathbb{R}G$-modules because

$-id: V \to V$ is an orientation reversing isomorphism. It is now easy to check that for any two oriented $\mathbb{R}G$-modules V and W, $V \oplus W \cong W \oplus V$.

We now define the graded ring $\tilde{R}_*(G)$ which is the analogue in the oriented case of $R_*(G)$ defined above. For $n \geqslant 1$ denote by $\tilde{R}_n(G)$ the free abelian group on the isomorphism classes of oriented $\mathbb{R}G$-modules of (real) dimension n modulo the subgroup generated by elements of the form $[V] + [-V]$; $[V]$ stands for the isomorphism class of the oriented $\mathbb{R}G$-module V. $\tilde{R}_0(G)$ is defined to be the free abelian group on $[0]$, the class of the 0-dimensional $\mathbb{R}G$-module. Let $\tilde{R}_*(G) = \sum_{n \geqslant 0} \tilde{R}_n(G)$ and define as before $[V] \cdot [W]$ to be $[V \oplus W]$, where $V \oplus W$ is given the direct sum orientation and diagonal G action. It is straightforward to check that this gives rise to a well-defined multiplication which makes $\tilde{R}_*(G)$ a commutative graded ring with unit $[0]$. Note that $2x = 0$ for all $x \in \tilde{R}_n(G)$ if n is odd. Let B be the set of all isomorphism classes of irreducible oriented $\mathbb{R}G$-modules, and let $B_i = \{x \in B : \dim x = i \bmod 2\}$, $i = 0, 1$. Then it can be shown that

$$\tilde{R}_*(G) \cong \mathbb{Z}[B]/\langle \{2b : b \in B_1\} \rangle,$$

the quotient of the polynomial ring over integers $\mathbb{Z}$ in the variable B by the ideal generated by $\{2b : b \in B_1\}$.

Now suppose that (M^n, ϕ), $n \geqslant 1$ is a smooth closed oriented G-manifold with a finite stationary point set S. Let $x \in S$, then the tangent space $T_x M$ at x to M, which is an oriented vector space, is an $\mathbb{R}G$-module. Since x is an isolated stationary point $T_x M$ does not contain any trivial $\mathbb{R}G$-submodule other than 0. To (M, ϕ) we associate the element $\tilde{\eta}(M, \phi) = \sum_{x \in S} [T_x M] \in \tilde{R}_n(G)$. For a 0-dimensional manifold X, the only G-action is the trivial one. We define $\tilde{\eta}(X, \mathit{trivial}) = |X| \cdot [0] \in \tilde{R}_0(G)$. We now state a result, which may be well-known to the experts but an explicit reference is not known and which says that the function $\tilde{\eta}$ behaves well with respect to G-cobordism relation.

PROPOSITION 2.1

Suppose (M, ϕ) and (M', ϕ') are equivariantly cobordant as oriented G-manifolds with finite stationary points. Then $\tilde{\eta}(M, \phi) = \tilde{\eta}(M', \phi')$ in $\tilde{R}_*(G)$. ■

The proof of the above result goes along the line of the proof of the corresponding result in unoriented case, (cf. § 32 of [3]). Thus by Proposition 2.1, we obtain a well-defined map $\tilde{\eta}: \mathscr{F}_*(G) \to \tilde{R}_*(G)$. It is straightforward to check that the map $\tilde{\eta}$ is a homomorphism of graded rings. Moreover, it can be shown that kernel of $\tilde{\eta}$ consists of elements having representatives (M, ϕ) where G acts without fixed point on M. In fact, for $G = S^1$, kernel of $\tilde{\eta}$ is precisely the inverse image of Torsion (MSO_*) under the map $\tilde{\varepsilon}$ (cf. [10]).

3. Action on Grassmann manifolds

Let $O(n)$ denote the orthogonal group of $n \times n$ matrices. The subgroup of $O(n)$ consisting of diagonal matrices can be identified with $(\mathbb{Z}_2)^n$. Let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$, and T_j be the involution

$$T_j(e_i) = \begin{cases} -e_i & \text{if } i = j \\ e_i & \text{if } i \neq j \end{cases}$$

Then there exists an action of $(\mathbb{Z}_2)^n$ on $\mathbb{R}^n$ given by the pairwise commuting actions of T_is. This action induces an action of $(\mathbb{Z}_2)^n$ on $G_{n,k}$, the real Grassmann manifold of k-dimensional subspaces in $\mathbb{R}^n$, and this action has finite stationary point set. A k-plane X in $\mathbb{R}^n$ is fixed by this action if and only if $X = \langle e_{i_1}, e_{i_2}, \ldots, e_{i_k} \rangle =: E_\alpha$ where $\alpha =: 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. Thus there are $\binom{n}{k}$ stationary points for this action. A Grassmann manifold $G_{n,k}$ along with this action of $(\mathbb{Z}_2)^n$ will be denoted by $(G_{n,k}, \phi)$.

