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Degree of approximation of functions in the Hölder metric by (e, c) means

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Abstract. Degree of approximation of functions by the (e, c) means of its Fourier series in the Hölder metric is studied.

Keywords. Fourier series; Hölder metric; Banach space.

1. Definitions and notations

Let f be a periodic function with period 2π and integrable in the sense of Lebesgue over $[-\pi, \pi]$. Let the Fourier series of f at $t = x$ be

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \tag{1}$$

Let

$$\Phi_x(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}. \tag{2}$$

Let $S_k(f; x)$ be the kth partial sum of the Fourier series (1). Then it is easily seen that (see [9], p. 50)

$$S_k(f; x) - f(x) = \frac{2}{\pi}\int_0^{\pi} \frac{\Phi_x(t)}{2\sin\frac{1}{2}t}\sin\left(k + \frac{1}{2}\right)t\,\mathrm{d}t. \tag{3}$$

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic and continuous functions defined on $[-\pi, \pi]$ under the sup-norm. For $0 < \alpha \leqslant 1$ and some positive constant K, the function space H_α is given by the following:

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leqslant K|x - y|^\alpha\}. \tag{4}$$

The space H_α is a Banach space [7] with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{x,y} \Delta^\alpha[f(x, y)] \tag{5}$$

where

$$\|f\|_c = \sup_{-\pi \leqslant x \leqslant \pi} |f(x)| \tag{6}$$

and

$$\Delta^\alpha f(x, y) = \frac{|f(x) - f(y)|}{|x - y|^\alpha}(x \neq y). \tag{7}$$

We shall use the convention that $\Delta^0 f(x, y) = 0$. The metric induced by the norm (5) on H_α is called a Hölder metric. It can be seen that $\|f\|_\beta \leqslant (2\pi)^{\alpha - \beta}\|f\|_\alpha$ for $0 \leqslant \beta < \alpha < 1$.

Thus $(H_\alpha, \|\cdot\|_\alpha)$ is a family of Banach space which decreases as a increases, i.e.

$$C_{2\pi} \supseteq H_\beta \supseteq H_\alpha \quad \text{for } 0 \leqslant \beta < \alpha < 1.$$

DEFINITION

An infinite series $\Sigma_{-\infty}^{\infty} c_n$ with partial sums $\{C_n\}$ is said to be summable $(e,c)(c>0)$ to sum S, if

$$\lim_{n\to\infty} \sqrt{\frac{c}{\pi n}} \sum_{-\infty}^{\infty} \exp\left(-\frac{ck^2}{n}\right) C_{n+k} = S \tag{8}$$

where it is understood that $C_{n+k} = 0$ when $n + k < 0$.

The (e,c) summability method which is a regular method of summation was introduced by Hardy and Littlewood [4] (cf. also [5]) as an auxiliary method to prove Tauberian theorem for Borel summability.

It is known [6] that, if $c_n = o(1)$ and

$$c = \frac{1}{2}\alpha = \frac{k}{2(1-k)} = \frac{1+q}{2q} \tag{9}$$

then summability of Σc_n by any one of the methods (e,c), Borel exponential method (B,α), Borel integral method $(B',\alpha), \alpha > 0$, Euler method $(E,q)(q>0)$ and circle method $(\gamma,k)(0<k<1)$ implies its summability to the same sum by any of the others.

2. Introduction

Alexits [1] studied the degree of approximation of function of H_α by the Cesàro mean of their Fourier series in the sup-norm. Since $C_{2\pi} \supseteq H_\alpha \supseteq H_\beta$ for $0 \leqslant \beta < \alpha \leqslant 1$, Prosdorff [7] obtained an estimate for $\|\sigma_n(f) - f\|_\beta$ for $f \in H_\alpha$, where $\sigma_n(f)$ is the Fejèr means of the Fourier series of f. Precisely he proved the following:

Theorem A ([7], Theorem 2). *Let $f \in H_\alpha (0 < \alpha \leqslant 1)$ and $0 \leqslant \beta < \alpha$. Then*

$$\|\sigma_n(f) - f\|_\beta = O(1) \begin{cases} n^{\beta-\alpha} & (0<\alpha<1) \\ \dfrac{1}{n(\log n)^{\beta-1}} & (\alpha = 1). \end{cases}$$

The case $\beta = 0$ of Theorem A is that of Alexits referred to earlier. Recently Chandra has studied the degree of approximation of functions in Hölder metric by Borel's means [3] and by Euler's means [2]. Precisely, he proved

Theorem B [3]. *Let $0 \leqslant \beta < \alpha \leqslant 1$ and let $f \in H_\alpha$. Then*

$$\|B_n(f) - f\|_\beta = O(n^{\beta-\alpha} \log n)$$

where $B_n(f)$ is the Borel exponential mean of $S_n(f;x)$.

