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Surface waves due to blasts on and above inviscid liquids of finite depth

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Abstract. For the problem of waves due to an explosion above the surface of a homogeneous ocean of finite depth, asymptotic expressions of the velocity potential and the surface displacement are determined for large times and distances from the pressure area produced by the incident shock. It is shown that the first item in Sakurai's approximation scheme for the pressure field inside the blast wave as well as the results of Taylor's point blast theory can be used to yield realistic expressions of surface displacement. Some interesting features of the wave motion in general are described. Finally some numerical calculations for the surface elevation were performed and included as a particular case.

Keywords. Surface waves; inviscid liquid; asymptotic expansion; blast theory; surface elevation.

1. Introduction

The problem of surface waves caused by the interaction of a blast-generated shock wave with an ideal incompressible fluid has been analysed by Rumiantsev [9], Kisler [4] and Sen [11], mainly when the fluid is infinitely deep. The problem of waves produced by explosions above the surface of a shallow liquid has also been touched upon by Kranzer and Keller [5] as an application of the asymptotic Caucny-Poisson wave theory for fluids of finite depth. This treatment, however, did not include the effects of the time variation of the pressure distribution on the surface. Choudhuri [1] and Wen [14] considered the case where the disturbance is over any arbitrary region of the free surface and the water is of uniform finite depth by the method of multiple Fourier transforms. In both the cases the method of stationary phase was applied to obtain the approximate expression for the potential function and surface elevation for large values of time and distance. Mondal and Mukherjee [8] considered the corresponding problem by Hankel transform method and finally the approximate expressions for the potential function and inertial surface elevation were obtained for large distances and times by the method of stationary phase.

The basic simplifying assumption in this problem is that the large difference between the densities of the gas and the fluid make the fluid displacements too small to affect the motion of the gas, which is supposed to be known. Here we present the three-dimensional problem of the generation of waves due to explosions above the surface of a fluid of constant finite depth due to the incident shock and of the area on which it acts. After deriving the formal solution of the problem in terms of infinite integrals in the usual manner, we use the known asymptotic expansions of the Bessel function and Kummer's confluent hypergeometric function alongwith the method of stationary phase to find approximate expressions of the velocity potential and the surface displacement ($=\zeta$) integrals of large times and distances from the pressure area. For the pressure field inside the blast wave, we first make use of an expression closely resembling the first term of Sakurai's [10] approximation scheme. It is also easy to see that the expressions of ζ in the form of infinite series may be obtained by the same methods as used by Sen [11], but these will not be deduced here. Instead, we describe the more tractable features of the asymptotic wave motion in its general form as well as special forms which use the results of the Taylor point blast theory, and then place our results on a more realistic footing.

2. Formulation of the problem

We assume that surface waves are excited when the spherical shock wave due to a point blast in the gas interacts with the fluid surface. An expanding circular region of pressure is formed on the free surface as a consequence. Using cylindrical coordinates (r, θ, z), we write the governing equations as follows:

For $t > 0$,

$$\left.\begin{aligned} p_0(r,t) &= f(r,t), \quad r < r_0(t), \\ &= 0, \quad r > r_0(t), \end{aligned}\right\} \tag{1}$$

$$\nabla^2 \varphi(r,z,t) = 0, \quad z < 0, \quad t > 0 \tag{2}$$

$$gp\zeta = -p_0(r,t) + \rho\left(\frac{\partial \varphi}{\partial t}\right)_{z=0} \tag{3}$$

$$\frac{\partial^2 \varphi}{\partial t^2} + g\frac{\partial \varphi}{\partial z} = \rho^{-1}\frac{\partial p_0}{\partial t}, \quad z = 0, \quad t > 0 \tag{4}$$

$$\varphi(r,0,0) = 0, \quad \varphi_t(r,0,0) = 0 \tag{5}$$

$$\frac{\partial \varphi}{\partial z} = 0 \quad \text{on } z = -h. \tag{6}$$

The conditions (5) are equivalent to the conditions

$$\varphi = 0, \quad \zeta = 0 \quad \text{at } t = 0,$$

since p_0 is finite and $r \to 0$ as $t \to 0+$.

3. Formal solution

We assume a solution of (2) of the form

$$\varphi = \int_0^\infty A(k,t) J_0(kr) \cosh k(z+h) \operatorname{sech} kh \, dk$$

so that (6) is satisfied.