In [8], [12], it was proved that $G_{n,k}$ bounds if and only if $v(n) > v(k)$ where for a positive integer n, $v(n)$ denotes the integer such that $2^{v(n)}$ divides n and $2^{v(n)+1}$ does not divide n. In this section, the Grassmann manifolds $(G_{n,k}, \phi)$, which bounds equivariantly, that is, $[G_{n,k}, \phi]_2 = 0$ in $Z_{k(n-k)}((\mathbb{Z}_2)^n)$ is determined completely. We need the following lemma.

Lemma 3.1. *Let G be a compact Lie group, X a closed smooth G-manifold. Let $t: X \to X$ be a smooth fixed point free involution on X such that $gt(x) = t(gx)$ for all $g \in G$. Then X bounds equivariantly. Moreover, if X is a smooth closed oriented G-manifold and $t: X \to X$ is a smooth fixed point free orientation reversing involution on X such that $gt(x) = t(gx)$ for all $g \in G$, then X is an oriented equivariant boundary.*

Proof. Let $W = X \times [-1, 1]/\sim$, where $\sim$ is given by $(x, s) \sim (t(x), -s)$. Then W is a compact manifold. An element of W is an equivalence class $[x, s]$, $x \in X$, $s \in [-1, 1]$. Define an action of G on W as follows. For $g \in G$, $[x, s] \in W$, $g[x, s] = [gx, s]$. Note that $(gt(x), -s) = (t(gx), -s) \sim (gx, s)$. Thus the above definition makes sense. Hence W is a smooth compact G-manifold with boundary and $\partial W = X \times \{-1, 1\}/\sim$ is G-diffeomorphic to X by the map $[x, s] \mapsto x$ when $s = 1$. Moreover, if X is oriented, then $(x, s) \mapsto (t(x), -s)$ is an orientation preserving fixed point free involution on $X \times [-1, 1]$ (as t is orientation reversing), hence W becomes an oriented G-manifold. ■

Proof of Theorem 1.1. (*a*) Suppose $n = 2k$. Then if X is a k-plane in $\mathbb{R}^{2k}$, $X^\perp$, the orthogonal complement is also a k-plane in $\mathbb{R}^{2k}$. Thus $X \mapsto X^\perp$ gives a smooth fixed point free involution on $G_{2k,k}$ which is easily seen to commute with each T_j, $j = 1, 2, \ldots, n$. The result follows by Lemma 3.1.

(*b*) Suppose $k \neq n/2$. Let λ be any subset of $\{1, 2, \ldots, n\}$ consisting of k elements, that is, $|\lambda| = k$. We shall write elements of λ in increasing order. Let $e_\lambda = \{e_i : i \in \lambda\}$. Thus for each such λ there correspond a stationary point of $G_{n,k}$ which is the k-plane E_λ spanned by the vectors in e_λ. Let $\gamma_{n,k}$ be the canonical k-plane bundle over $G_{n,k}$. Then the tangent bundle $\tau G_{n,k}$ has the following description [5], $\tau G_{n,k} \cong \gamma_{n,k} \otimes \gamma_{n,k}^\perp$. Thus the tangent space at any point $X \in G_{n,k}$ is $X \otimes X^\perp$, where $X^\perp$ is the orthogonal complement of X in $\mathbb{R}^n$. Let $X_\lambda := T_{E_\lambda} G_{n,k}$ denote the tangent space at the fixed point corresponding to λ. Then the standard basis of the tangent space X_λ is given by $k(n-k)$ vectors $\{e_{i_r j} = e_{i_r} \otimes e_j\}_{r=1,2,\ldots,k}$, where $i_1 < i_2 < \cdots < i_k$ are elements of λ and $j \in \{1, 2, \ldots, n\} - \lambda$. Note that the action of $(\mathbb{Z}_2)^n$ on X_λ is given by the pairwise commuting actions of the involutions T_α, $\alpha = 1, 2, \ldots, n$, thus,

$$T_\alpha(e_{i_r j}) = \begin{cases} e_{i_r j} & \text{if } \alpha \neq i_r, j \\ -e_{i_r j} & \text{if } \alpha = i_r \text{ or } j. \end{cases}$$

