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A proof of Howard's conjecture in homogeneous parallel shear flows – II: Limitations of Fjortoft's necessary instability criterion

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Abstract. The present paper on the linear instability of nonviscous homogeneous parallel shear flows mathematically demonstrates the correctness of Howard's [4] prediction, for a class of velocity distributions specified by a monotone function U of the altitude y and a single point of inflexion in the domain of flow, by showing not only the existence of a critical wave number $k_c > 0$ but also deriving an explicit expression for it, beyond which for all wave numbers the manifesting perturbations attain stability. An exciting conclusion to which the above result leads to is that the necessary instability criterion of Fjortoft has the seeds of its own destruction in the entire range of wave numbers $k > k_c$—a result which is not at all evident either from the criterion itself or from its derivation and has thus remained undiscovered ever since Fjortoft enunciated [3].

Keywords. Shear flows;

1. Introduction

The point of inflexion theorem of Rayleigh [5] and the semicircle theorem of Howard [4] impose necessary restrictions on the basic velocity field $U(y)$ and the complex wave velocity field $c = c_r + ic_i$ which are accessible to an arbitrary unstable ($c_i > 0$) wave in the linear instability of nonviscous homogeneous parallel shear flows and it is of interest to have a similar restriction on the growth rate kc_i possible for such an unstable wave, k being the wave number and y being the altitude. In his pioneering contribution (1961; henceforth referred to as Ho), Howard established one such estimate in the form

$$k^2 c_i^2 \leqslant \underset{\text{Flow domain}}{\text{Max}} \left(\frac{\mathrm{d}U}{\mathrm{d}y}\right)^2, \tag{1}$$

and considering its inability to provide the correct qualitative result for the case of plane Couette flow with $\mathrm{d}U/\mathrm{d}y$ constant, which is known to be neutrally stable with $kc_i \to 0$ as $k \to \infty$ remarked "This estimate is not usually sharp—for example, the Couette flow with $\mathrm{d}U/\mathrm{d}y$ constant, is known to be neutrally stable—but in most cases it will probably give the correct order of magnitude of the maximum growth rate. It is sufficient to show that c_i must approach zero as wavelength decreases to zero given the boundedness of $\mathrm{d}U/\mathrm{d}y$; but there is likelihood that infact $kc_i \to 0$ as $k \to \infty$, and with sufficient assumptions the still stronger statement that all waves shorter than some critical wavelength are stable is probably true, as illustrated by the examples of Drazin and Holmboe cited in I".

A rigorous mathematical proof of the first part of this conjectural assertion of Howard, namely that $kc_i \to 0$ as $k \to \infty$, was given in an earlier paper by Banerjee *et al* [1] under the restriction of the boundedness of $\mathrm{d}^2 U/\mathrm{d}y^2$ in the concerned domain of flow and the present paper which is in continuation to the earlier one mathematically demonstrates the correctness of the latter part of this assertion, namely that all waves shorter than some critical wavelength are stable, that is $c_i = 0$ when $k > k_c$ where k_c is some critical value of k for the class of velocity distributions specified by a monotone function U of the altitude y and having a single point of inflexion in the domain of flow [2].

An exciting conclusion to which this latter part of Howard's assertion leads to is that the basic assumption $c_i \neq 0$ in Fjortoft's derivation of his necessary instability criterion breaks down, for the class of velocity distributions as specified in the preceding paragraph, in the wave number range $k > k_c$ where k_c has the same meaning as given in the abstract, thus rendering the derivation of the criterion invalid. This invalidity assumes striking proportions for the wave with wave length zero, that is $k \to \infty$, in which case Fjortoft's necessary criterion of instability is actually a sufficient criterion of stability as will be shown later. What is really surprising is that it has taken such a long time to discover this wave number dependence of Fjortoft's necessary instability criterion but it may, possibly, be expected on the ground that neither the Fjortoft's discriminant $(\mathrm{d}^2 U/\mathrm{d}y^2)(U - U_s)$ which is to be negative somewhere in the domain of flow for any general velocity distribution $U(y)$ and negative everywhere in the domain of flow except being zero at the point of inflexion of $U(y)$ in the present context, involves any wave number implicitly or explicitly nor the derivation of the criterion itself shows any restrictivity with respect to some wave number in the set of all admissible wave numbers $k \geqslant 0$ where $U_s = U(y_s)$, $y_1 < y_s < y_2$ and $\mathrm{d}^2 U/\mathrm{d}y^2 = 0$ at $y = y_s$ with U being twice continuously differentiable in $y_1 \leqslant y \leqslant y_2$.