Substituting for φ in (4), we obtain the following differential equation for $A(k,t)$:

$$\ddot{A}+\sigma^2 A = k\rho^{-1}\frac{\partial}{\partial t}\int_0^{r_0(t)} \alpha f(\alpha,t)J_0(k\alpha)\mathrm{d}\alpha$$

where

$$\sigma^2 = gk\tanh kh.$$

The solution of this equation is

$$A = A_0(k)\cos(\sigma t+\varepsilon_k)+(k/\rho\sigma)\int_0^t \sin[\sigma(t-s)]\frac{\partial}{\partial s}\int_0^{r_0(s)} \alpha f(\alpha,s)J_0(k\alpha)\mathrm{d}\alpha\,\mathrm{d}s.$$

By (5), $A_0 = 0$.

An integration by parts then gives

$$A=(k/\rho)\int_0^t \cos[\sigma(t-s)]\mathrm{d}s\int_0^{r_0(s)} \alpha f(\alpha,s)J_0(k\alpha)\mathrm{d}\alpha. \quad (7)$$

the velocity potential is therefore

$$\varphi=\rho^{-1}\int_0^\infty kJ_0(kr)\frac{\cosh k(z+h)}{\cosh kh}\mathrm{d}k\int_0^t \cos\sigma(t-s)\mathrm{d}s\int_0^{r_0(s)} \alpha f(\alpha,s)J_0(k\alpha)\mathrm{d}\alpha. \quad (8)$$

The surface displacement is then determined by (3):

$$\zeta=-(g\rho)^{-1}\int_0^\infty \sigma kJ_0(kr)\mathrm{d}k\int_0^t \sin\sigma(t-s)\mathrm{d}s\int_0^{r_0(s)} \alpha f(\alpha,s)J_0(k\alpha)\mathrm{d}\alpha. \quad (9)$$

4. Asymptotic representation of φ and ζ for a uniformly expanding pressure area

We adopt the following model for $f(r,t)$ because it closely resembles the first term of Sakurai's [8] approximation scheme for the determination of the pressure field inside the blast wave

$$f(r,t)=(t+t_1)^{-n}F(r/r_0(t)),\quad r<r_0(t) \quad (10)$$

where $n(>1)$ is non-integral, and t_1 is the time taken (from the moment of the explosion) by the shock front to just reach the surface. Also, at high pressures, $R(t+t_1)\equiv$ the radius of the shock front at time $(t+t_1)\propto(t+t_1)^{2/5}$ Ref. [6]

From this result, one can obtain the expression for $r_0(t)$:

$$r_0(t)=[\{R(t+t_1)\}^2-\{R(t_1)\}^2]^{1/2}.$$

Here, however, we assume, alongwith the model (10), that

$$r_0(t) = ut, \quad u = \text{constant}, \tag{11}$$

for convenience of analysis.

Then, from (8)

$$\varphi = \rho^{-1} \int_0^1 \alpha F(\alpha)\mathrm{d}\alpha \int_0^\infty kJ_0(kr) \frac{\cosh k(z+h)}{\cosh kh} \mathrm{d}k$$
$$\times \int_0^t r_0^2(s)(t_1+s)^{-n} \cos\sigma(t-s) J_0(k\alpha r_0(s))\mathrm{d}s. \tag{12}$$

To evaluate (12) asymptotically for large r and t, we first replace $J_0(k\alpha r_0(s))$ by its integral representation,

$$J_0(k\alpha r_0(s)) = (2/\pi) \int_0^{\pi/2} \cos(k\alpha us \sin\theta)\mathrm{d}\theta, \tag{13}$$

and $J_0(kr)$ by the first term of its asymptotic expansion for large kr,

$$J_0(kr) \simeq (2/\pi kr)^{1/2} \cos(kr - \pi/4). \tag{14}$$

The resulting s-integral is expressed in terms of a function $T_n(ia, t, t_1)$ defined as follows:

$$\begin{aligned} T_n(ia,t,t_1) &= e^{iat_1} \int_0^t s^2(t_1+s)^{-n} e^{ias}\mathrm{d}s \\ &= [(3-n)^{-1} s^{3-n} {}_1F_1(3-n; 4-n; ias) \\ &\quad - 2(2-n)^{-1} t_1 s^{2-n} {}_1F_1(2-n; 3-n; ias) \\ &\quad + (1-n)^{-1} t_1^2 s^{1-n} {}_1F_1(1-n; 2-n; ias)]_{s=t_1}^{s=(t+t_1)}. \end{aligned} \tag{15}$$

Here

$$a \equiv a_j = -\{\sigma + (-1)^j ku\alpha \sin\theta\}, \quad j = 1, 2. \tag{16}$$

and ${}_1F_1$ denotes Kummer's confluent hypergeometric function.