These give the representation class $X(\lambda)$ of $(\mathbb{Z}_2)^n$ on X_λ. Let $\omega \subset \{1, 2, \ldots, n\}$ be given by $\omega = \{1, 2, \ldots, k\}$. We claim that the representation class $X(\omega)$ never occurs at any other stationary point; in other words, if $\lambda \neq \omega$ then the representation of $(\mathbb{Z}_2)^n$ at X_λ is not equivalent to the representation at X_ω. Suppose, $\lambda \neq \omega$. We can choose $\alpha \in \omega$ such that $\alpha \notin \lambda$. Now a basis at X_ω is given by $\{e_{ij}\}$, $i \in \{1, 2, \ldots, k\}$ and $j \in \{k+1, \ldots, n\}$, where the span of e_{ij} is a $(\mathbb{Z}_2)^n$-module for any $i \leqslant k$ and $j > k$. Thus the action of T_α on X_ω has (-1)-eigen space of dimension $n-k$, whereas the action of T_α on X_λ has (-1)-eigen space of dimension k. If there exists a $(\mathbb{Z}_2)^n$-isomorphism between X_ω and X_λ, then we must have $k = n-k$, which is impossible as $k \neq n/2$. Thus $X(\lambda)$ is distinct from $X(\omega)$, as claimed. Hence the element $\sum_\lambda X(\lambda) \in S_{k(n-k)}((\mathbb{Z}_2)^n)$ is not zero. It follows from § 2 that $[G_{n,k}, \phi]_2 \neq 0$. ■

Remark 3.2 1. The proof of part (b) actually shows that the representation classes $X(\lambda)$ and $X(\mu)$ are distinct if $\lambda \neq \mu$, $\lambda, \mu \subset \{1, 2, \ldots, n\}$, as we have not made any use of the special choice ω.
2. Note that $v(n) > v(k)$ is a necessary condition for $(G_{n,k}, \phi)$ to bound equivariantly. For if $[G_{n,k}, \phi_2] = 0$ in $Z_{k(n-k)}((\mathbb{Z}_2)^n)$ then $[G_{n,k}]_2 = 0$ in $MO_{k(n-k)}$, hence $v(n) > v(k)$ by Theorem 1.1 of [8]. Moreover, note that the above theorem produces elements in the kernel of the homomorphism ε, for $[G_{n,k}, \phi]_2$ belongs to kernel of ε whenever $v(k) < v(n)$ and $k \neq n/2$.
3. In the case (a), that is when $n = 2k$, if $\lambda \subset \{1, 2, \ldots, n\}$ such that $|\lambda| = k$, then $\lambda' = \{1, 2, \ldots, n\} - \lambda$ has cardinality k. In this case, one can check alternatively, that $X(\lambda) = X(\lambda')$, so that each representation class $X(\lambda)$ occurs twice. As a result, $\sum_\lambda X(\lambda) = 0$. It follows from Stong's theorem that $[G_{n,k}, \phi]_2 = 0$.

Next, we consider certain natural S^1 action on complex Grassmann manifolds $\mathbb{C}G_{n,k}$. For $w \in S^1$, let $\phi_w : \mathbb{C}^n \to \mathbb{C}^n$ denote the unitary map defined by

$$\phi_w(z_1, z_2, \ldots, z_n) = (wz_1, w^2 z_2, \ldots, w^n z_n).$$

This induces an action of S^1 on the complex Grassmann manifold $\mathbb{C}G_{n,k}$ of k-dimensional complex subspaces of $\mathbb{C}^n$. Let $e_1, e_2, \ldots, e_n$ denote the standard basis of $\mathbb{C}^n$. This action of S^1 on $\mathbb{C}G_{n,k}$ has finite stationary point set and the stationary points are given by $\{\langle e_{i_1}, e_{i_2}, \ldots, e_{i_k}\rangle : 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n\}$, where $\langle e_{i_1}, e_{i_2}, \ldots, e_{i_k}\rangle$ is the space spanned by $\{e_{i_1}, \ldots, e_{i_k}\}$ (cf. [9], § 4). We denote this action by ϕ. We now prove

Theorem 3.3. a) *If k or $n-k$ is even then $(\mathbb{C}G_{n,k}, \phi)$ does not bound equivariantly.*
b) *If n is even and k is odd then $(\mathbb{C}G_{n,k}, \phi)$ bounds equivariantly.*

Proof. (*a*) In [7] it was proved that if k or $n-k$ is even then the signature of $\mathbb{C}G_{n,k}$ is non-zero and $[\mathbb{C}G_{n,k}]$ generates an infinite cyclic group of $MSO_{2k(n-k)}$. It follows immediately that $[\mathbb{C}G_{n,k}, \phi] \neq 0$ in $\mathscr{F}_{2k(n-k)}(S^1)$. Alternatively, one can check that $\sum_\lambda X(\lambda) \neq 0$ in $\tilde{R}_*(S^1)$, just as in 1.1, and get the result, as $\tilde{\eta} : \mathscr{F}_*(S^1) \to \tilde{R}_*(S^1)$ is a homomorphism. Here, $X(\lambda)$ denote the oriented representation class at $X_\lambda = T_{E_\lambda} \mathbb{C}G_{n,k}$.

