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A note on the growth of topological Sidon sets

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Abstract. We give an estimate for the number of elements in the intersection of topological Sidon sets in $\mathbf{R}^n$ with compact convex subsets and deduce a necessary and sufficient conditions for an orbit of a linear transformation of $\mathbf{R}^n$ to be a topological Sidon set.

Keywords. Topological Sidon sets; growth of sets.

Given a locally compact abelian group G, a subset Λ of the dual group X is called a topological Sidon set if any $b \in l^\infty(\Lambda$, namely any bounded complex-valued functions on Λ, is the restriction to Λ of the Fourier transform of a complex bounded Radon measure on G. These sets play an important role in harmonic analysis ([LR], [M]). When G is compact, X is discrete and the notion of topological Sidon sets coincides with that of Sidon sets. ([LR], [M].)

For any topological Sidon set Λ as above there exist $c \geqslant 1$ and a compact subset K of G such that any $b \in l^\infty(\Lambda)$ is the Fourier transform of a measure which is supported on K and has norm at most $c\|b\|_\infty$. When this condition holds for a $c \geqslant 1$ and a compact subset K, Λ is called a (c, K) topological Sidon set.

Sidon sets are known to be 'thin' set ([LR], [M], [P]). Further, estimates are known for the number of elements in intersections of Sidon sets with finite subsets (see Theorem 3). The purpose of this note is to give the similar estimate for the number of elements in intersections of topological Sidon sets in $\mathbf{R}^m$ with compact convex subsets. Let l denote the Lebesgue measure on $\mathbf{R}^m$. For a set E we denote by $|E|$ the cardinality of E. Then our result shows in particular the following.

Theorem 1. *Let $m \in \mathbf{N}$. Then for any compact set $K \subset \mathbf{R}^m$ and $c \geqslant 1$, there exist a $d > 0$ and a neighbourhood U of $0 \in \mathbf{R}^m$ such that for any (c, K) topological Sidon set Λ of $\mathbf{R}^m$ and any convex subset A of $\mathbf{R}^m$ we have*

$$|\Lambda \cap A| \leqslant d \log(l(A + U)/l(U)).$$

We deduce from the theorem the following criterion for orbits of linear transformations to be topological Sidon sets.

COROLLARY

Let $A: \mathbf{R}^m \mapsto \mathbf{R}^m$ be a linear transformation and $v \in \mathbf{R}^m$. Then $\{A^n(v) | n \in \mathbf{N}\}$ is an infinite topological Sidon set if and only if v is not contained in any A-invariant subspace of $\mathbf{R}^m$ on which all the eigenvalues are of absolute value at most 1.

While the estimate as in the theorem is adequate for the above corollary, it seems worthwhile to note that our argument below gives not just existence of a neighbourhood U, but a concrete way of choosing such a neighbourhood. This is of some interest since the right hand side would typically be big when U is small and so for getting a better estimate one would be interested in choosing U as big as may be allowable. We shall prove the following stronger version of theorem 1.

Theorem 2. *Let $m\in\mathbf{N}$. Then for any $c\geqslant 1$ there exists a $d>0$ such that the following holds: for any compact set K of $\mathbf{R}^m$, any (c,K) topological Sidon set Λ of $\mathbf{R}^m$ and any convex subset A of $\mathbf{R}^m$ we have $|\Lambda\cap A|\leqslant d\log(l(A+3U)/l(U))$, where $U=\{\lambda\in\mathbf{R}^m|\sup_{x\in K\cup B}|\Sigma_{i=1}^m\lambda_i x_i|\leqslant 1/4\pi c\}$, B being any basis of $\mathbf{R}^n$.*

We shall now recall a result from [LR], on which our proof of Theorem 2 is based, prove some preparatory results and then proceed to prove the theorem.

A finite subset A of a discrete topological group X is said to be a *test set of order M*, where $M\geqslant 1$, if $|A^2A^{-1}|\leqslant M|A|$.

