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# Generation and propagation of *SH*-type waves due to stress discontinuity in a linear viscoelastic layered medium

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**Abstract.** In this paper the generation and propagation of *SH*-type waves due to stress discontinuity in a linear viscoelastic layered medium is studied. Using Fourier transforms and complex contour integration technique, the displacement is evaluated at the free surface in closed form for two special types of stress discontinuity created at the interface. The numerical result for displacement component is evaluated for different values of non-dimensional station (distance) and is shown graphically. Graphs are compared with the corresponding graph of classical elastic case.

**Keywords.** *SH*-type waves; stress discontinuity;

## 1. Introduction

The usefulness of surface waves and its investigations in isotropic elastic medium have been well recognised in the study of earthquake waves, seismology and geophysics. Wave propagation in a layered medium has been studied extensively by many people, especially in the last two decades. Various approximate theories have been proposed to predict the dynamic response of layered medium and one of them is due to Sun *et al* [10]. Nag and Pal [7] have considered the disturbance of *SH*-type waves due to shearing stress discontinuity in an isotropic elastic medium. In another paper, Pal and Debnath [8] have considered the propagation of *SH*-type waves due to uniformly moving stress discontinuity at the interface of anisotropic elastic layered media.

Due to the effect of viscosity, gravity plays an important role in the propagation of surface waves (Love, Rayleigh, etc.). The viscoelastic behaviour of the material is described by the mechanical behaviour of solid materials with small voids. The linear viscoelasticity generally displayed by linear elastic materials is termed as 'standard linear solid', if elastic materials are having voids.

Kanai [5] has discussed the Love-type waves propagating in a singly stratified viscoelastic layer residing on the semi-infinite viscoelastic body under the conditions of the surface of discontinuity. Sarkar [9] considered the effect of body forces and stress discontinuity on the motion of *SH*-type waves in a semi-infinite viscoelastic medium. The propagation of *SH*-waves in nonhomogeneous viscoelastic layer over a semi-infinite voigt medium due to irregularity in the crustal layer has been discussed by Chattopadhyay [1]. He has followed the perturbation technique as indicated by

Eringen and Samuels [4]. The viscoelastic behaviour of linear elastic materials with voids has been considered by Cowin [2].

The present paper considers the generation and propagation of $SH$-type wave due to shearing stress discontinuity at the interface of two homogeneous viscoelastic media. Fourier transform method combined with complex contour integration technique is used to evaluate the displacement function at the free surface for two different types of stress discontinuity. Numerical results are obtained for a case only with the aid of viscoelastic model as considered by Martineĉk [6]. Results are shown graphically and are found to be in good agreement with classical elastic case.

Since the material of the earth is viscoelastic of a standard linear type, certain seismic observations and calculations may be explained on this basis. Thus the problem considered here is of interest in the theory of seismology.

## 2. Formulation of the problem and basic equations

Let us consider a viscoelastic layer of standard linear type (I) of thickness $h$ lying over a viscoelastic half-space (II). The origin of the rectangular co-ordinate system is taken at the interface. The wave-generating mechanism is a shearing stress discontinuity which is assured to be created suddenly at the interface. The geometry of the problem is depicted in figure 1. As the $SH$-type of motion is being considered here, we have $u = w = 0$ and $v = v(x, z, t)$. The displacement $v$ is also assumed to be continuous, bounded and independent of $y$. The only equation of motion in two-layered viscoelastic media in terms of stress components is given by

$$\rho_i \frac{\partial^2 v_i}{\partial t^2} = \frac{\partial}{\partial x}(\tau_{xy})_i + \frac{\partial}{\partial z}(\tau_{yz})_i, \quad i = 1, 2 \tag{2.1}$$

where

$$(\tau_{xy})_i = \left(\mu_i + \mu_i' \frac{\partial}{\partial t}\right) \frac{\partial v_i}{\partial x}$$

$$(\tau_{yz})_i = \left(\mu_i + \mu_i' \frac{\partial}{\partial t}\right) \frac{\partial v_i}{\partial z} \tag{2.2}$$

$\mu_i$ are related to shear moduli and $\mu_i'$ to viscoelastic parameters. Substituting (2.2) in (2.1), the resulting equations of motion become

$$\rho_i \frac{\partial^2 v_i}{\partial t^2} = \left(\mu_i + \mu_i' \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 v_i}{\partial x^2} + \frac{\partial^2 v_i}{\partial z^2}\right). \tag{2.3}$$

Assuming that the stress functions are harmonic and decrease with time, we have

$$\tau_{xy} = \theta(x, z) e^{-\omega t}, \quad \tau_{yz} = \psi(x, z) e^{-\omega t}, \tag{2.4}$$

and correspondingly

$$v(x, z) = V(x, z) e^{-\omega t}, \tag{2.5}$$

where $\omega$ is the frequency parameter.