Proof of Howard's Conjecture. To facilitate reference to Ho, we shall make use of the same notation here and denote the basic velocity field by $U(y)$ while the Rayleigh stability equation that governs the linear instability of nonviscous homogeneous parallel shear flows is

(Ho; equation (5.1) with $\beta = 0$ and $n = 1$)

$$\frac{\mathrm{d}^2 H}{\mathrm{d}y^2} - k^2 H - \frac{\left(\dfrac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right) H}{U - c} = 0. \tag{2}$$

The boundary conditions are that H must vanish on the rigid walls which may recede to $\pm\infty$ in the limiting cases and thus

$$H(y_1) = H(y_2) = 0. \tag{3}$$

Multiplying equation (2) by H^* (the complex conjugate of H) throughout and integrating the resulting equation over the vertical range of y with the help of the boundary conditions (3), we derive

$$\int_{y_1}^{y_2} (|DH|^2 + k^2|H|^2)\mathrm{d}y + \int_{y_1}^{y_2} \frac{\left(\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right)|H|^2}{U - c}\,\mathrm{d}y = 0, \tag{4}$$

where D stands for d/dz.

Equating the real and the imaginary parts of both sides of equation (4), we obtain

$$\int_{y_1}^{y_2} (|DH|^2 + k^2|H|^2)\mathrm{d}y + \int_{y_1}^{y_2} \frac{\left(\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right)(U - c_r)|H|^2}{(U - c_r)^2 + c_i^2}\,\mathrm{d}y = 0, \tag{5}$$

and

$$c_i \int_{y_1}^{y_2} \frac{\left(\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right)|H|^2}{(U - c_r)^2 + c_i^2}\,\mathrm{d}y = 0. \tag{6}$$

Rayleigh's theorem, which states that a necessary criterion of instability ($c_i > 0$) is that the velocity distribution $U(y)$ must have at least one point of inflexion at some $y = y_s$ where $y_1 < y_s < y_2$ and $U_s = U(y_s)$ follows from equation (6) while Fjortoft's more stronger theorem, which states that a necessary criterion of instability is that

$$\left(\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right)(U - U_s) < 0 \quad \text{at some point } y = y_q \neq y_s \text{ (obviously)}$$

$$\text{where } y_1 < y_q < y_2 \text{ and } U_s = U(y_s), \tag{7}$$

follows from equation

$$\int_{y_1}^{y_2} (|DH|^2 + k^2|H|^2)\mathrm{d}y + \int_{y_1}^{y_2} \frac{\left(\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right)(U - U_s)|H|^2}{(U - c_r)^2 + c_i^2}\,\mathrm{d}y = 0, \tag{8}$$

which is obtained by multiplying equation (6) throughout by the constant factor $(c_r - U_s)$ after cancelling $c_i > 0$ from both sides of it and then adding the resulting equation to equation (5).

Further multiplying equation (2) by $\mathrm{d}^2 H^*/\mathrm{d}y^2$ throughout, we get

$$\frac{\mathrm{d}^2 H^*}{\mathrm{d}y^2}\left(\frac{\mathrm{d}^2 H}{\mathrm{d}y^2} - k^2 H\right) - \frac{\mathrm{d}^2 H^*}{\mathrm{d}y^2}\cdot\frac{\left(\frac{\mathrm{d}^2 U}{\mathrm{d}y^2} H\right)}{U - c} = 0, \tag{9}$$

and substituting for $\mathrm{d}^2 H^*/\mathrm{d}y^2$ from equation (2) in the last term of equation (9), we derive upon integrating this latter resulting equation over the range of y with the help of the boundary conditions (3)

$$\int_{y_1}^{y_2} (|D^2 H|^2 + k^2|DH|^2)\mathrm{d}y - k^2 \int_{y_1}^{y_2} \frac{\left(\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right)|H|^2}{U - c} -$$

$$\int_{y_1}^{y_2} \frac{\left(\frac{d^2 U}{dy^2}\right)^2 |H|^2}{(U-c_r)^2+c_i^2} dy = 0. \tag{10}$$

