On the zeros of \( \zeta^{(l)}(s) - a \) (on the zeros of a class of a generalized Dirichlet series – XVII)*

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Abstract. Some very precise results (see Theorems 4 and 5) are proved about the \( \alpha \)-values of the \( l \)th derivative of a class of generalized Dirichlet series, for \( l > l_0 = l_0(a) \) \( (l_0 \) being a large constant). In particular for the precise results on the zeros of \( \zeta^{(l)}(s) - a \) (\( a \) any complex constant and \( l > l_0 \) see Theorems 1 and 2 of the introduction.

Keywords. Riemann zeta function; generalized Dirichlet series; derivatives; distribution of zeros.

1. Introduction

The object of this paper is to prove the following two theorems.

Theorem 1. Let \( \delta = \log \left( \frac{\log m}{\log 2} \right) \left( \frac{\log 3}{2} \right)^{\frac{1}{2}} \). There exists an effective constant \( \epsilon_0 > 0 \) such that if \( \epsilon \) is any constant satisfying \( 0 < \epsilon < \epsilon_0 \), then the rectangle

\[
\left\{ \sigma \geq l(\delta - \epsilon), T_{0} - \pi (\log 15)^{-1} \leq t \leq T_{0} + \pi (\log 15)^{-1} \right\}
\]

contains precisely one zero of \( \zeta^{(l)}(s) \), provided \( l \) exceeds a constant \( l_0 = l_0(\epsilon) \) depending only on \( \epsilon \). This zero is a simple zero. Moreover this zero does not lie on the boundary of this rectangle and further lies in

\[
\sigma \leq l(\delta + \epsilon).
\]

Here as usual \( s = \sigma + it \) and \( k \) is any integer, positive negative or zero.

Theorem 2. Let \( \delta = (\log \log 15)(\log 15)^{-1} \) and \( a \) any non-zero complex constant. There exists an effective constant \( \epsilon_0 > 0 \) such that if \( \epsilon \) is any constant satisfying \( 0 < \epsilon < \epsilon_0 \), then the rectangle

\[
\left\{ \sigma \geq l(\delta - \epsilon), T_{0} - \pi (\log 15)^{-1} \leq t \leq T_{0} + \pi (\log 15)^{-1} \right\}
\]

where \( T_{0} = (\Im \log \frac{1}{a} + \pi l + 2k \pi)(\log 15)^{-1} \), contains precisely one zero of \( \zeta^{(l)}(s) - a \), provided \( l \) exceeds an effective constant \( l_0 = l_0(a, \epsilon) \) depending only on \( a \) and \( \epsilon \). This zero is a simple zero. Moreover this zero does not lie on the boundary of this rectangle and further lies in

\[
\sigma \leq l(\delta + \epsilon).
\]

* Dedicated to Prof. Paul Erdös on his eighty-first birthday
Here $k$ is any integer, positive negative or zero.

**Remark.** In [1] we dealt with slightly different questions on the zeros in $\sigma > \frac{1}{2}$ of $\zeta(\sigma) - a$ where $a$ is any complex constant and $l$ is any fixed positive integer. Interested reader may consult this paper. However the results of the present paper deal with large $l$ and are more precise.

The main ingredient of the proof of Theorems 1 and 2 (and the more general results to be stated and proved in §3 and §4) is the following theorem (see Theorem 3.42 on page 116 on [2]).

**Theorem 3.** (Rouché's Theorem). If $f(z)$ and $g(z)$ are analytic inside and on a closed contour $C$, and $|g(z)| < |f(z)|$ on $C$ then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside $C$.

**Remark 1.** In what follows we use $s$ in place of $z$.

**Remark 2.** It is somewhat surprising that we can prove (with the help of Theorem 3) Theorems 4 and 5, which are much more general than Theorems 1 and 2. These will be stated in §3 and §4 respectively.