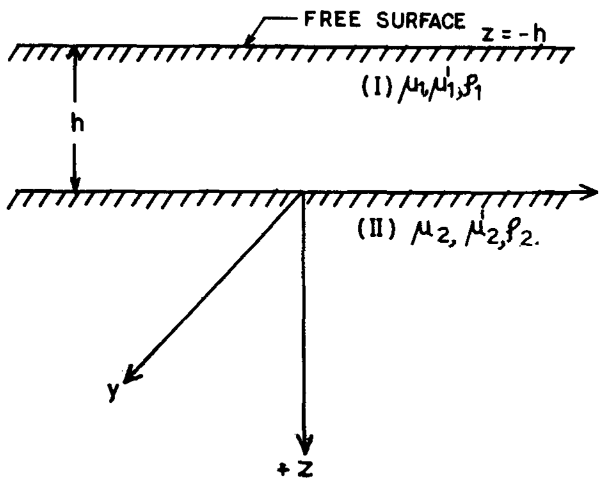


**Figure 1.** Standard linear viscoelastic layered model.

With the help of (2.5), (2.3) becomes

$$\omega^2 V = \sigma_j^2 \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2} \right), \tag{2.6}$$

where

$$\sigma_j^2 = \frac{(\mu_j - \mu_j' \omega)}{\rho_j}, \quad j = 1, 2.$$

## 3. Method of solution

Let us define the Fourier transform $\bar{V}(\xi, z)$ of $V(x, z)$ by

$$\bar{V}(\xi, z) = \int_{-\infty}^{\infty} V(x, z) e^{-i\xi x} \mathrm{d}x. \tag{3.1}$$

Therefore

$$V(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{V}(\xi, z) e^{i\xi x} \mathrm{d}\xi. \tag{3.2}$$

Applying the above transformation into (2.6), it is found that $\bar{V}(\xi, z)$ satisfies the equation

$$\omega^2 \bar{V}(\xi, z) = \sigma_j^2 \left( -\xi^2 \bar{V} + \frac{\partial^2 \bar{V}}{\partial z^2} \right)$$

or

$$\frac{\partial^2 \bar{V}}{\partial z^2} - \eta_j^2 \bar{V} = 0,$$

where

$$\eta_j^2 = \xi^2 + \frac{\omega^2}{\sigma_j^2}, \quad j = 1, 2. \tag{3.3}$$

Thus for the layers (I) and (II) we have

$$V_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} (A_1 \cosh \eta_1 z + B_1 \sinh \eta_1 z) e^{i\xi x} \mathrm{d}\xi, \tag{3.4}$$

$$V_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} B_2 e^{(-\eta_2 z + i\xi x)} \mathrm{d}\xi. \tag{3.5}$$

The boundary conditions of the problem under consideration are

(i) stress component must vanish on the free surface i.e.

$$(\tau_{yz})_1 = 0 \text{ at } z = -h \quad \text{for all } t > 0 \tag{3.6}$$

(ii) displacements must be continuous at the interface i.e.

$$V_1 = V_2 \text{ at } z = 0 \quad \text{for } t > 0 \tag{3.7}$$

(iii) stress components (shearing) must be discontinuous at the interface $z = 0$ i.e.

$$(\tau_{yz})_1 = (\tau_{yz})_2 = S(x) e^{-\omega t} \text{ at } z = 0, \quad \text{for all } x \text{ and } t, \tag{3.8}$$

where $S(x)$ is some continuous function of $x$ to be chosen later.