(*b*) Let k be odd and first assume that $k = n/2$. In this case $X \mapsto X^\perp$ gives a smooth involution of $\mathbb{C}G_{n,k}$, without fixed point. This commutes with the given action of S^1, as this action preserves innerproduct. We claim that this involution is orientation reversing. To see this, note that $H^2(\mathbb{C}G_{2k,k}; \mathbb{Z})$ is generated by the first Chern class $c_1(\gamma_{2k,k})$ of the canonical k-plane bundle over $\mathbb{C}G_{2k,k}$. Let $\theta : H^*(\mathbb{C}G_{2k,k}; \mathbb{Z}) \to$

$H^*(\mathbb{C}G_{2k,k};\mathbb{Z})$ denote the isomorphism induced by $\perp:\mathbb{C}G_{2k,k}\to\mathbb{C}G_{2k,k}$. Note that the involution $\perp$ is covered by the bundle map which sends $\gamma_{2k,k}$ to $\gamma_{2k,k}^{\perp}$ and hence $\theta(c_1(\gamma_{2k,k}))=-c_1(\gamma_{2k,k})$. Now, $c_1^{k^2}(\gamma_{2k,k})\in H^{2k^2}(\mathbb{C}G_{2k,k};\mathbb{Z})$ is a non-zero element, therefore there exists a unique $\alpha\in\mathbb{Z}-\{0\}$ such that $c_1^{k^2}(\gamma_{2k,k})=\alpha\cdot u$, where u is a generator of $H^{2k^2}(\mathbb{C}G_{2k,k};\mathbb{Z})$. But as k is odd and $\theta(c_1(\gamma_{2k,k}))=-c_1(\gamma_{2k,k})$, we have $\theta(c_1^{k^2}(\gamma_{2k,k}))=(-1)c_1^{k^2}(\gamma_{2k,k})$. This implies $\theta(u)=-u$. Hence the involution $\perp$ is orientation reversing. The result follows from Lemma 3.1.

Next consider the case k is odd, n is even and $k\neq n/2$. Let $n=2m$. We regard $\mathbb{C}^n$ as an m-dimensional right $\mathbb{H}$-space, where $\mathbb{H}$ is the division ring of quaternions. If (z_1,z_2) is a pair of complex numbers then it can be considered as a quaternion z_1+z_2j. Since $jz=\bar{z}j$ for any complex number z, we have $j(z_1+z_2j)=-\bar{z}_2+\bar{z}_1j$. Write elements of $\mathbb{C}^n$ as $(z_1,z_2,\ldots,z_n)$ with respect to the basis $e_1,e_{2m},e_2,e_{2m-1},\ldots,e_m,e_{m+1}$ and consider $\mathbb{C}^n$ as the m-tuple of quaternions $\mathbb{H}^m$ with basis $e_1+e_{2m}j,\ e_2+e_{2m-1}j,\ldots,e_m+e_{m+1}j$. Then we can define a map $j:\mathbb{C}^n\to\mathbb{C}^n$ by $j(z_1,z_2,\ldots,z_n)=(-\bar{z}_2,\bar{z}_1,\ldots,-\bar{z}_n,\bar{z}_{n-1})$. Note that j is conjugate linear and hence if X is a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$ then $j(X)$ is again a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$. Moreover, we have $j^2=-id$. Thus j induces an involution J on $\mathbb{C}G_{n,k}$, clearly J is a smooth involution on $\mathbb{C}G_{n,k}$. We claim that J is a fixed point free involution. For suppose, $J(X)=X$, $X\in\mathbb{C}G_{n,k}$. Then X is a left $\mathbb{H}$-space and $J^2=id$, so $\dim_{\mathbb{C}}X=k$ must be even, as $\dim_{\mathbb{H}}X=(1/2)\dim_{\mathbb{C}}X$; which is a contradiction as k is odd by our assumption. Next, we claim that the action ϕ on $\mathbb{C}G_{n,k}$ commutes with J. To see this, note that for each e_i and $w\in S^1$, $\phi_w(e_i)=w^ie_i$. Thus if $(z_1,z_2,\ldots,z_{2m})$ is the coordinate of a point in $\mathbb{C}^n$ with respect to the basis $e_1,e_{2m},e_2,\ldots,e_m,e_{m+1}$, then

$$w^{2m+1}j\phi_w(z_1,z_2,\ldots,z_{2m})=\phi_wj(z_1,z_2,\ldots,z_{2m}).$$

Hence the induced maps J and ϕ_w on $\mathbb{C}G_{n,k}$ commutes with each other for each $w\in S^1$. Next, we show that the involution J is orientation reversing. Since $\mathbb{C}G_{n,k}$ is path-connected, it is enough to check it at one point. Note that the orientation of $\mathbb{C}G_{n,k}$ as a real manifold is given by the orientation of each k-plane in $\mathbb{C}^n$ considered as an oriented real vector subspace of $\mathbb{R}^{2n}$, with the standard orientation on $\mathbb{R}^{2n}$. The oriented real basis of $\langle e_{i_1},e_{i_2},\ldots,e_{i_k}\rangle$ is $\{e_{i_1},\ldots,e_{i_k},\sqrt{-1}e_{i_1},\ldots,\sqrt{-1}\,e_{i_k}\}$, where $1\leqslant i_1<i_2<\cdots<i_k\leqslant n$. Moreover note that $j(e_r)=e_{n+1-r}$; hence $J(\langle e_{i_1},\ldots,e_{i_k}\rangle)=\langle e_{n+1-i_1},\ldots,e_{n+1-i_k}\rangle$. From this one can show that J is orientation reversing, as k is odd. The result now follows again by Lemma 3.1. ■