Equating the real part of both sides of equation (10), it follows that

$$\int_{y_1}^{y_2} (|D^2H|^2 + k^2|DH|^2)dy - k^2 \int_{y_1}^{y_2} \frac{\left(\frac{d^2 U}{dy^2}\right)(U-c_r)|H|^2}{(U-c_r)^2+c_i^2} - \int_{y_1}^{y_2} \frac{\left(\frac{d^2 U}{dy^2}\right)^2 |H|^2}{(U-c_r)^2+c_i^2} dy = 0, \tag{11}$$

and adding to equation (11), the equation

$$k^2(U_s - c_r) \int_{y_1}^{y_2} \frac{\left(\frac{d^2 U}{dy^2}\right) |H|^2}{(U-c_r)^2+c_i^2} dy = 0, \tag{12}$$

which follows from equation (6) since $c_i > 0$, we obtain

$$\int_{y_1}^{y_2} (|D^2H|^2 + k^2|DH|^2)dy - k^2 \int_{y_1}^{y_2} \frac{\left(\frac{d^2 U}{dy^2}\right)(U-U_s)|H|^2}{(U-c_r)^2+c_i^2} dy - \int_{y_1}^{y_2} \frac{\left(\frac{d^2 U}{dy^2}\right)^2 |H|^2}{(U-c_r)^2+c_i^2} dy = 0, \tag{13}$$

U_s being the value of U at $y = y_s$ where $y_1 < y_s < y_2$. Writing equation (13) in the form

$$\int_{y_1}^{y_2} (|D^2H|^2 + k^2|DH|^2)dy - k^2 \int_{y_1}^{y_2} \frac{\left[\left(\frac{d^2 U}{dy^2}\right)(U-U_s) + \left(\frac{d^2 U}{dy^2}\right)^2 \Big/ k^2\right]}{(U-c_r)^2+c_i^2} |H|^2 dy = 0, \tag{14}$$

we derive that a necessary criterion of instability is that

$$\left(\frac{d^2 U}{dy^2}\right)(U-U_s) + \frac{\left(\frac{d^2 U}{dy^2}\right)^2}{k^2} > 0 \quad \text{at some point } y = y_p \neq y_s \text{ (obviously)}$$
$$\text{where } y_1 < y_p < y_2. \tag{15}$$

The necessary instability criterion expressed by inequality (15) imposes another independent restriction, one being imposed by Fjortoft on Fjortoft's discriminant $(d^2 U/dy^2)(U - U_s)$, and is valid for any general velocity distribution $U(y)$.

We shall presently show the importance of this necessary instability criterion inestablishing the conjecture of Howard for a specific class of velocity distributions.

Consider the class of velocity distributions specified by a monotone function U of the altitude y and a single point of inflexion in the domain of flow $y_1 \leqslant y \leqslant y_2$. If instability is to manifest in such flows then Rayleigh's criterion implies that $y_1 < y_s < y_2$ and Fjortoft's more stronger criterion implies that

$$\left(\frac{d^2 U}{dy^2}\right)(U - U_s) \leqslant 0 \quad \text{everywhere in } y_1 \leqslant y \leqslant y_2, \tag{16}$$

with equality only where $y = y_s$ [2]. It may be noted that for a $U(y)$ belonging to this class $(d^2 U/dy^2)(U - U_s)$ can either be $\leqslant 0$ or $\geqslant 0$ everywhere in the domain of flow with equality only where $y = y_s$, and it is Fjortoft's criterion which shows that only those flows can possibly be unstable for which $(d^2 U/dy^2)(U - U_s) \leqslant 0$ everywhere in the domain of flow with equality only where $y = y_s$. Thus, a necessary criterion of instability can be derived from inequalities (15) and (16) in the form

$$-\left|\frac{d^2 U}{dy^2}\right| |U - U_s| + \frac{\left(\frac{d^2 U}{dy^2}\right)^2}{k^2} > 0 \quad \text{at some point } y = y_p \neq y_s \text{ (obviously)}$$

$$\text{where } y_1 < y_p < y_2. \tag{17}$$

Hence, if

$$k^2 > k_c^2 = \underset{y(\neq y_s)\in \text{Flow Domain}}{\text{Max}} \left[\frac{\left(\frac{d^2 U}{dy^2}\right)^2}{\left|\frac{d^2 U}{dy^2}\right| |U - U_s|}\right], \tag{18}$$

then the basic assumption $c_i > 0$ is not tenable and we must have $c_i = 0$ which implies stability since Rayleigh's equation (2) and boundary conditions (3) are invariant under complex conjugation.

It is clear from the above mathematical analysis that the conjecture of Howard remains valid even for a larger class of velocity distributions $U(y)$ which have a single point of inflexion at some $y = y_s$ where $y_1 < y_s < y_2$ and for which $(d^2 U/dy^2)(U - U_s) \leqslant 0$ everywhere in $y_1 \leqslant y \leqslant y_2$ with equality only where $y = y_s$.