The above boundary conditions determine the unknown constants $A_1$, $A_2$ and $B_2$. After simplifying we have at the free surface ($z = -h$)

$$V_1(x, -h) = \int_{-\infty}^{\infty} \exp(-\eta_1 h + i\xi x) \frac{U(\xi)}{\eta_1} [\Sigma\{K^m e^{-2m\eta} s_1^h + K^{m+1} e^{-(2m+1)\eta} s_1^h\}] \mathrm{d}\xi, \tag{3.9}$$

where $U(\xi)$ is an unknown function related to $S(x)$ by

$$U(\xi) = \frac{1}{2\pi\sigma_1^2 \rho_1} \int_{-\infty}^{\infty} S(x) \exp(i\xi x) \mathrm{d}x \tag{3.10}$$

and

$$K = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} < 1$$

which is associated with the reflection coefficient in the two media.

## 4. Determination of unknown function $U(\xi)$

We now consider two different forms of the function $S(x)$ to determine $U(\xi)$.

*Case* I. Let

$$S(x) = P, \quad |x| \leqslant a$$
$$= 0, \quad \text{elsewhere.} \tag{4.1}$$

This case implies that the stress discontinuity is created in the region $-a \leqslant x \leqslant a$. Hence

$$U(\xi) = \frac{P}{2\pi\sigma_1^2\rho_1}\int_{-a}^{a}\exp(-i\xi x)\mathrm{d}x \tag{4.2}$$

$$= \frac{Pi}{2\pi\sigma_1^2\rho_1}\left(\frac{e^{i\xi a} - e^{-i\xi a}}{\xi}\right).$$

From (3.9) and (4.2) we have

$$V_1(x, -h) = \frac{P}{\pi\sigma_1^2\rho_1} I_m \int_0^{\infty}\left(\frac{e^{i\xi x_1} - e^{i\xi x_2}}{\eta_1\xi}\right)e^{-\eta_1 h}[\Sigma K^m e^{-2m\eta_1 h} + K^{m+1}e^{-(2m+1)\eta_1 h}]\mathrm{d}\xi$$

$$(x_1 = x - a, x_2 = x + a). \tag{4.3}$$

Here we wish to evaluate the integral for a few values of $m$, say $m = 0, 1, 2$ only. So we have

$$V_1(x, -h) = \frac{P}{\pi\sigma_1^2\rho_1}[I_0 + I_1 + I_2 + \cdots], \tag{4.4}$$

where

$$I_0 = \int_0^{\infty}\left(\frac{\sin\xi x_1}{\xi\eta_1} - \frac{\sin\xi x_2}{\xi\eta_1}\right)e^{-\eta_1 h}\mathrm{d}\xi + \int_0^{\infty}K\left(\frac{\sin\xi x_1}{\xi\eta_1} - \frac{\sin\xi x_2}{\xi\eta_1}\right)e^{-3\eta_1 h}\mathrm{d}\xi$$

$$= I_{01} + I_{02};\ \text{(say)} \tag{4.5}$$

$$I_1 = \int_0^{\infty}K\left(\frac{\sin\xi x_1}{\xi\eta_1} - \frac{\sin\xi x_2}{\xi\eta_1}\right)e^{-5\eta_1 h}\mathrm{d}\xi + \int_0^{\infty}K^2\left(\frac{\sin\xi x_1}{\xi\eta_1} - \frac{\sin\xi x_2}{\xi\eta_1}\right)e^{-7\eta_1 h}\mathrm{d}\xi$$

$$= I_{11} + I_{12};\ \text{(say)} \tag{4.6}$$

$$I_2 = \int_0^{\infty}K^2\left(\frac{\sin\xi x_1}{\xi\eta_1} - \frac{\sin\xi x_2}{\xi\eta_1}\right)e^{-9\eta_1 h}\mathrm{d}\xi + \int_0^{\infty}K^3\left(\frac{\sin\xi x_1}{\xi\eta_1} - \frac{\sin\xi x_2}{\xi\eta_1}\right)e^{-11\eta_1 h}\mathrm{d}\xi$$

$$= I_{21} + I_{22};\ \text{(say)}. \tag{4.7}$$

To evaluate $I_{02}, I_{11}, I_{12}, I_{21}, I_{22}$, we use the method of contour integration and $I_{01}$ is directly evaluated from the Table of Integral Transforms by Eradelyi [3].