It is proved in [9] that $\mathbb{C}G_{n,k}$ is an oriented boundary if n is even and k is odd. Thus the above action does not give any non-trivial element in the kernel of $\tilde{\varepsilon}$. However, we can perturb the above action of S^1 on $\mathbb{C}G_{n,k}$ in a suitable way to generate infinitely many elements in the kernel of $\tilde{\varepsilon}$. Before we do that, let us have a close look at the representation class of $(\mathbb{C}G_{n,k},\phi)$ in the case n is even and k is odd. Since $\tilde{\eta}$ is a homomorphism, it is clear from the above theorem that $\tilde{\eta}([\mathbb{C}G_{n,k},\phi])=0$. Let us establish this, alternatively, by analysing the tangential representations at stationary points. This description will be useful in proving the next theorem. The tangent bundle of $\mathbb{C}G_{n,k}$ has the following description [5]: $\tau\mathbb{C}G_{n,k}\cong\bar{\gamma}_{n,k}\otimes\gamma_{n,k}^{\perp}$, where $\gamma_{n,k}$ is the canonical k-plane bundle, $\gamma_{n,k}^{\perp}$ its orthogonal complement and $\bar{\gamma}_{n,k}=\mathrm{Hom}_{\mathbb{C}}(\gamma_{n,k},\mathbb{C})$ is its conjugate. Let $\lambda=\{r_1,r_2,\ldots,r_k\}\subset\{1,2,\ldots,n\}$ and E_λ be the stationary point corresponding to λ. Let X_λ be the tangent space at E_λ. Then $X_\lambda=\langle\bar{e}_{r_1},\ldots,\bar{e}_{r_k}\rangle\otimes\langle e_j:j\neq r_1,r_2,\ldots,r_k\rangle$, where $\{\bar{e}_{r_1},\ldots,\bar{e}_{r_k}\}$ is a basis of $\mathrm{Hom}_{\mathbb{C}}(E_\lambda;\mathbb{C})$. Note that for each $w\in S^1$, $\phi_w(e_j)=w^je_j$ and the induced action on $\bar{e}_j$ is $\phi_w(\bar{e}_j)=w^{-j}\bar{e}_j$. Note that a natural complex basis of X_λ is given by $\bar{e}_i \otimes e_j$, $i \in \lambda$, $j \notin \lambda$, written in dictionary ordering with respect to the subscripts. In fact, $\{\bar{e}_i \otimes e_j, i \in \lambda, j \notin \lambda\}$, forms a basis of eigen vectors for $\phi_w: X_\lambda \to X_\lambda$, $w \in S^1$. Clearly, the complex representation of S^1 at X_λ is the sum of 1-dimensional irreducible complex representations of S^1 with corresponding eigen values w^{j-r}. Note that since n is even and k is odd, the number of stationary points is even, moreover if $\lambda = \{r_1, r_2, \ldots, r_k\}$, then $\lambda' = \{n+1-r_1, \ldots, n+1-r_k\}$ is distinct from λ. It is now easy to check that, the assignment

$$\bar{e}_r \otimes e_j \mapsto \bar{e}_{n+1-r} \otimes e_{n+1-j}, \quad r \in \lambda, j \notin \lambda,$$

extends to a conjugate linear isomorphism between X_λ and $X_{\lambda'}$, which preserves the group action. Since $\dim_{\mathbb{C}} X_\lambda$ is odd, it follows that there is an orientation reversing S^1-equivariant isomorphism $X_\lambda \cong X_{\lambda'}$. Consequently, according to our definition of $\tilde{R}_*(G)$, $[X_\lambda] + [X_{\lambda'}] = 0$. Since $\lambda \subset \{1, \ldots, n\}$ is arbitrary, it follows that $\tilde{\eta}(\mathbb{C}G_{n,k}, \phi) = 0$.