**Remark 3.** Theorems 4 and 5 can be generalized to include derivatives of $\zeta$ and $L$ functions and also of $\zeta$ function of ray classes of any algebraic number field and so on. But we have not done so.

2. **Notation**

$\{\lambda_n\}$ ($n = 1, 2, 3, \ldots$) will denote any sequence of real numbers with $\lambda_1 = 1$ and $\frac{1}{2} \leq \lambda_{n+1} - \lambda_n \leq A$ where $A (\geq 1)$ is any fixed constant. $\{a_n\}$ ($n = 1, 2, 3, \ldots$) will denote any sequence of complex numbers with $a_1 = 1$ and $|a_n| \leq n^d$. $k$ will be any integer, positive negative or zero. $\delta_n (n \geq 2)$ will denote $(2 \log \lambda_n)(2 \log \lambda_n)^{-1}$.

3. **A generalization of Theorem 1**

**Theorem 4.** Let $n_0 > 1$ be any integer, $|a_{n_0}| > A^{-1}, |a_{n_0 + 1}| > A^{-1}$ and $\delta = \left( \log \frac{\log \lambda_{n_0 + 1}}{\log \lambda_{n_0}} \right)^{-1}$. Also let $\lambda_{n+1} < \lambda_n^2$ for all $n > 1$. There exists an effective constant $\varepsilon_0$ such that if $\varepsilon$ is any constant satisfying $0 < \varepsilon \leq \varepsilon_0$, then the rectangle

$$\{\sigma \geq 1(\delta - \varepsilon), T_0 + 2k\pi \left( \log \frac{\lambda_{n_0 + 1}}{\lambda_{n_0}} \right)^{-1} \leq t \leq T_0 + (2k + 2)\pi \left( \log \frac{\lambda_{n_0 + 1}}{\lambda_{n_0}} \right)^{-1}\}$$

where $T_0 = \left( \text{Im} \log \left( \frac{a_{n_0 + 1}}{a_{n_0}} \right) \right) \left( \log \frac{\lambda_{n_0 + 1}}{\lambda_{n_0}} \right)^{-1}, \textbf{contains precisely one zero of the analytic function}$

$$\sum_{n \geq n_0} a_n (\log \lambda_n)^{\lambda_n^{-s}}.$$
provided \( I \) exceeds an effective positive constant \( I_0 = I_0(A, \varepsilon, n_0) \) depending only on the parameters indicated. This zero is a simple zero. Moreover this zero does not lie on the boundary of this rectangle and further lies in
\[ \sigma \leq I(\delta + \varepsilon). \]

**Remark.** Theorem 1 follows by taking \( n_0 = 2, \lambda_n = n \) and \( a_n = 1 \) for all \( n \).

The following lemma will be used in this section and also while applying Theorem 5 of §4 to deduce Theorem 2.

**Lemma 1.** For any \( \delta > 0 \) the function \( (\log x)x^{-\delta} \) (of \( x \) in \( x \geq 1 \)) is increasing for \( 1 \leq x \leq \exp(\delta^{-1}) \) and decreasing for \( x \geq \exp(\delta^{-1}) \). It has precisely one maximum at \( x = \exp(\delta^{-1}) \).

**Remark.** The maximum value is \((e\delta)^{-1}\). The proof of this lemma is trivial and will be left as an exercise.

To prove Theorem 4 we apply Theorem 3 to
\[
f(s) = 1 + \left( \frac{a_n + 1}{a_n} \right) \left( \frac{\log \lambda_n + 1}{\log \lambda_n} \right)^s \left( \frac{\lambda_n + 1}{\lambda_n} \right)
\]
and
\[
g(s) = \sum_{n \geq n_0 + 2} a_n' \left( \frac{\log \lambda_n}{\log \lambda_n} \right)^s \left( \frac{\lambda_n}{\lambda_n} \right)
\]
where \( a_n' = a_n(a_n)^{-1} \). It suffices to prove that \( f(s) + g(s) \) has its zeros as claimed in Theorem 4.