Thus, we have

$$I_{01} = \int_0^{x_1}\mathbb{K}_0\left[\frac{\omega h}{\sigma_1}\sqrt{\frac{x^2}{h^2} + \frac{1}{4}}\right]\mathrm{d}x + \int_0^{x_2}\mathbb{K}_0\left[\frac{\omega h}{\sigma_1}\sqrt{\frac{x^2}{h^2} + \frac{1}{4}}\right]\mathrm{d}x \tag{4.8}$$

where $\mathbb{K}_0(\theta)$ is a modified Bessel's function of argument $\theta$ and of order zero.

$$I_{02} = -2\int_{\omega h/\sigma_1}^{\omega h/\sigma_2} \frac{(\sin\xi\bar{x}_1 - \sin\xi\bar{x}_2)}{\zeta\left(\dfrac{\omega^2 h^2}{\sigma_1^2} - \zeta^2\right)^{1/2}} e^{-(3/2)[(\omega^2 h^2/\sigma_1^2) - \zeta^2]^{1/2}} \mathrm{d}\zeta \tag{4.9}$$

where $\bar{x}_1 = (x_1/h), \bar{x}_2 = (x_2/h)$ and $\omega h/\sigma_1 > \omega h/\sigma_2$

$$I_{12} = 8\int_{\omega h/\sigma_1}^{\omega h/\sigma_2} \frac{(\sin\xi\bar{x}_1 - \sin\xi\bar{x}_2)}{\zeta\left(\dfrac{\omega^2 h^2}{\sigma_1^2} - \zeta^2\right)^{1/2}} e^{-(5/2)[(\omega^2 h^2/\sigma_1^2) - \zeta^2]^{1/2}} \mathrm{d}\zeta \tag{4.10}$$

$$I_{02} = -2\int_{\omega h/\sigma_1}^{\omega h/\sigma_2} \frac{(\sin\xi\bar{x}_1 - \sin\xi\bar{x}_2)}{\zeta\left(\dfrac{\omega^2 h^2}{\sigma_1^2} - \zeta^2\right)} e^{-(7/2)[(\omega^2 h^2/\sigma_1^2) - \zeta^2)]^{1/2}} D(\zeta)\mathrm{d}\zeta \tag{4.11}$$

where

$$D(\zeta) = \frac{\left(\dfrac{\omega^2 h^2}{\sigma_1^2} + \dfrac{\omega^2 h^2}{\sigma_2^2} - 2\zeta^2\right)\left(\zeta^2 - \dfrac{\omega^2 h^2}{\sigma_2^2}\right)^{1/2}}{\zeta\left[\left(\dfrac{\omega^2 h^2}{\sigma_1^2} + \dfrac{\omega^2 h^2}{\sigma_2^2} - 2\zeta^2\right)^2 + 4\left(\dfrac{\omega^2 h^2}{\sigma_1^2} - \zeta^2\right)\left(\zeta^2 - \dfrac{\omega^2 h^2}{\sigma_2^2}\right)\right]} \tag{4.12}$$

where $\omega h/\sigma_1 > \omega h/\sigma_2$.

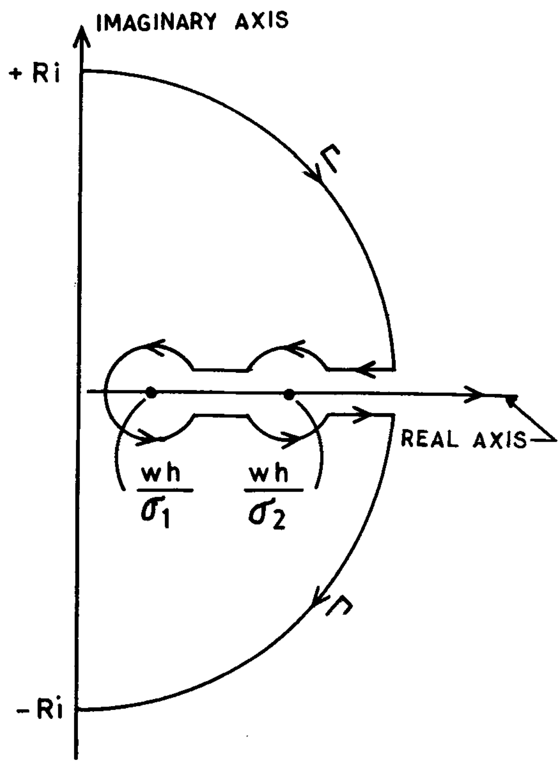


**Figure 2.** Complex contour integration in $\zeta$-plane.

The integrals in $I_{01}, I_{02}, I_{11}, I_{12}, \ldots$ have branch points at $\zeta = \pm\omega h/\sigma_1$, $\pm\omega h/\sigma_2$ and a simple pole at $\zeta = 0$. The path of the contour integration is shown in figure 2. Hence

$$v_1(x, -h, t) = \frac{e^{-\omega t}P}{\pi\sigma_1^2\rho_1}[I_{01} + I_{02} + I_{11} + I_{12} + \cdots]. \tag{4.13}$$

*Case* II. Let

$$S(x) = Ph\delta(x), \quad -\infty \leqslant x < \infty. \tag{4.14}$$

Factor $h$ is multiplied on the right side because both sides should maintain the dimension of stress.