Theorem 3 [LR]. *If $E\subset X$ is a Sidon set with Sidon constant $\kappa\geqslant 1$, then $|A\cap E|\leqslant 2\kappa^2 eM\log|A|$ for test sets of order M such that $|A|\geqslant 2$.*

The following proposition signifies that any countable set close to a topological Sidon set is again a topological Sidon set. It is just a higher dimensional version of Lemma 3 of Ch. VI of [M] and is deduced analogously, as indicated below.

PROPOSITION 1

Let $\Lambda=\{\lambda_n\}_{n=1}^\infty$ be a (c,K) topological Sidon set in $\mathbf{R}^m$ and $\rho>1$ be given. Let ε be such that $(1-\varepsilon c)^{-1}=\rho$ and let $0<\theta<\varepsilon$ and $W=\{\lambda\in\mathbf{R}^m|\sup_{x\in K}|_{i=1}^m\lambda_i x_i|\leqslant\theta/4\pi\}$. For each n let $\lambda_n'\in\lambda_n+W$. Then $\Lambda'=\{\lambda_n'\}_{n=1}^\infty$ is a topological Sidon set and further any function b in $l^\infty(\Lambda')$ is the restriction to Λ' of the Fourier transform of a measure $\mu\in M(\mathbf{R}^m)$ with $\|\mu\|<\rho c\|b\|_\infty$.

Proof. Since Λ is a topological Sidon set, Λ is a coherent set of frequencies. (cf: [M], Theorem I of Ch. VI for a proof in the case $m=1$. The proof actually holds in general.) We now argue as in the proof of the assertion (a)$\Rightarrow$(c) in Theorem X of Ch. IV of [M]: The argument there shows that for the set W as above $\{\lambda_n+W\}_{n=1}^\infty$ are mutually disjoint and if $H:\Lambda+W\to\Lambda\times W$ is the (well-defined) map such that $H(\lambda_n+u)=(\lambda_n,u)$, for all $n\in\mathbf{N}$, $u\in W$, then for each $g\in B(\Lambda\times W)$, $g\circ H\in B(\Lambda+W)$ and $\|g\circ H\|_{B(\Lambda+W)}<\rho\|g\|_{B(\Lambda\times W)}$.

Let $b\in l^\infty(\Lambda')$ be given. Let $\tilde f(\lambda_n+u)=b(\lambda_n')$, $\forall n,\forall u\in W$ and let f be the restriction of $\tilde f$ to Λ. Since Λ is a (c,K) topological Sidon set, there exists a measure $\mu\in M(\mathbf{R}^m)$ such that $\hat u=f$ on Λ and $\|\mu\|\leqslant c\|f\|_\infty$. Then $\mu\times\delta_0$ yields an element of $B(\Lambda\times W)$; we denote it by g and put $\tilde v=g\circ H\in B(\Lambda+W)$. Then

$$\|\tilde v\|_{B(\Lambda+W)}<\rho\|g\|_{B(\Lambda\times W)}\leqslant\rho\|\mu\times\delta_0\|\leqslant\rho c\|f\|_0=\rho c\|\tilde f\|_\infty.$$

Hence there exists a measure $v\in M(\mathbf{R}^m)$ such that $\|v\|<\rho c\|\tilde f\|_\infty$ and $\tilde v=\tilde f$ on $\Lambda+W$; in particular $\|v\|<\rho c\|b\|_\infty$ and $\hat v=b$ on Λ'.

Let $\{x_1,\ldots,x_m\}$ be any linearly independent set in $\mathbf{R}^m$ with m elements. Then any translate of the set $\{\Sigma_{i=1}^m t_i x_i|0\leqslant t_i\leqslant 1\}$ is called a *parallelopiped* in $\mathbf{R}^m$; further if $\{x_1,\ldots,x_m\}$ is an orthogonal set, then such a parallelopiped is called a *box*.