**Lemma 2.** The zeros of \( f(s) \) are all simple and are given by \( s = s_0 \) where
\[
s_0 = \left( \log(-a_n' + 1) + \log \left( \frac{\log \lambda_n + 1}{\log \lambda_n} \right) \right) \left( \frac{\lambda_n + 1}{\lambda_n} \right)^{-1},
\]
for all possible values of \( \log(-a_n' + 1) \). If \( s_0 = \sigma_0 + it_0 \) then
\[
\sigma_0 = \left( \log(|a_n' + 1|) + \log \left( \frac{\log \lambda_n + 1}{\log \lambda_n} \right) \right) \left( \frac{\lambda_n + 1}{\lambda_n} \right)^{-1},
\]
and
\[
t_0 = (\text{Im} \log(-a_n' + 1)) \left( \log \left( \frac{\lambda_n + 1}{\lambda_n} \right) \right)^{-1}
\]
Also
\[
f(s) = 1 - \left( \frac{\lambda_n + 1}{\lambda_n} \right)^{-s + s_0}
\]

**Proof.** The proof is trivial.

**Lemma 3.** For \( \sigma \geq 200A \), we have
\[
|g(s)| \leq \left( \frac{\lambda_n + 1}{\lambda_n} \right)^{-\sigma + s_0} S
\]
where

\[ S = \sum_{n > n_0 + 2} |a_n||a_{n+1}|^{-1} \left( \frac{\log \lambda_n}{\log \lambda_{n+1}} \right)^t \left( \frac{\lambda_n}{\lambda_{n+1}} \right)^{-\sigma}. \]

**Proof.** The proof follows from

\[ |g(s)| \leq \sum_{n > n_0 + 2} |a'_n| \left( \frac{\log \lambda_n}{\log \lambda_{n_0}} \right)^t \left( \frac{\lambda_n}{\lambda_{n_0}} \right)^{-\sigma} \]

and the fact that

\[ \left( \frac{\lambda_{n_0} + 1}{\lambda_{n_0}} \right)^{\sigma_0} |a'_{n_0 + 1}| \left( \frac{\log \lambda_{n_0} + 1}{\log \lambda_{n_0}} \right)^t. \]

**Remark.** Hereafter we write \( \sigma_0 = \delta_0 t \) and

\[ \delta_0 = l^{-1}(\log|a'_{n_0 + 1}|) \left( \log \frac{\lambda_{n_0} + 1}{\lambda_{n_0}} \right)^{-1} + \delta. \]

Also we remark that the condition \( \sigma \geq l(\delta - \epsilon) \) is the same as \( \sigma \geq l(\delta - \epsilon) \) with a change of \( \epsilon \).

**Lemma 4.** Let \( S = S(\sigma) \). Then for \( \sigma \geq l(\delta - \epsilon) \) we have,

\[ S(\sigma) < \frac{1}{1000}, \]

provided \( l \geq l_0 = l_0(\Lambda, \epsilon, n_0) \), which is effective.

To prove this lemma it suffices to prove that

\[ S(l(\delta - \epsilon)) < \frac{1}{1000}. \]

This will be done in two stages. We have (by Lemma 3)

\[ S(l(\delta - \epsilon)) = \sum_{n > n_0 + 2} |a_n||a_{n+1}|^{-1} \left\{ \left( \frac{\log \lambda_n}{\log \lambda_{n+1}} \right) \left( \frac{\lambda_n}{\lambda_{n+1}} \right)^{-\delta + \epsilon} \right\}. \]

In Lemma 5 we prove that \( \exp(\delta^{-1}) < \lambda_{n_0} \) and so by Lemma 1 it follows that \( (\log \lambda_n)_{n_0}^{-\delta} \) is decreasing for \( n \geq n_0 + 2 \). Hence it suffices to prove that

\[ \left( \frac{\log \lambda_{n_0 + 2}}{\log \lambda_{n_0 + 1}} \right) \left( \frac{\lambda_{n_0} + 2}{\lambda_{n_0 + 1}} \right)^{-\delta + \epsilon} < 1. \]