Now

$$U(\xi) = \frac{Ph}{2\pi\sigma_1^2\rho_1}. \tag{4.15}$$

Therefore, in this case, we have

$$\begin{aligned} V_1(x, -h) &= \frac{Ph}{\pi\sigma_1^2\rho_1}\mathrm{Re}\int_0^\infty \frac{e^{i\xi x}e^{-\eta_1 h}}{\eta_1}[\Sigma K^m e^{-2m\eta_1 h} + K^{m+1}e^{-(2m+1)\eta_1 h}]\mathrm{d}\xi \\ &= \frac{Ph}{\pi\sigma_1^2\rho_1}\int_0^\infty \frac{\cos\xi x\, e^{-\eta_1 h}}{\eta_1}[\Sigma K^m e^{-2m\eta_1 h} + K^{m+1}e^{-(2m+1)\eta_1 h}]\mathrm{d}\xi \end{aligned} \tag{4.16}$$

In this case also, we evaluate the integral on the right-hand side of (4.16) for a few values of $m$ only, say $m = 0, 1, 2$. Hence

$$V_1(x, -h) = \frac{Ph}{\pi\sigma_1^2\rho_1}[I_0 + I_1 + I_2 + \ldots], \tag{4.17}$$

$$\begin{aligned} I_0 &= \int_0^\infty \frac{\cos\xi x e^{-\eta_1 h}}{\eta_1}[1 + Ke^{-\eta_1 h}]\mathrm{d}\xi \\ &= I_{01} + I_{02}\ \text{(say)} \end{aligned} \tag{4.18}$$

$$\begin{aligned} I_1 &= \int_0^\infty \frac{\cos\xi x e^{-\eta_1 h}}{\eta_1}[Ke^{-2\eta_1 h} + K^2 e^{-3\eta_1 h}]\mathrm{d}\xi \\ &= I_{11} + I_{12}\ \text{(say)} \end{aligned} \tag{4.19}$$

$$\begin{aligned} I_2 &= \int_0^\infty \frac{\cos\xi x e^{-\eta_1 h}}{\eta_1}[K^2 e^{-4\eta_1 h} + K^3 e^{-5\eta_1 h}]\mathrm{d}\xi \\ &= I_{21} + I_{22}\ \text{(say)}. \end{aligned} \tag{4.20}$$

Just like case I, we can evaluate $I_{01}, I_{02}, I_{11}, I_{12}, \ldots$ as follows:

$$I_{01} = K_0\left[\frac{\omega h}{\sigma_1}\sqrt{\bar{x}^2 + \frac{1}{4}}\right] \quad (\bar{x} = x/h) \tag{4.21}$$

$$I_{02} = -4 \int_{\omega h/\sigma_1}^{\omega h/\sigma_2} \frac{\left[ e^{-\zeta \bar{x}} e^{-(3/2)[(\omega^2 h^2/\sigma_1^2) - \zeta^2]^{1/2} [\zeta^2 - [\omega^2 h^2/\sigma_2^2]^{1/2}} \right]}{\left[ \dfrac{\omega^2 h^2}{\sigma_1^2} + \dfrac{\omega^2 h^2}{\sigma_1^2} \right]} \mathrm{d}\zeta \tag{4.22}$$

$$I_{11} = -4 \int_{\omega h/\sigma_1}^{\omega h/\sigma_2} \frac{\left[ e^{-\zeta \bar{x}} e^{-5/2[(\omega^2 h^2/\sigma_1^2) - \xi^2]^{1/2} [\xi^2 - (\omega^2 h^2/\sigma_2^2)]^{1/2}} \right]}{\left[ \dfrac{\omega^2 h^2}{\sigma_1^2} + \dfrac{\omega^2 h^2}{\sigma_1^2} \right]} \mathrm{d}\zeta \tag{4.23}$$

$$I_{12} = -8 \int_{\omega h/\sigma_1}^{\omega h/\sigma_2} \frac{\left[ e^{-\xi \bar{x}} e^{-7/2[\omega^2 h^2/\sigma_1^2) - \xi^2]^{1/2} [\xi^2 - (\omega^2 h^2/\sigma_2^2)]^{1/2}} \right]}{\left[ \dfrac{\omega^2 h^2}{\sigma_1^2} + \dfrac{\omega^2 h^2}{\sigma_1^2} \right]} \mathrm{d}\zeta \tag{4.24}$$

etc.