In place of (12), we have now

$$(\pi\rho u^{-2}/2)\varphi \simeq (8\pi r)^{-1/2} \mathrm{Re} \int_0^1 \alpha F(\alpha)\mathrm{d}\alpha \int_0^{\pi/2} \mathrm{d}\theta$$
$$\times \int_K^\infty k^{1/2} \cosh[k(z+h)] \operatorname{sech} kh \sum_{j=1}^2 T_n(ia_j(k), t, t_1)$$
$$\times [\exp\{irP_j(k)\} + \exp\{irQ_j(k)\}]\mathrm{d}k, \tag{17}$$

where

$$\begin{aligned} P_j(k) &= \sigma(t+t_1)r^{-1} + (-1)^j ku\alpha t_1 r^{-1} \sin[\theta - k + (\pi/4r)], \\ Q_j(k) &= \sigma(t+t_1)r^{-1} + (-1)^j ku\alpha t_1 r^{-1} \sin[\theta + k - (\pi/4r)] \end{aligned} \tag{18}$$

and $K > 0$ is such that $Kr \gg 1$, and the stationary point(s), if any, lies in (K, ∞).

The function $Q_j(k)$ has no stationary point for $j = 2$ in $0 < k < \infty$, and none either in the same interval for $j = 1$, since $ut_1 \ll r$. Therefore, the part of the k-integral arising from $\exp\{irQ_j(k)\}$ in (17) is $O(r^{-1})$, as $r \to \infty$. The function $P_j(k)$, on the other hand, has one and only one stationary point, $k = k_j$ (say), when

$$\tau \equiv [(t + t_1)\sqrt{gh}/2r] > \frac{1}{2} + \frac{ut_1}{2r}\delta_{j1} \tag{19}$$

where δ_{j1} is Kronecker's delta function. To show this, we note that

(i) $P_j'(k)$ is continuous in $0 < k < \infty$,

(ii) $P_j'(k)$ is strictly monotone decreasing in $0 < k < \infty$, since $P_j''(k) < 0$ therein,

(iii) $P_j'(k) \to 2\tau - 1 + (-1)^j u\alpha t_1 r^{-1} \sin\theta \equiv a_1(\text{say})$ as $k \to 0+$

$P_j'(k) \to (-1)^j u\alpha t_1 r^{-1} \sin\theta - 1 \equiv a_2(\text{say})$, as $k \to \infty$,

< 0 for both j, since $ut_1 r^{-1} \ll 1$.

These conditions make $P_j'(k)$ vanish once and only once in $0 < k < \infty$, when $(a_1)_{\min} > 0$, that is, when $\tau > \dfrac{1}{2} + \dfrac{ut_1}{2r}\delta_{j1}$, as stated above.

A similar argument shows that the equation $[P_j'(k)]_{u=0} = 0$ has one and only one non-negative real root $k = k_0$ (say), independent of j, when $\tau > 1/2$ and hence, under the condition (19) as well.

Since $ut_1 r^{-1} \ll 1$, an approximate value of k_j may be obtained by putting

$$k_j = k_0 + \varepsilon_j \tag{20}$$

in the equation $P_j'(k) = 0$, whence

$$\varepsilon_j \simeq (-1)^{j+1} u\alpha t_1 r^{-1} \sin\theta / P_j''(k_0). \tag{21}$$

Applying the method of stationary phase to evaluate the k-integral of (17), we obtain

$$\begin{aligned}(\pi\rho u^{-2}/2)\varphi \simeq & \int_0^1 \alpha F(\alpha)\mathrm{d}\alpha \int_0^{\pi/2} \mathrm{d}\theta \cdot (2r)^{-1}(k_j/|P''(k_j)|)^{1/2} \\ & \times \cosh k_j(z + h)\operatorname{sech} k_j h \\ & \times \operatorname{Re}\sum_j T_n(ia_j(k_j), t, t_1)\exp i\{rP_j(k_j) - \pi/4\} \\ & \geqq O(r'^{-3/2}), \quad r' \equiv (r/h) \to \infty. \end{aligned} \tag{22}$$

The asymptotic expansions of the functions ${}_1F_1$ for large arguments [Erdélyi, [2] I, 6.13.1 (2)] show that

$$T_n(ia_j(k_j), t, t_1) \sim \frac{t^2}{ia_j(k_j)}(t + t_1)^{-n} e^{ia_j(k_j)(t+t_1)}, \tag{23}$$

where we suppose $n < 2$, a restriction required for the Taylor point blast theory.