This will be done in Lemma 6. This would complete the proof of Lemma 4 since for all large \( n \)

\[ \left( \frac{\log \lambda_n}{\log \lambda_{n+1}} \right) \left( \frac{\lambda_n}{\lambda_{n+1}} \right)^{-\delta + \epsilon} \]
is less than a negative constant power of \( \lambda_n \).

**Lemma 5.** We have
\[
\exp(\delta^{-1}) < \lambda_{n_0 + 1}.
\]

**Proof.** Since for \( 0 < x < 1 \) we have \(-\log(1 - x) > x\), it follows that
\[
\delta = \left(-\log\left(1 - \left(1 - \frac{\log \lambda_{n_0}}{\log \lambda_{n_0 + 1}}\right)\right)\right)\left(\frac{\log \lambda_{n_0 + 1}}{\log \lambda_{n_0}}\right)^{-1}
\]
\[
> \left(1 - \frac{\log \lambda_{n_0}}{\log \lambda_{n_0 + 1}}\right)\left(\frac{\log \lambda_{n_0 + 1}}{\log \lambda_{n_0}}\right)^{-1}
\]
\[
= (\log \lambda_{n_0 + 1})^{-1}.
\]

This proves the lemma.

**Lemma 6.** We have
\[
\left(\frac{\log \lambda_{n_0 + 2}}{\log \lambda_{n_0 + 1}}\right)\left(\frac{\lambda_{n_0 + 1}}{\lambda_{n_0 + 2}}\right) < 1.
\]

**Proof.** We have \( \lambda_{n_0 + 2} < \lambda_{n_0 + 1}^2 \) and also for \( 0 < x < 1 \) we have \( \log(1 + x) < x \). Using these we obtain
\[
\left(1 + \left(\frac{\log \lambda_{n_0 + 2}}{\log \lambda_{n_0 + 1}}\right)\left(\log \lambda_{n_0 + 1}\right)^{-1}\right)^{\log \lambda_{n_0 + 1}} < \frac{\lambda_{n_0 + 2}}{\lambda_{n_0 + 1}}
\]
and so
\[
\left(\frac{\log \lambda_{n_0 + 2}}{\log \lambda_{n_0 + 1}}\right)\left(\frac{\lambda_{n_0 + 1}}{\lambda_{n_0 + 2}}\right)^{\log \lambda_{n_0 + 1}} < 1
\]
and since \( (\log \lambda_{n_0 + 1})^{-1} < \delta \), we obtain Lemma 6. Lemmas 2 and 4 complete the proof of Theorem 4.

4. A generalization of Theorem 2

**Theorem 5.** Let \( \delta_n \) be the maximum of \( \delta_n \) taken over all \( n \) for which \( a_n \neq 0 \) and \( n > 1 \). Suppose that for all \( n \neq 1, n_1 \) we have \( \delta_n - \delta \geq A^{-1} \) and also \( \lambda_n - 1 \geq A^{-1} \). We further suppose that \( |a_n| \geq A^{-1} \) and put \( \delta_n = \delta \). There exists an effective constant \( \varepsilon_0 \) such that for all \( \varepsilon \) satisfying \( 0 < \varepsilon \leq \varepsilon_0 \), the rectangle
\[
\{ \sigma \geq 1(\delta - \varepsilon), T_0 - \pi(\log \lambda_n)^{-1} \leq t \leq T_0 + \pi(\log \lambda_n)^{-1} \}
\]
where \( T_0 = (\text{Im} \log(-a_n) + 2k\pi)(\log \lambda_n)^{-1} \), contains precisely one zero of the analytic function
\[
1 + \sum_{n=2}^{\infty} a_n(\log \lambda_n)^t \lambda_n^{-s}
\]
provided \( l \) exceeds an effective constant \( l_0 = l_0(A, \varepsilon, n_1) \) depending only on the parameters indicated. This zero is a simple zero. Moreover this zero does not lie on the boundary of
this rectangle and further lies in
\[ \sigma \leq l(\delta + \epsilon). \]