Next, we consider a different action of S^1 as follows. We choose distinct integers $v_1, v_2, \ldots, v_n$ such that $|v_i - v_j| \neq |v_k - v_l|$ for any $i \neq j$, $k \neq l$ and $\{i, j\} \neq \{k, l\}$. For each $w \in S^1$ define $\psi_w: \mathbb{C}^n \to \mathbb{C}^n$ by

$$\psi_w(z_1, z_2, \ldots, z_n) = (w^{v_1} z_1, w^{v_2} z_2, \ldots, w^{v_n} z_n).$$

As before, this induces an action ψ of S^1 on $\mathbb{C}G_{n,k}$. We claim that this action of S^1 has finite number of stationary points of $\mathbb{C}G_{n,k}$. Since $\psi_w(e_i) = w^{v_i} e_i$, it is clear that for any $\lambda = \{r_1, r_2, \ldots, r_k\} \subset \{1, 2, \ldots, n\}$, $E_\lambda = \langle e_{r_1}, \ldots, e_{r_k} \rangle$ is a stationary point. We shall show that these are the only stationary points of this action. Let X be a k-dimensional subspace of $\mathbb{C}^n$ such that $\psi_w(X) = X$ for all $w \in S^1$. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for X. Write each v_i as a linear combination $\sum_{a_i e_i}$ of the canonical basis vectors. Let $\lambda = \{i_1, i_2, \ldots, i_l\} \subset \{1, 2, \ldots, n\}$ be such that e_{i_r}, $i_r \in \lambda$, appears in the representation of v_j as above for at least one j. Clearly, $l = |\lambda| \geqslant k$. If we show that e_{i_r} belongs to X for each $i_r \in \lambda$, then it will follow that $l = k$ and $X = \langle e_{i_1}, \ldots, e_{i_k} \rangle$. So let $v = v_i$ and $v = \sum_{r=1}^{l} a_r e_{i_r}$, we may assume without any loss of generality that $a_r \neq 0$ for each $r = 1, 2, \ldots, l$. Since $\psi_w(X) = X$ for all $w \in S^1$ and $v \in X$, $\psi_w(v) = \sum_{r=1}^{l} a_r w^{v_{i_r}} e_{i_r} \in X$. We may choose $w_1, w_2, \ldots, w_l \in S^1$ such that $\det W \neq 0$, where W is the $l \times l$ matrix, $W = (w_s^{v_{i_r}})$. In fact, $\det W =$ Vandermonde determinant $\times$ a certain Schur function and we can choose $w_1, w_2, \ldots, w_l$, algebraically independent over Q (the field of rationals) so that the Schur function is never zero, (cf. [6]). Set $u_j = \psi_{w_j}(v) = \sum_{r=1}^{l} a_r w_j^{v_{i_r}} e_{i_r} \in X$, $1 \leqslant j \leqslant l$. Then we have an $l \times l$ matrix $(\alpha_{rs}) = (a_r w_s^{v_{i_r}})$. Clearly, $\det(\alpha_{rs}) = a_1 \cdot a_2 \cdots a_l \det W \neq 0$. Since $\det(\alpha_{rs}) \neq 0$, it is now straightforward to check that for each $i_r \in \lambda$, there exist β_1, $\beta_2, \ldots, \beta_l$, not all zero, such that $e_{i_r} = \sum_{j=1}^{l} \beta_j u_j$. Thus $e_{i_r} \in X$. Therefore, the action ψ on $\mathbb{C}G_{n,k}$ has finite stationary point set. We now prove with ψ as above,

Theorem 3.4. *$[\mathbb{C}G_{n,k}, \psi] = 0$ in $\mathscr{F}_{2k(n-k)}(S^1)$ if $n = 2k$ and k is odd and $[\mathbb{C}G_{n,k}, \psi] \neq 0$ in $\mathscr{F}_{2k(n-k)}(S^1)$ otherwise.*

Proof. If k or $n-k$ is even or if $n = 2k$ and k is odd, then the proof is same as the corresponding cases of 3.3. So we assume that n is even, k is odd and $k \neq n/2$. Since $\tilde{\eta}: \mathscr{F}_*(S^1) \to \tilde{R}_*(S^1)$ is a homomorphism, it is enough to prove that $\tilde{\eta}(\mathbb{C}G_{n,k}, \psi) \neq 0$. Let $\lambda \subset \{1, 2, \ldots, n\}$ and E_λ be the corresponding stationary point. Then from the discussion following Theorem 3.3, it is clear that the complex representation at X_λ is decomposed into irreducible 1-dimensional complex representations characterized by the corresponding eigen values $w^{v_j - v_i}$, $i \in \lambda$, $j \notin \lambda$. If $\lambda' \subset \{1, 2, \ldots, n\}$ is distinct from λ then we can always choose $p \in \lambda'$, $q \notin \lambda'$ such that for every $i \in \lambda, j \notin \lambda$, $\{p,q\} \neq \{i,j\}$. By our choice $|v_q - v_p| \neq |v_j - v_i|$. Therefore, unlike the previous action, there does not exist an S^1-equivariant orientation reversing isomorphism between the $\mathbb{R}S^1$-modules X_λ and $X_{\lambda'}$. As a result, there will be no cancellation. Hence $\tilde{\eta}(\mathbb{C}G_{n,k}, \psi) \neq 0$. ∎