Theorem C [2]. *Let $0 \leqslant \beta < \alpha \leqslant 1$ and let $f \in H_\alpha$. Then*

$$\|E_n^q(f) - f\|_\beta = O(n^{\beta-\alpha} \log n)$$

where $E_n^q(f)$ is the Euler (E,q), $q > 0$ mean of $S_n(f;x)$.

The object of this paper is to find the degree of approximation of functions by the (e, c)-mean of its Fourier series in the Hölder metric. Denoting the (e, c)-mean of $S_n(f; x)$

$$e_n(f, x) = e_n^c(f; x) = \sqrt{\frac{c}{\pi n}} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{ck^2}{n}\right) S_{n+k}(f; x), \tag{10}$$

where $S_{n+x}(f; x) = 0$, $n + k < 0$, we prove the following theorems:

Theorem 1. *Let* $0 < \alpha \leqslant 1$ *and* $0 \leqslant \beta < \alpha$. *Let* $f \in H_\alpha$. *Then*

$$\|e_n(f) - f\|_\beta = O(1) \begin{cases} \dfrac{\log n}{n^{\alpha-\beta}} & (0 < \alpha - \beta \leqslant \frac{1}{2}) \\ \dfrac{1}{n^{1/2}} & (\frac{1}{2} < \alpha - \beta \leqslant 1). \end{cases}$$

Theorem 2. *Let* $0 < \alpha \leqslant 1$ *and* $0 \leqslant \beta < \alpha$ *and let* $f \in H_\alpha$. *Further, if*

$$\int_{2\pi/2n+1}^{\pi \log n/n^{1/2}} \frac{|\Phi_x(t + (2\pi/2n + 1)) - \Phi_x(t)|}{t} \exp(-nt^2/4c)\mathrm{d}t = O\left(\frac{1}{n^\alpha}\right) \tag{11}$$

then

$$\|e_n(f) - f\|_\beta = O(1) \begin{cases} \dfrac{(\log n)^{\beta/\alpha}}{n^{\alpha-\beta}}, & 0 < \alpha - \beta \leqslant \frac{1}{2} \\ \dfrac{1}{n^{1/2}}, & \frac{1}{2} < \alpha - \beta \leqslant 1. \end{cases}$$

3. Additional notations and estimates

We use the following additional notations:

$$e_n(t) = \sqrt{\frac{c}{\pi n}} \sum_{k=-n}^{\infty} \exp\left(-\frac{ck^2}{n}\right) \sin\left(n + \frac{1}{2}\right)t \tag{12}$$

$$K_n(t) = \sqrt{\frac{c}{\pi n}} \left\{1 + 2\sum_{k=1}^{n} \exp\left(-\frac{ck^2}{n}\right) \cos kt\right\} \tag{13}$$

$$L_n(t) = \sqrt{\frac{c}{\pi n}} \left\{\sum_{k=n+1}^{\infty} \exp\left(-\frac{ck^2}{n}\right) \sin\left(n + k + \frac{1}{2}\right)t\right\} \tag{14}$$

$$\theta = \theta(n) = \sqrt{\frac{c}{\pi n}} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{ck^2}{n}\right) \tag{15}$$

$$\eta = \eta(n) = \frac{2\pi}{2n + 1} \tag{16}$$

$$N = N(n) = \frac{\pi \log n}{n^{1/2}} \tag{17}$$

$$\lambda = \frac{1}{4c} \tag{18}$$

$$F(t) = \Phi_x(t) - \Phi_y(t) \tag{19}$$

Estimates. We need the following estimates:

If $f\in H_\alpha$, $0<\alpha\leqslant 1$, then

$$F(t)=\begin{cases} O(|t|^\alpha) & (20)\\ O(|x-y|^\alpha) & (21)\end{cases}$$

and

$$F(t)-F(t_1)=O(|t-t_1|^\alpha) \tag{22}$$

$$\exp(-n\lambda(t+\eta)^2)-\exp(-n\lambda t^2)=O(t+\eta)\exp(-n\lambda t^2) \tag{23}$$

$$K_n(t)=\exp(-n\lambda t^2)+\psi(n), \quad \text{where } \psi(n)=O(\mathrm{e}^{-\delta n}), \quad c>\delta>0 \tag{24}$$

$$L_n(t)=O(t\mathrm{e}^{-\delta n}), \quad (c>\delta>0) \tag{25}$$

$$\theta(n)-1=O(n^{-1/2}). \tag{26}$$

If there is no confusion, we shall write throughout δ as a suitably chosen positive constant not necessarily the same at each occurrence.

Proof of the estimates. Estimates (20) and (21) follow immediately from the definition of $\Phi_x(t)$ and H_α. Now

$$\begin{aligned}2(F(t)-F(t_1))&=2[(\Phi_x(t)-\Phi_y(t))-(\Phi_x(t_1)-\Phi_y(t_1)]\\&=2[(\Phi_x(t)-\Phi_x(t_1))-(\Phi_y(t)-\Phi_y(t_1))]\end{aligned}$$

and

$$\begin{aligned}2|\Phi_x(t)-\Phi_x(t_1)|&\leqslant|f(x+t)-f(x+t_1)|+|f(x-t)-f(x-t_1)|\\&=O(|t_1-t|^\alpha)\quad \text{as } f\in H_\alpha.\end{aligned}$$

Hence (22) follows at once.

Proof of (23). We put $g(x)=\exp(-n\lambda x^2)$. By mean value theorem for some $0<\xi<1$

$$\exp(-n\lambda(t+\eta)^2)-\exp(-n\lambda t^2)=g(t+\eta)-g(t)=\eta g'(t+\xi\eta),$$

from which (23) follows at once.

Proof of (24) is contained in (Siddiqui [8], p. 122), and proof (26) can be found in (Hardy [6]; p. 205).

Proof of (25). We have

$$\begin{aligned}L_n(t)&=\sqrt{\frac{c}{n\pi}}\sum_{k=n+1}^{\infty}\exp\left(-\frac{ck^2}{n}\right)\sin\left(n+k+\frac{1}{2}\right)t\\&=O(n^{-1/2}t)\sum_{k=n+1}^{\infty}k\exp\left(-\frac{ck^2}{n}\right)\\&=O(n^{-1/2}t)\int_{n+1}^{\infty}x\exp\left(-\frac{cx^2}{n}\right)\mathrm{d}x\\&=O(\sqrt{n}t)\int_{n+1}^{\infty}\frac{x}{n}\exp\left(-\frac{cx^2}{n}\right)\mathrm{d}x\end{aligned}$$

$$= O(\sqrt{n}t)\left[\exp\left(-\frac{cx^2}{n}\right)\right]_n^\infty$$

$$= O(\sqrt{n}te^{-cn}) = O(te^{-\delta n}) \quad (0 < \delta < c).$$

Proof of Theorem 1. From (3), (10) and (15), we get taking $S_{n+k}(f;x) = 0$, when $k < -n$

$$e_n(f;x) - f(x) = \sqrt{\frac{c}{\pi n}} \sum_{k=-n}^{\infty} \exp\left(-\frac{ck^2}{n}\right) S_{n+k}(f;x) + (\theta(n) - 1)f(x)$$

$$= \frac{2}{\pi}\frac{c}{\pi n}\int_0^\pi \frac{\Phi_x(t)}{2\sin\frac{1}{2}t}\left(\sum_{k=-n}^{\infty}\exp\left(-\frac{ck^2}{n}\right)\sin\left(n+k+\frac{1}{2}\right)t\right)dt$$

$$+ (\theta(n) - 1)f(x).$$

Let

$$l_n(x) - e_n^c(f;x) - f(x).$$

Then using (12) and (19), we obtain

$$l_n(x) = l_n(y) = \frac{2}{\pi}\int_0^\pi \frac{F(t)}{2\sin\frac{1}{2}t} e_n(t)dt + (\theta(n) - 1)(f(x) - f(y)). \tag{27}$$