The integrals in (4.22), (4.23), (4.24) are valid only when $\omega h/\sigma_1 > \omega h/\sigma_2$.

Hence, in this case the displacement component on the free surface $z = -h$ is given by

$$v_1(x, -h, t) = \frac{e^{-\omega t} Ph}{\pi \sigma_1^2 \rho_1} [I_{01} + I_{02} + I_{11} + I_{12} + \cdots]. \tag{4.25}$$

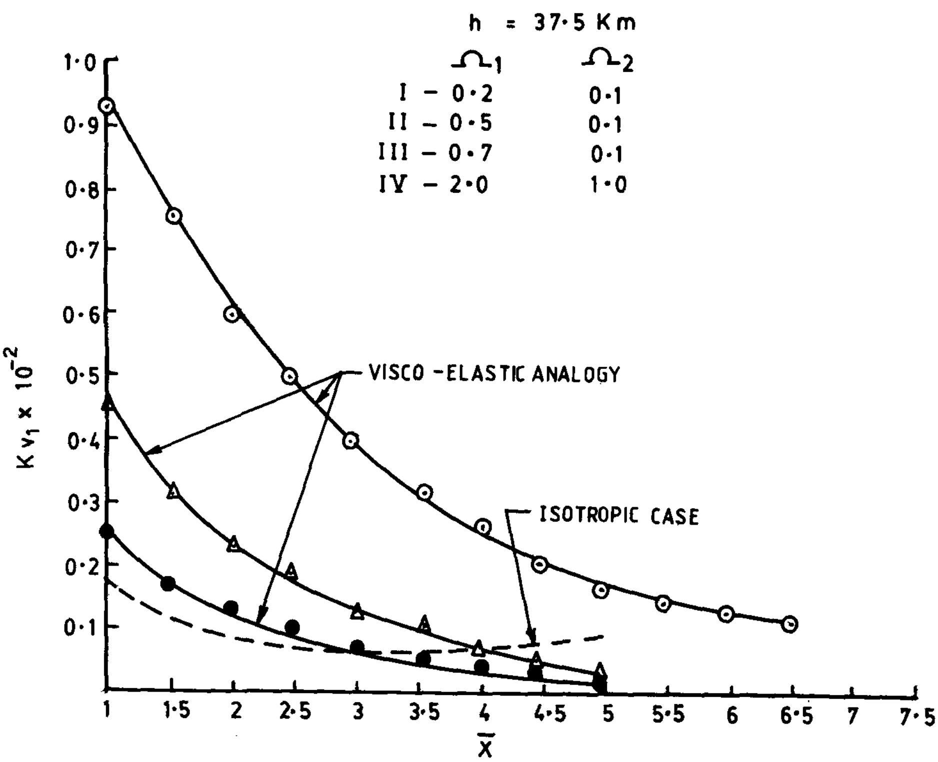


**Figure 3.** Variation of displacement with distance from the source.

## 5. Numerical results and discussion

Numerical calculations are performed here for case II only using Gauss quadrature formula and the table of integral transforms (Eradelyi [3]). The values of $Kv_1 \times 10^{-2}$, where $K = \pi\sigma_1^2\rho_1 e^{\omega t}/P$ are tabulated for different values of $\bar{x}$ and $\Omega_1 = \omega h/\sigma_1$ and keeping $\Omega_2 = \omega h/\sigma_2$ constant. The values of non-dimensional parameters $\Omega_1$ and $\Omega_2$ are taken from a viscoelastic model considered by Martineĉk [6]. For comparison a graph corresponding to isotropic case is drawn (figure 3) and is found to be in good agreement with viscoelastic analogy up to a certain value of $\bar{x}$. From the curves so drawn, it is inferred that the displacement $v_1$ decreases as $\bar{x}$ increases and the rate of decrease slows down after a certain distance.

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