Remark 3.5. By Theorem 3.1(ii) of [9], $[\mathbb{C}G_{n,k}] = 0$ if n is even and k is odd. Therefore Theorem 3.4 implies that $[\mathbb{C}G_{n,k}, \psi]$ belongs to kernel of $\tilde{\varepsilon}$ whenever n is even, k is odd and $k \neq n/2$. It is also interesting to note that in the case when n is even, k is odd and $k \neq n/2$, we can choose integers $v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n$, in an arbitrary way, satisfying the mentioned condition so that $[\mathbb{C}G_{n,k}, \psi] \neq [\mathbb{C}G_{n,k}, \psi']$, where ψ' is same as ψ, replacing v_i by v'_i. Thus we have infinitely many nontrivial elements in the kernel of $\tilde{\varepsilon}$.

For any $n \geqslant 3$ we can choose a sequence $\{v^r\}$ of finite sequences $v^r = \{v^r_1, v^r_2, \ldots, v^r_n\}$ of length n so as to satisfy $|v^r_i - v^r_j| \neq |v^s_p - v^s_q|$, for $i \neq j$, $p \neq q$ and $\{i,j\} \neq \{p,q\}$ and for any r and s (including the case $r = s$) and $|v^r_i - v^r_j| \neq |v^s_i - v^s_j|$ for any $r, s, r \neq s$, and $i, j, i \neq j$. For instance, choose natural numbers $p_1, p_2, \ldots, p_r, \ldots, p_1 > 1$, $p_r > p^n_{r-1}$ for $r \geqslant 2$, and set $v^r = \{p_r, p^2_r, \ldots, p^n_r\}$. Now for $k < n$, let ψ_r denote the action of S^1 on $\mathbb{C}G_{n,k}$ defined by v^r as above. We exclude the case when $n = 2k$ and k is odd. Then for any such choice of $\{v^r\}$, $[\mathbb{C}G_{n,k}, \psi_r] \neq [\mathbb{C}G_{n,k}, \psi_s]$ for $r \neq s$, as mentioned in the above remark and moreover, any finite number of these classes $[\mathbb{C}G_{n,k}, \psi_r]$, $r \geqslant 1$, are linearly independent over $\mathbb{Z}$. This can be seen easily by applying the homomorphism $\tilde{\eta}$ and comparing the monomials in $\mathbb{Z}[B_0]$ (cf. § 2 and note that all irreducible real representations of S^1 are 2-dimensional). In particular, we can take $k = 1$ and $d > 1$ and consider $\mathbb{C}G_{d+1,1}$ so that $\dim_{\mathbb{R}} \mathbb{C}G_{d+1,1} = 2d$. This yields,

Theorem 3.6. *For any $d > 1$, rank $\mathscr{F}_{2d}(S^1)$ is not finite.* ∎

4. Action on flag manifolds

Let $G(n_1, n_2, \ldots, n_s)$, $n = n_1 + n_2 + \cdots + n_s$, $s \geqslant 3$ denote the real flag manifold of all flags $(A_1, A_2, \ldots, A_s)$ where A_i is a left vector subspace of $\mathbb{R}^n$, $A_i \perp A_j$, for $i \neq j$, $\dim_{\mathbb{R}} A_i = n_i$, $1 \leqslant i, j \leqslant s$, $G(n_1, n_2, \ldots, n_s)$ is a smooth manifold of dimension $\sum_{1 \leqslant i < j \leqslant s} n_i n_j$. Alternatively, it can be described as the homogeneous space $O(n)/O(n_1) \times \cdots \times O(n_s)$. The group $(\mathbb{Z}_2)^n$ acts on $G(n_1, n_2, \ldots, n_s)$ by pairwise commuting involutions T_α, $\alpha = 1, 2, \ldots, n$, having finite stationary point set. This action is induced from the actions of T_αs on $\mathbb{R}^n$ as described in the last section. The number of stationary points of the action of $(\mathbb{Z}_2)^n$ on $G(n_1, n_2, \ldots, n_s)$ is $n!/n_1! \cdots n_s!$. We denote this action by $(G(n_1, n_2, \ldots, n_s), \phi)$.

Proof of Theorem 1.2. Suppose $n_i = n_j$ for some $i \neq j$. In this case there exists an obvious smooth fixed point free involution which interchanges the ith and the jth component of each flag in $G(n_1, n_2, \ldots, n_s)$ which is easily seen to commute with each T_α. Hence by Lemma 3.1 $[G(n_1, n_2, \ldots, n_s), \phi]_2 = 0$.