We have

$$e_n(t) = \sqrt{\frac{c}{\pi n}} \sum_{k=-n}^{\infty} \exp\left(-\frac{ck^2}{n}\right)\sin\left(n+k+\frac{1}{2}\right)t$$

$$= \sqrt{\frac{c}{\pi n}}\left[\sum_{k=-n}^{n} \exp\left(-\frac{ck^2}{n}\right)\sin\left(n+k+\frac{1}{2}\right)t\right.$$

$$\left.+ \sum_{k=n+1}^{\infty} \exp\left(-\frac{ck^2}{n}\right)\sin\left(n+k+\frac{1}{2}\right)t\right]$$

$$= \sqrt{\frac{c}{\pi n}}\left[\left(1 + 2\sum_{k=1}^{n} \exp\left(-\frac{ck^2}{n}\right)\cos kt\right)\sin\left(n+\frac{1}{2}\right)t\right.$$

$$\left.+ \sum_{k=n+1}^{\infty} \exp\left(-\frac{ck^2}{n}\right)\sin\left(n+k+\frac{1}{2}\right)t\right]$$

$$= K_n(t)\sin\left(n+\frac{1}{2}\right)t + L_n(t), \tag{28}$$

using (13) and (14).

From (27), (28) and (24), we get

$$l_n(x) - l_n(y) = \frac{2}{\pi}\int_0^\pi \frac{F(t)}{2\sin\frac{1}{2}t} K_n(t)\sin\left(n+\frac{1}{2}\right)t\,dt$$

$$+ \frac{2}{\pi}\int_0^\pi \frac{F(t)}{2\sin\frac{1}{2}t} L_n(t)dt + (\theta(n) - 1)(f(x) - f(y)) \tag{29}$$

$$= \frac{2}{\pi}\int_0^\pi \frac{F(t)}{2\sin\frac{1}{2}t} e^{-n\lambda t^2} \sin\left(n+\frac{1}{2}\right)t\,dt$$

$$+ \frac{2}{\pi}\psi(n)\int_0^\pi \frac{F(t)}{2\sin\frac{1}{2}t} \sin\left(n+\frac{1}{2}\right)t\,dt$$

$$+ \frac{2}{\pi}\int_0^\pi \frac{F(t)}{2\sin\frac{1}{2}t} L_n(t)dt + (\theta(n)-1)(f(x)-f(y))$$

$$= I + J + K + L, \text{ say.} \tag{30}$$

Using (20), (21) and (24), we get

$$J = O(e^{-\delta n})\begin{cases} \int_0^\pi t^{\alpha-1}dt \\ |x-y|^\alpha \int_0^\pi n\,dt \end{cases} \tag{31}$$

$$= O(1)\begin{cases} e^{-\delta n} \\ e^{-\delta n}|x-y|^\alpha. \end{cases} \tag{32}$$

Using (31) and (32)

$$J = J^{1-\beta/\alpha}J^{\beta/\alpha} = O(1)e^{-\delta n(1-\beta/\alpha)}(|x-y|^\alpha)^{\beta/\alpha}$$

$$= O(1)(e^{-\delta n})^{1-\beta/\alpha}(|x-y|^\alpha)^{\beta/\alpha} = O(1)e^{-\delta n}|x-y|^\beta. \tag{33}$$

As

$$f(x)-f(y) = O(1) \quad \text{and} \quad f(x)-f(y) = O(|x-y|^\alpha),$$

using (26), we get

$$L = O(1)\begin{cases} \dfrac{1}{n^{1/2}} \\ \dfrac{|x-y|^\alpha}{n^{1/2}}. \end{cases}$$

Similarly (argue as in J)

$$L = O(1)\frac{|x-y|^\beta}{n^{1/2}}. \tag{34}$$

We write

$$I = \frac{2}{\pi}\left[\int_0^\eta + \int_\eta^N + \int_N^\pi\right] \equiv I_1 + I_2 + I_3, \text{ say.} \tag{35}$$