PROPOSITION 2

Let A be a compact, convex subset of $\mathbf{R}^m$ with nonempty interior. Then A contains a parallelopiped P such that $l(A) \leqslant (2m)^m l(P)$.

Proof. By a suitable translation, we can assume that $0 \in A$. We define an orthogonal set $\{x_1, \ldots, x_m\}$ in $\mathbf{R}^m$ and a linearly independent subset $\{y_1, \ldots, y_m\}$ of A by induction as follows. Let $x_1 = y_1 \in A$ be an element of maximum norm. Assume that for some $k \leqslant m-1$, an orthogonal set $\{x_1, \ldots, x_k\}$ and a linearly independent subset $\{y_1, \ldots, y_k\}$ are chosen. Let $P_k: \mathbf{R}^m \to \langle x_1, \ldots, x_k \rangle^{\perp}$ be the orthogonal projection map onto the subspace of $\mathbf{R}^m$ orthogonal to $\{x_1, \ldots, x_k\}$. Choose x_{k+1} to be an element of maximum norm in $P_k(A)$; since A has nonempty interior $x_{k+1} \neq 0$. Let $y_{k+1} \in A$ be such that $P_k(y_{k+1}) = x_{k+1}$. Clearly $\{x_1, \ldots, x_{k+1}\}$ is an orthogonal set and $\{y_1, \ldots, y_{k+1}\}$ are linearly independent. By induction this yields the sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ as desired.

Let I be the box generated by $\{x_1, \ldots, x_m\}$, i.e. $I = \{\Sigma_{i=1}^m t_i x_i | 0 \leqslant t_i \leqslant 1\}$. Let $J = \{\Sigma_{i=1}^m t_i x_i | -1 \leqslant t_i \leqslant 1\}$. Then $l(J) = 2^m l(I)$. If $a \in A$ and $(a_1, \ldots, a_m)$ are the coordinates of a with respect to the vectors $x_1, \ldots, x_m$, then $|a_i| \leqslant \| x_i \| \, \forall i$ and hence $a \in J$. Therefore $A \subseteq J$ and consequently $l(A) \leqslant l(J) = 2^m l(I)$. Let P be the parallelopiped generated by $\{y_1/m, \ldots, y_m/m\}$, i.e. $P = \{\Sigma_{i=1}^m t_i y_i/m | 0 \leqslant t_i \leqslant 1\}$. Since A is convex and $0 \in A$ it follows that $P \subseteq A$. The matrix of the transformation $x_i \leftrightarrow y_i$ is lower triangular with diagonal entries equal to 1. Therefore $l(P) = m^m l(I)$. Hence we get $l(A) \leqslant (2m)^m l(P)$.

Proof of Theorem 2. Write $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$. Let B be any basis of $\mathbf{R}^m$. Let $\theta \in (0, 1/c)$ be arbitrary and let $\varepsilon \in (\theta, 1/c)$. Let $U_\theta = \{\lambda \in \mathbf{R}^m | \sup_{x \in K \cup B} |\Sigma_{i=1}^m \lambda_i x_i| \leqslant \theta/4\pi\}$. We apply Proposition 1 to $\rho = (1 - \varepsilon c)^{-1}$ and ε, θ and U_θ as above. Clearly, U_θ is a convex, compact and symmetric neighbourhood of 0. Applying Proposition 2 to U_θ, we get a parallelopiped $P \subseteq U_\theta$ such that $l(U_\theta) \leqslant (2m)^m l(P)$. Let $\{z_1, \ldots, z_m\}$ be such that P is a translate of $\{\Sigma t_i z_i | 0 \leqslant t_i \leqslant 1\}$. Let L be the lattice generated by $\{z_1, \ldots, z_m\}$. If we choose $\lambda'_n \in (\lambda_n + U_\theta) \cap L$, then $\Lambda' = \{\lambda'_n\}_{n=1}^{\infty}$ is a coherent set of frequencies with respect to $(1, F)$, where F is a fundamental domain of the annihilator L° of L. By Proposition 1, Λ' is a topological Sidon set and any $b \in L^\infty(\Lambda')$ is the restriction to Λ' of the Fourier transform of a measure $\mu \in M(\mathbf{R}^m)$ with $\|\mu\| \leqslant \rho c \|b\|_\infty$. This implies that Λ' is a $(\rho c, F)$ topological Sidon set ([M]). Since $\hat{L} = \mathbf{R}^m / L^\circ$ and F is a fundamental domain for L° in $\mathbf{R}^m$, this is equivalent to saying that Λ' is a Sidon set in L with Sidon constant ρc.