The following two theorems are, thus true:

Theorem 1. *All nonviscous homogeneous parallel shear flows, with velocity distributions specified by a monotone function U of the altitude y and a single point of inflexion in the domain of flow, are stable against all infinitesimally small perturbations in the wave number range*

$$k > k_c = \underset{y(\neq y_s)\in \text{Flow Domain}}{\text{Max}} \sqrt{\left[\frac{\left(\frac{d^2 U}{dy^2}\right)^2}{\left|\frac{d^2 U}{dy^2}\right| |U - U_s|}\right]}.$$

Theorem 2. *All nonviscous homogeneous parallel shear flows with velocity distributions $U(y)$ specified by a single point of inflexion in the domain of flow and the constraint $(\mathrm{d}^2U/\mathrm{d}y^2)(U-U_s)\leqslant 0$ everywhere in $y_1\leqslant y\leqslant y_2$ with equality only where $y=y_s$ are stable against all infinitesimally small perturbations in the wave number range*

$$k>k_c=\max_{y(\neq y_s)\in\text{Flow domain}}\sqrt{\left[\frac{\left(\dfrac{\mathrm{d}^2U}{\mathrm{d}y^2}\right)^2}{\left|\dfrac{\mathrm{d}^2U}{\mathrm{d}y^2}\right||U-U_s|}\right]}.$$

An Example. Consider a sinusoidal flow with $U(y)=\sin y(y_1\leqslant y\leqslant y_2)$ such that $y_1<0<y_2$. Rayleigh's necessary instability criterion is thus satisfied and hence we cannot draw any conclusion regarding stability or otherwise of the flow.

Now, let $y_2-y_1<\pi$. Then, since

$$\left(\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right)(U-U_s)=-\sin y(\sin y-\sin 0)=-\sin^2 y\leqslant 0$$

everywhere in $y_1\leqslant y\leqslant y_2$, with equality only where $y=y_s=0$ (origin being the only point of inflexion in the flow domain) Fjortoft's necessary instability criterion, in addition to Rayleigh's, is also satisfied and hence we cannot draw any conclusion, regarding stability or otherwise of the flow, as before.

Further, since according to the present criterion

$$\left(\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right)(U-U_s)+\frac{\left(\dfrac{\mathrm{d}^2U}{\mathrm{d}y^2}\right)^2}{k^2}=-(\sin^2 y)\left(1-\frac{1}{k^2}\right)$$

must be greater than zero at some point, other than the point of inflexion obviously, as a necessary criterion of instability, we see that it is satisfied only for $k^2<1$. Hence, for $k^2>1$ the flow must be stable. This simple counter-example to Rayleigh's necessary instability criterion was given by Tollmien [6] and incidentally it also serves the purpose of a counter-example to Fjortoft's necessary instability criterion in the light of our present work.

For velocity distributions $U(y)$ belonging to the class for which Theorem 1 is valid, we obtain a necessary criterion of instability for the wave with wave length zero (that is $k\to\infty$) from inequality (15) as

$$\left(\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right)(U-U_s)>0 \quad \text{at some point } y=y_p\neq y_s \text{ (obviously)} \quad \text{where } y_1<y_p<y_2, \tag{19}$$

and hence if

$$\left(\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right)(U-U_s)\leqslant 0 \quad \text{everywhere in } y_1\leqslant y\leqslant y_2 \text{ with equality only where } y=y_s, \tag{20}$$

then we must have $c_i = 0$ which implies stability. It is to be noted that Fjortoft's necessary criterion of instability, which is given by

$$\left(\frac{d^2 U}{dy^2}\right)(U - U_s) \leqslant 0 \quad \text{everywhere in } y_1 \leqslant y \leqslant y_2 \text{ with}$$
$$\text{equality only where } y = y_s, \tag{21}$$

in the present context, has actually become a sufficient criterion of stability for the wave with $k \to \infty$ and this is in accordance with Banerjee *et al*'s [1] theorem on the rate of growth of an arbitrary unstable perturbation. We state this result in the form of a mathematical theorem as follows:

Theorem 3. *Fjortoft's necessary criterion of instability, for all nonviscous homogeneous parallel shear flows with velocity distributions specified by a monotone U function U of the altitude y and a single point of inflexion in the domain of flow, is actually a sufficient condition of stability for the wave with* $k \to \infty$ *and this result is in accordance with the prediction of Howard* [4] *and its subsequent confirmation by Banerjee et al* [1].

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