Using (20), we get

$$I_1 = \int_0^\eta \frac{F(t)}{2\sin\frac{1}{2}t} e^{-n\lambda t^2} \sin\left(n+\frac{1}{2}\right)t\,dt$$

$$= O(1)\int_0^\eta t^{\alpha-1}e^{-n\lambda t^2}dt$$

$$= O(1)\int_0^\eta t^{\alpha-1}dt = 0(n^{-\alpha}). \tag{36}$$

Using (21), we get

$$I_1 = O(1)\int_0^\eta |x-y|^\alpha n e^{-n\lambda t^2}dt$$

$$= O(1)|x-y|^\alpha n\int_0^\eta dt = O(|x-y|^\alpha). \tag{37}$$

Using (20), we get

$$I_3 = \frac{2}{\pi}\int_N^\pi \frac{F(t)}{2\sin\frac{1}{2}t}e^{-n\lambda t^2}\sin\left(n+\frac{1}{2}\right)t\,dt$$

$$= O(1)\int_N^\pi t^{\alpha-1}e^{-n\lambda t^2}dt$$

$$= O(1)e^{-n\lambda N^2}\int_N^\pi t^{\alpha-1}dt \quad (\text{as } e^{-n\lambda t^2} \text{ is decreasing})$$

$$= O(e^{-n\lambda N^2})$$

$$= O(1)(e^{-\lambda\pi^2(\log n)^2})$$

$$= O(1)(e^{-\Delta\log n}) \quad (\Delta > 0 \text{ however large})$$

$$= O(1)\left(\frac{1}{n^\Delta}\right). \tag{38}$$

Using (21), we get

$$I_3 = O(|x-y|^\alpha)\int_N^\pi \frac{e^{-\lambda nt^2}}{t}dt$$

$$= O(|x-y|^\alpha)e^{-n\lambda N^2}\int_N^\pi \frac{dt}{t}$$

$$= O(|x-y|^\alpha e^{-\lambda\pi^2(\log n)^2}\log N$$

$$= O(|x-y|^\alpha)\left(\frac{1}{n^\Delta}\right) \quad (\Delta > 0 \text{ however large}) \tag{39}$$

as in (38).

Now

$$I_2 = \frac{2}{\pi}\int_\eta^N \frac{F(t)}{2\sin\frac{1}{2}t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt$$

$$= \frac{2}{\pi}\int_\eta^N \frac{F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt$$

$$+ \frac{2}{\pi}\int_\eta^N F(t)\left[\frac{1}{2\sin\frac{1}{2}t}-\frac{1}{t}\right]e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt$$

$$= I_{2,1} + I_{2,2}, \text{ say}. \tag{40}$$

Using (20) and the fact that

$$\left[\frac{1}{2\sin\frac{1}{2}t}-\frac{1}{t}\right]=O(t),$$

we get

$$\begin{aligned}I_{2,2}&=\frac{2}{\pi}\int_{\eta}^{N}F(t)\left[\frac{1}{2\sin\frac{1}{2}t}-\frac{1}{t}\right]e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\&=O(1)\int_{\eta}^{N}t^{\alpha+1}e^{-\lambda nt^2}dt\\&=O(n^{-1})\int_{\eta}^{N}t^{\alpha}(nte^{-\lambda nt^2})dt\\&=O(n^{-1})\int_{\eta}^{N}t^{\alpha}\frac{d}{dt}(e^{-\lambda nt^2})dt\\&=O(n^{-1-\alpha})\quad\text{(integrating by parts).}\end{aligned}\tag{41}$$