Now let A be a compact, convex subset of $\mathbf{R}^m$. Put $A + U_\theta = B$ and $B + U_\theta = C$. We shall prove that $C \cap L$ is a test set with associated constant $(18m)^m$. We have

$$|(C \cap L) + (C \cap L) - (C \cap L)| \leqslant l(C + C - C + U_\theta)/l(P)$$

$$\leqslant l((C + U_\theta) + (C + U_\theta) - (C + U_\theta))/l(P).$$

$C + U_\theta$ is a convex, compact subset of $\mathbf{R}^m$ with nonempty interior. Applying Proposition 2 we get a parallelopiped $P_1 \subset C + U_\theta$ such that

$$l((C + U_\theta) + (C + U_\theta) - (C + U_\theta)) \leqslant (6m)^m l(P_1).$$

Then

$$(6m)^m l(P_1) \leqslant (6m)^m l(C + U_\theta) = (6m)^m l(B + U_\theta + U_\theta) \leqslant (6m)^m 3^m l(B),$$

because B contains a translate of U_θ. These inequalities and the fact that U_θ contains

P yields that

$$\begin{aligned}|(C\cap L)+(C\cap L)-(C\cap L)| &\leqslant (18m)^m l(B)/l(P)\\ &\leqslant (18m)^m|L\cap(B+U_\theta)| = (18m)^m|C\cap L|.\end{aligned}$$

This proves that $C\cap L$ is a test set as claimed. By applying Theorem 3 to $C\cap L$ we now get that

$$|\Lambda\cap A| \leqslant |\Lambda\cap(A+U_\theta)| \leqslant |\Lambda'\cap(A+2U_\theta)| \leqslant d_1 \log|L\cap(A+2U_\theta)|,$$

where $d_1 = 2e(\rho c)^2(18m)^m$. Then

$$|\Lambda\cap A| \leqslant d_1 \log(l(A+2U_\theta+U_\theta)/l(P)) \leqslant d\log(l(A+3U_\theta)/l(U_\theta))$$

where d is a constant depending on c, ε and m. By letting $\theta\to 1/c$ we get the required result.

The following theorem is analogous to Theorem II in Ch. VI of [M].

Theorem 4. *If $\{\lambda_n\}_{n=1}^\infty$ is a sequence in $\mathbf{R}^m$ such that for some $\alpha>1$ we have for all large n, $\|\lambda_{n+1}\| \geqslant \alpha\|\lambda_n\|$ then $\{\lambda_n\}_{n=1}^\infty$ is a topological Sidon set.*

This can be deduced from the following lemma in the same way as Theorem II in Ch. VI of [M] from the analogous lemma there.

Lemma. If $\{\lambda_n\}_{n=1}^\infty$ is a sequence in $\mathbf{R}^m$ such that $\|\lambda_{n+1}\| \geqslant 6\|\lambda_n\|$, $\forall n$, and if $\{b_n\}_{n=1}^\infty$ is any sequence in $\mathbf{T}$, then there exists a point $s\in\mathbf{R}^m$ such that $\|s\| \leqslant 1/\|\lambda_1\|$ and $|\langle s,\lambda_n\rangle - b_n| \leqslant 1, \forall n$.