Remark. Theorem 2 follows by taking \( \lambda_n = n \) and \( a_n = (-1)^{l+1} a^{-1} \) for all \( n \geq 2 \). Note that the maximum of \( \delta_n \) occurs when \( n = 15 \). It is necessary to check that \( \delta_{15} > \delta_{16} \). In fact we have
\[ e^\epsilon = 15.21 \ldots, \log_{10} \delta_{15}^{-1} = 0.434357 \ldots \text{ and } \log_{10} \delta_{16}^{-1} = 0.434455 \ldots, \]
by using tables.

To prove Theorem 5 we apply Theorem 3 to
\[ f(s) = 1 + a_n \log \lambda_n \lambda_n^{-s} \]
and
\[ g(s) = \sum a_n \log \lambda_n \lambda_n^{-s} \]
where the asterisk denotes the restrictions \( n \neq 1, n_1 \).

Lemma 1. The zeros of \( f(s) \) are all simple and are given by \( s = s_0 \) where
\[ s_0 = (\log(-a_n) + l\log \log \lambda_n)(\log \lambda_n)^{-1} \]
for all possible values of \( \log(-a_n) \). If \( s = \sigma_0 + it_0 \), then
\[ \sigma_0 = (\log|a_n| + l\log \log \lambda_n)(\log \lambda_n)^{-1} \]
and
\[ t_0 = (\operatorname{Im} \log(-a_n))(\log \lambda_n)^{-1}. \]
Also
\[ f(s) \leq 1 - \lambda_n^{-s + s_0}. \]

Remark. We write \( \sigma = \delta_0 l \) and \( \delta_0 = l^{-1}(\log|a_n|)(\log \lambda_n)^{-1} + \delta \). The condition \( \sigma \geq l(\delta - \epsilon) \) is the same as \( \sigma \geq l(\delta - \epsilon) \) with a change of \( \epsilon \).

Proof. The proof is trivial.

Lemma 2. For \( \sigma \geq l(\delta - \epsilon) \), we have
\[ |g(s)| \leq \sum a_n |\log \lambda_n| \lambda_n^{-l \sigma + l \epsilon}. \]

Proof. LHS is trivially not more than
\[ \sum a_n |\log \lambda_n| \lambda_n^{-\sigma} \]
for all \( \sigma \geq 200 A \). This proves the lemma.

Lemma 3. We have for \( \sigma \geq l(\delta - \epsilon) \),
\[ |g(s)| \leq \frac{1}{1000}. \]
Proof. Using $\log \lambda_n = (\lambda_n)^{\delta_n}$ we obtain, by Lemma 2,

$$|g(s)| \leq \sum a_n n (\lambda_n^{-\delta_n} + \epsilon)^t.$$ 

By the hypothesis of Theorem 5 we see that $\delta - \delta_n \geq A^{-1}$ (note also that $\lambda_n - e \geq A^{-1}$ so that $\delta \geq \log\log(e + A^{-1})$) if $\lambda_n \leq e$ and so Lemma 3 is proved.

Lemmas 1 and 3 complete the proof of Theorem 5.

Open questions

1) How much can one generalize Theorems 1 and 2?
2) Whatever the integer constant $l \geq 1$ and whatever the complex constant $a$, prove that $\zeta^{(l)}(s) - a$ has infinity of simple zeros in $\sigma > \frac{1}{2}$, (more precisely $\gg T$ simple zeros in $(\sigma \geq \frac{1}{2} + \delta, T \leq t \leq 2T)$ for some absolute constant $\delta > 0$).

References