Next, we write

$$\begin{aligned}2I_{2,1}&=2\int_{\eta}^{N}\frac{F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\&=\left(\int_{\eta}^{N}+\int_{2\eta}^{N+\eta}+\int_{\eta}^{2\eta}-\int_{N}^{N+\eta}\right)\frac{F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\&=\int_{\eta}^{N}\left(\frac{F(t)}{t}e^{-\lambda nt^2}-\frac{F(t+\eta)}{t+\eta}e^{-\lambda n(t+\eta)^2}\right)\sin\left(n+\frac{1}{2}\right)t\,dt\\&\quad+\int_{\eta}^{2N}\frac{F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\&\quad-\int_{N}^{N+\eta}\frac{F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\&\qquad\qquad\text{(since }\sin(n+\tfrac{1}{2})(t+\eta)=-\sin(n+\tfrac{1}{2})t)\\&=-\int_{\eta}^{N}\frac{F(t+\eta)-F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\&\quad+\int_{\eta}^{N}F(t+\eta)\left(\frac{1}{t}-\frac{1}{t+\eta}\right)e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\&\quad+\int_{\eta}^{N}\frac{F(t+\eta)}{t+\eta}[e^{-\lambda nt^2}-e^{-\lambda n(t+\eta)^2}]\sin\left(n+\frac{1}{2}\right)t\,dt\\&\quad-\int_{\eta}^{2\eta}\frac{F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\&\quad-\int_{N}^{N+\eta}\frac{F(t)}{t}e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt\\&=M_1+M_2+M_3+M_4+M_5,\text{ say.}\end{aligned}\tag{42}$$

Using (22), we have

$$M_1 = O(\eta^\alpha)\int_\eta^N \frac{e^{-\lambda n t^2}}{t}dt = O(\eta^\alpha)\int_\eta^N \frac{dt}{t} = O\left(\frac{\log n}{n^\alpha}\right). \tag{43}$$

Using (20) and (23)

$$\begin{aligned} M_3 &= O(1)\int_\eta^N (t+\eta)^{\alpha-1}e^{-\lambda n t^2}(t+\eta)dt \\ &= O(1)\int_\eta^N t^\alpha e^{-n\lambda t^2}dt \\ &= O(n^{-1})\int_\eta^N t^{\alpha-1}\frac{d}{dt}(e^{-\lambda n t^2})dt \\ &= O(n^{-\alpha}) \quad \text{(integrating by parts).} \end{aligned} \tag{44}$$

Using (20), we get

$$M_4 = O(1)\int_\eta^{2\eta} t^{\alpha-1}e^{-\lambda n t^2}dt = O(n^{-\alpha}) \tag{45}$$

and

$$\begin{aligned} M_5 &= O(1)\int_N^{N+\eta} t^{\alpha-1}e^{-\lambda n t^2}dt \\ &= O(1)e^{-\lambda n N^2}\int_N^{N+\eta} t^{\alpha-1}dt \\ &= O(1)e^{-\lambda\pi^2(\log n)^2}N^\alpha \\ &= O(1)n^{-\Delta}\left(\frac{\log n}{n^{1/2}}\right)^\alpha \quad (\Delta \text{ however large}) \\ &= O(1)(n^{-\Delta}). \end{aligned} \tag{46}$$

Now, we write

$$\begin{aligned} 2M_2 &= \int_\eta^N F(t+\eta)\left(\frac{1}{t}-\frac{1}{t+\eta}\right)e^{-\lambda n t^2}\sin\left(n+\frac{1}{2}\right)t\,dt \\ &= \left(\int_\eta^N + \int_{2\eta}^{N+\eta} + \int_\eta^{2\eta} - \int_N^{N+\eta}\right)F(t+\eta)\left(\frac{1}{t}-\frac{1}{t+\eta}\right)e^{-\lambda n t^2} \\ &\quad\times \sin\left(n+\frac{1}{2}\right)t\,dt \\ &= \int_\eta^N \left\{F(t+\eta)\left(\frac{1}{t}-\frac{1}{t+\eta}\right)e^{-\lambda n t^2} - F(t+2\eta)\left(\frac{1}{t+\eta}-\frac{1}{t+2\eta}\right)\right. \\ &\quad\left.\times e^{-n\lambda(t+\eta)^2}\right\}\sin\left(n+\frac{1}{2}\right)t\,dt \\ &\quad+ \int_\eta^{2\eta} F(t+\eta)\left(\frac{1}{t}-\frac{1}{t+\eta}\right)e^{-n\lambda t^2}\sin\left(n+\frac{1}{2}\right)t\,dt \end{aligned}$$

$$-\int_N^{N+\eta} F(t+\eta)\left(\frac{1}{t}-\frac{1}{t+\eta}\right)e^{-n\lambda t^2}\sin\left(n+\frac{1}{2}\right)t\,dt$$