Next suppose that $n_i \neq n_j$ for $i \neq j$. We may without loss of generality always write n_i in increasing order.

$$\left.\begin{array}{c}\text{Let } \lambda = (\lambda^1, \lambda^2, \ldots, \lambda^s) \text{ be a partition of } \{1, 2, \ldots, n\},\\ \text{where the subset } \lambda^i \text{ has cardinality } n_i.\\ \text{We shall write elements of } \lambda^i \text{ in increasing order.}\end{array}\right\} \quad (1)$$

Let $e_{\lambda^i} = \{e_k : k \in \lambda^i\}$. Then the fixed points of $G(n_1, n_2, \ldots, n_s)$ are

$$\{(\langle e_{\lambda^1}\rangle, \langle e_{\lambda^2}\rangle, \ldots, \langle e_{\lambda^s}\rangle) : \text{for all partition } \lambda = (\lambda^1, \lambda^2, \ldots, \lambda^s) \text{ as stated in (1)}\},$$

where $\langle e_{\lambda^i}\rangle = E_{\lambda^i}$ (say) is the space spanned by e_{λ^i}. Thus for each λ as stated in (1) there exists a fixed point of $(G(n_1, n_2, \ldots, n_s), \phi)$ and as before, we shall denote by X_λ the tangent space to $G(n_1, n_2, \ldots n_s)$ at the stationary point corresponding to λ. Then by [5]

$$X_\lambda = \oplus_{1 \leqslant i < j \leqslant s} E_{\lambda^i} \otimes E_{\lambda^j} \tag{2}$$

A basis of this is given by $\{e_{\lambda^i \lambda^j}, 1 \leqslant i < j \leqslant s\}$, where $e_{\lambda^i \lambda^j} = \{e_k \otimes e_l : k \in \lambda^i, l \in \lambda^j\}$. The representation of $(\mathbb{Z}_2)^n$ on X_λ is given by its action on the basis element:

$$T_\alpha(e_k \otimes e_l) = \begin{cases} -e_k \otimes e_l & \text{if } \alpha = k \text{ or } l \\ e_k \otimes e_l & \text{otherwise} \end{cases} \tag{3}$$

Let us now consider the partition $\omega = (\omega^1, \omega^2, \ldots, \omega^s)$, where

$$\omega^1 = \{1, 2, \ldots, n_1\},$$

$$\omega^2 = \{n_1 + 1, \ldots, n_1 + n_2\}, \ldots,$$

$$\omega^s = \{n_1 + n_2 + \cdots + n_{s-1} + 1, \ldots n_1 + \cdots + n_s\}.$$

Then

$$TE_\omega = \oplus_{1 \leqslant i < j \leqslant s} E_{\omega^i} \otimes E_{\omega^j} \tag{4}$$

We claim that if $\lambda \neq \omega$ then $X(\lambda)$ is distinct from $X(\omega)$, where $X(\lambda)$ is the representation class of $(\mathbb{Z}_2)^n$ at $X(\lambda)$. To see this, suppose $\lambda \neq \omega$. Then $\omega^i \neq \lambda^i$ for some i. Choose $\alpha \in \omega^i$ such that $\alpha \notin \lambda^i$. Let $\alpha \in \lambda^j, i \neq j$. Then from (3) and (4) it follows that the action of T_α on X_ω has (-1)-eigen space of dimension $n_1 + \cdots + n_{i-1} + n_{i+1} + \cdots + n_s = n - n_i$, whereas from (2) and (3) it follows that the action of T_α on X_λ has (-1)-eigen space of dimension $n_1 + \cdots + n_{j-1} + n_{j+1} + \cdots + n_s = n - n_j$. If there exist an equivariant linear isomorphism $X_\lambda \cong X_\omega$, then we must have $n - n_i = n - n_j$, that is $n_i = n_j$ for $i \neq j$, which is impossible. Thus the representation class $X(\omega)$ does not occur at any other stationary point. In other words $\sum X(\lambda) \in S_d((\mathbb{Z}_2)^n)$ is non-zero, where $d = \sum_{1 \leqslant i < j \leqslant s} n_i n_j$ is the dimension of $G(n_1, \ldots, n_s)$. Hence $[G(n_1, \ldots, n_s), \phi]_2 \neq 0$. This completes the proof. ■

Remark 4.1. (*a*) In [9] it was proved (Theorem 2.2(a)) that $[G(n_1, \ldots, n_s)]_2 = 0$ if $n_i = n_j$ for some $i \neq j$, $1 \leqslant i, j \leqslant s$, or for some $v(n_i) < v(n)$, where $v(n)$ is as in § 3. Thus Theorem 1.2 implies that $[G(n_1, n_2, \ldots, n_s), \phi]_2$ is a nontrivial element of kernel of ε if $n_i \neq n_j$ for $i \neq j$ and $v(n_i) < v(n)$ for some i.
(*b*) To get a complete answer to the question 'Which flag manifolds bound?' it would be enough to determine whether $[G(n_1, \ldots, n_s), \phi]_2$ belongs to kernel of ε or not, in the case when n is odd and $n_i s$ are distinct.

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