Proof. Let $\lambda=(a_1,\dots,a_m)$ be a nonzero element in $\mathbf{R}^m$. Let B be a ball in $\mathbf{R}^m$ with radius $1/\|\lambda\|$ and centre at x_0. Let $\beta=(a_1/\|\lambda\|^2,\dots,a_m/\|\lambda\|^2)$. Then the points $x_0\pm\beta$ are contained in the boundary of B and each of the two line segments joining x_0 to $x_0\pm\beta$ is mapped onto $\mathbf{T}$ by the map $x\to\langle x,\lambda\rangle$. Therefore given any $b\in\mathbf{T}$ we can find a point $y\in B$ such that $\langle y,\lambda\rangle = b$ and $B(y,1/2\|\lambda\|)\subseteq B$. By induction we choose balls B_n and points $y_n\in B_n$ such that $B_{n+1}\subset B(y_n,1/6\|\lambda_n\|)\subset B_n$, $\forall n$ as follows: Let B_1 be the ball with centre at 0 and radius $=1/\|\lambda_1\|$. Let $y_1\in B_1$ be such that $\langle y_1,\lambda_1\rangle = b_1$ and $B(y_1,1/6\|\lambda_1\|)\subset B_1$. Suppose B_n and y_n have been chosen satisfying the above conditions. Let B_{n+1} be the ball with centre at y_n and radius $=1/\|\lambda_{n+1}\|$. Then $B_{n+1}\subset B(y_n,1/6\|\lambda_n\|)\subseteq B_n$. Choose $y_{n+1}\in B_{n+1}$ such that $\langle y_{n+1},\lambda_{n+1}\rangle = b_{n+1}$ and $B(y_{n+1},1/6\|\lambda_{n+1}\|)\subset B_{n+1}$. Let s be the point of intersection of $\{B_n\}$. Then $s\in\cap_1^\infty B(y_n,1/6\|\lambda_n\|)$ also. Then for all n, $\|s-y_n\| \leqslant 1/6\|\lambda_n\|$ and hence $|\langle s,\lambda_n\rangle - b_n| = |\langle s,\lambda_n\rangle - \langle y_n,\lambda_n\rangle| \leqslant 1$; which proves the lemma.

Proof of the Corollary. There exists a unique largest A-invariant subspace V of $\mathbf{R}^m$ such that all eigenvalues of A on V are of absolute value at most 1. Suppose $v\in V$. Using Jordan decomposition it is easy to see that there exists a $c>0$ such that $\|A^n(v)\| \leqslant cn^{m-1}$ for all n. Let $r_n = cn^{m-1}$ and B_n the ball with centre at 0 and radius r_n. If $\{A^n(v)\}_{n=1}^\infty = \Lambda$ is an infinite topological Sidon set then $A^n(v)$, $n\in\mathbf{N}$, are all distinct and hence by Theorem 2 above, we have, $n\leqslant|B_n\cap\Lambda| \leqslant d\log(l(B_n+3U)/l(U))$ for some compact neighbourhood U of 0. Therefore there exists a constant D such that $n\leqslant D\log r_n$ for all n. Since $r_n = cn^{m-1}$ this implies that $n/\log n$ is bounded which is a contradiction.

Now suppose that $v \notin V$. Using Jordan decomposition one can see that there exists a $c > 1$ and an integer $k \geqslant 1$ such that $\|A^{n+k}(v)\| \geqslant c\|A^n(v)\|$, for all large n. It follows from Theorem 4 that Λ is a finite union of topological Sidon sets. Since Λ is uniformly discrete it is a topological Sidon set.

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References

[LR] Lopez J M and Ross K A, Sidon Sets, *Lecture Notes in Pure and Applied Mathematics*, **13**; (New York: Marcel Dekker)

[M] Meyer Y, *Algebraic numbers and harmonic analysis* (Amsterdam, London: North-Holland Publishing Company) (1972)

[P] Pisier Gilles, Arithmetic characterization of Sidon sets; *Bull. Am. Math. Soc.* **8** (1983), 87–89