$$= P + Q - R, \text{ say.} \tag{47}$$

Using (20) and the fact that

$$\frac{1}{t}-\frac{1}{t+\eta} = O(\eta/t^2)$$

it can be proved employing the argument used in proving (45) and (46) that

$$Q = O(n^{-\alpha}) \tag{48}$$

$$R = O\left(\frac{1}{n^{\Delta}}\right). \tag{49}$$

By formal computation, we get

$$P = \int_\eta^N (F(t+\eta)-F(t+2\eta))\left(\frac{1}{t}-\frac{1}{t+\eta}\right)e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt$$

$$+\int_\eta^N F(t+2\eta)\left[\frac{1}{t}-\frac{2}{t+\eta}+\frac{1}{t+2\eta}\right]e^{-\lambda nt^2}\sin\left(n+\frac{1}{2}\right)t\,dt$$

$$+\int_\eta^N F(t+2\eta)\left(\frac{1}{t+\eta}-\frac{1}{t+2\eta}\right)(e^{-\lambda nt^2}-e^{-\lambda n(t+\eta)^2})\sin\left(n+\frac{1}{2}\right)t\,dt$$

$$= P_1 + P_2 + P_3, \text{ say.} \tag{50}$$

Using (22), we get

$$P_1 = O(\eta^\alpha)\int_\eta^N \frac{\eta}{t(t+\eta)}e^{-\lambda nt^2}\,dt$$

$$= O(\eta^{1+\alpha})\int_\eta^N \frac{dt}{t^2} = O(n^{-\alpha}). \tag{51}$$

As

$$\left[\frac{1}{t}-\frac{2}{t+\eta}+\frac{1}{t+2\eta}\right] = \frac{2\eta^2}{t(t+\eta)(t+2\eta)}.$$

we obtain using (20)

$$P_2 = O(1)\int_\eta^N \frac{\eta^2}{t^3}(t+2\eta)^\alpha e^{-\lambda nt^2}\,dt$$

$$= O(\eta^2)\int_\eta^N \frac{dt}{t^{3-\alpha}} = O(n^{-\alpha}). \tag{52}$$

Lastly using (20) and (23), we get

$$P_3 = O(\eta)\int_\eta^N t^{\alpha-1}\,dt = O\left(\frac{(\log n)^\alpha}{n^{1+\alpha/2}}\right). \tag{53}$$

Collecting the results of (42)–(53), we get

$$I_{2,1} = O\left(\frac{\log n}{n^\alpha}\right). \tag{54}$$

From (40), (41) and (54), we have

$$I_2 = O\left(\frac{\log n}{n^\alpha}\right). \tag{55}$$

Using (21), we also get

$$\begin{aligned} I_2 &= O(|x-y|^\alpha)\int_\eta^N \frac{e^{-\lambda n t^2}}{2\sin\frac{1}{2}t}\mathrm{d}t \\ &= O(|x-y|^\alpha)\int_\eta^N \frac{\mathrm{d}t}{t} = O(|x-y|^\alpha \log n). \end{aligned} \tag{56}$$

Writing

$$I_k = I_k^{1-\beta/\alpha} I_k^{\beta/\alpha} \quad (k=1,2,3)$$

and using the estimates (36), (37) for I_1, (55), (56) for I_2 and (38), (39) for I_3 we get

$$I_1 = O(|x-y|^\beta n^{\beta-\alpha}) \tag{57}$$

$$I_2 = O(|x-y|^\beta n^{\beta-\alpha}\log n) \tag{58}$$

$$I_3 = O\left(|x-y|^\beta \frac{1}{n^\Delta}\right), \quad \Delta > 0, \quad \text{however large.} \tag{59}$$

From (35), (57), (58) and (59), we get

$$I = O(1)|x-y|^\beta n^{\beta-\alpha}\log n. \tag{60}$$

Using (25), we get

$$\begin{aligned} K &= \frac{2}{\pi}\int_0^\pi \frac{F(t)}{2\sin\frac{1}{2}t} L_n(t)\mathrm{d}t \\ &= O(e^{-\delta n})\int_0^\pi |F(t)|\mathrm{d}t = \begin{cases} O(e^{-\delta n}) \\ O(|x-y|^\alpha e^{-\delta n}) \end{cases}. \end{aligned} \tag{61}$$

From (61), we get (writing $K = K^{1-\beta/\alpha}K^{\beta/\alpha}$)

$$K = O(e^{-\delta n}|x-y|^\beta). \tag{62}$$

Collecting the results of (30), (33), (34), (52) and (62) we get

$$l_n(x) - l_n(y) = O(1)\begin{cases} |x-y|^\beta n^{\beta-\alpha}\log n, & 0 < \alpha-\beta \leqslant \frac{1}{2} \\ |x-y|^\beta \frac{1}{n}, & \frac{1}{2} < \alpha-\beta \leqslant 1 \end{cases}.$$

Hence

$$\sup_{\substack{x,y \\ x\neq y}} \Delta^\beta l_n(x,y) = O(1)\begin{cases} n^{\beta-\alpha}\log n, & 0 < \alpha-\beta \leqslant \frac{1}{2} \\ \frac{1}{\sqrt{n}}, & \frac{1}{2} < \alpha-\beta \leqslant 1 \end{cases}. \tag{63}$$

Again $f \in H_\alpha \Rightarrow \Phi_x(t) = O(|t|^\alpha)$ and so proceeding as above, we get

$$\|l_n\|_c| = \sup_{-\pi \leqslant x \leqslant \pi} |l_n(x)| = \begin{cases} \dfrac{\log n}{n^\alpha}, & 0 < \alpha \leqslant \frac{1}{2} \\ \dfrac{1}{\sqrt{n}}, & \frac{1}{2} < \alpha \leqslant 1 \end{cases} \tag{64}$$

Theorem 1 is completely proved by combining (63) and (64).

Proof of Theorem 2. We proceed as in the proof of Theorem 1 and retain all the estimates of J, K and L. As regards I, we retain all the estimates of the components of I except the one given in (43) for M_1 which contributes the estimation $O(\log n/n^\alpha)$. By (11) of the hypothesis of Theorem 2

$$\begin{aligned} M_1 &= \int_\eta^N \frac{F(t+\eta) - F(t)}{t} e^{-\lambda n t^2} \sin\left(n + \frac{1}{2}\right) t \, dt \\ &= O(1) \int_\eta^N \frac{|F(t+\eta) - F(t)|}{t} e^{-\lambda n t^2} \left|\sin\left(n + \frac{1}{2}\right) t\right| dt \\ &= O(1) \int_\eta^N \frac{|F(t+\eta) - F(t)|}{t} e^{-\lambda n t^2} dt = O(n^{-\alpha}). \end{aligned} \tag{65}$$

Using (65) instead of (43), it can be proved that

$$I_2 = O(n^{-\alpha}). \tag{66}$$

Now using (56) and (66)

$$\begin{aligned} I_2 &= I_2^{\beta/\alpha} I_2^{1-\beta/\alpha} \\ &= O(1)(|x-y|^\alpha \log n)^{\beta/\alpha} (n^{-\alpha})^{1-\beta/\alpha} \\ &= O(1)|x-y|^\beta (\log n)^{\beta/\alpha} n^{\beta-\alpha}. \end{aligned} \tag{67}$$

Proceeding as in Theorem 1 and using (67) and the estimates of I_1 and I_3 from (57), we obtain

$$\sup_{\substack{x,y \\ x \neq y}} \Delta^\beta l_n(x,y) = O(1) \begin{cases} n^{\beta-\alpha} (\log n)^{\beta-\alpha}, & 0 < \alpha - \beta \leqslant \frac{1}{2} \\ \dfrac{1}{\sqrt{n}}, & \frac{1}{2} < \alpha - \beta \leqslant 1 \end{cases}. \tag{68}$$

Arguing as in Theorem 1 and using (11) as employed above in the estimation of I_2, it can be shown that

$$\|l_n\|_c = \begin{cases} O(n^{-\alpha}), & 0 < \alpha \leqslant \frac{1}{2} \\ O\left(\dfrac{1}{\sqrt{n}}\right), & \frac{1}{2} < \alpha \leqslant 1 \end{cases}. \tag{69}$$

Now Theorem 2 follows at once from (68) and (69).

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