Using the approximations (20) and (23), we get for (22), the expression

$$(\pi\rho u^{-2}/2)\varphi \simeq \int_0^1 \alpha F(\alpha)\mathrm{d}\alpha \int_0^{\pi/2} \mathrm{d}\theta \frac{t^2}{2r(t+t_1)^n}(g\tanh k_0 h|P_j''(k_0)|)^{-1/2}$$
$$\times \cosh[k_0(z+h)]\operatorname{sech} k_0 h$$
$$\times \sum_j \sin\{k_0 r + (-1)^j k_0 u\alpha t \sin\theta\}. \quad (24)$$

By [Erdélyi, [2], I, 7.12. (45)], we have

$$(2/\pi)\int_0^{\pi/2} \sin\{k_0 r + (-1)^j k_0 u t\alpha \sin\theta\}\mathrm{d}\theta$$
$$= \sin(k_0 r)J_0(k_0 u t\alpha) + (-1)^j(2/\pi)\cos(k_0 r)s_{0,0}(k_0 u t\alpha) \quad (25)$$

when $\tau > 1/2 + \dfrac{ut_1}{2r}$ so that both k_1 and k_2 exist, the Lommel function $s_{0,0}(k_0 u t\alpha)$ cancels out in the j-sum of (24). The asymptotic expression for φ thus finally becomes

$$\rho\varphi \simeq \frac{u^2 t^2}{r(t+t_1)^n}\{g\tanh(k_0 h)|P_j''(k_0)|\}^{-1/2}\frac{\cosh k_0(z+h)}{\cosh k_0 h}\bar{F}(k_0 ut)\sin(k_0 r), \quad (26)$$

where

$$\bar{F}(x) = \int_0^1 \alpha F(\alpha)J_0(\alpha x)\mathrm{d}\alpha. \quad (27)$$

The result (26) holds under the conditions

$$r \gg r_0(t) \gg r_0(t_1), \quad k_0 r \gg 1, \quad \tau > \frac{1}{2} + \frac{ut_1}{2r} \quad (28)$$

A similar process applied to (9) gives for the surface displacement the asymptotic expression

$$\zeta \simeq -(g\rho)^{-1}\frac{u^2 t^2}{r}(t+t_1)^{-n}k_0^{1/2}|P_j''(k_0)|^{-1/2}\bar{F}(k_0 ut)\cos(k_0 r) \quad (29)$$

under the same conditions (28).

If $F(x) = D$, $0 < x < 1$, the limiting value of ζ, as $h \to \infty$, equals the corresponding value of ζ for the case of infinite depth [Sen [11], eqn. (69)].

4.1 *An illustrative case*

When a concentrated explosion of constant total energy E takes place in a still atmosphere of density ρ_0, Taylor's formula for the maximum pressure (which happens to be on the shock front) is

$$p_{max} = 0{\cdot}141\{E^2\rho_0^3(t+t_1)^{-6}\}^{1/5} \tag{30}$$

when the ratio of specific heats of air is about 1.4.

If we adopt this law of pressure for an approximation in the present case while retaining the hypothesis $r_0(t) = ut$ for a relatively small spread of the pressure area, we have

$$n = 6/5, \quad \text{and } F(R) = 0{\cdot}141\ (E^2\rho_0^3)^{1/5} \quad \text{for all } R$$

so that

$$\bar{F}(k) = 0{\cdot}141(E^2\rho_0^3)^{1/5}k^{-1}J_1(k).$$

Equation (29) then gives

$$\zeta \simeq -0{\cdot}141\frac{(E^2\rho_0^3)^{1/5}}{g\rho k_0^{1/2}}\cdot\frac{ut}{r(t+t_1)^{6/5}}\cdot|P_j''(k_0)|^{-1/2}J_1(k_0ut)\cos(k_0r) \tag{31}$$

subject to the conditions (28).

5. Wave elevation due to a Taylor point blast above the fluid surface

At the outset, we transform the general expression (9) for ζ as follows:

$$\zeta = -(g\rho)^{-1}t\,\mathrm{Im}\int_0^\infty \sigma kJ_0(kr)e^{i\sigma(st+t_1)}\mathrm{d}k$$
$$\times\int_0^1 r_0^2(st)\cdot(t_1+st)^{-n}e^{-i\sigma(st+t_1)}\bar{F}(kr_0(st))\mathrm{d}s. \tag{32}$$

Writing

$$T_n(k,t,t_1) = e^{-i\sigma t_1}\int_0^1 r_0^2(st)(t_1+st)^{-n}e^{-i\sigma st}\bar{F}(kr_0(st))\mathrm{d}s \tag{33}$$

We follow the same procedure as shown in §4, it being assumed that the function $\bar{F}(kr_0(st))$ is sufficiently well behaved, and it does not make T_n strongly oscillatory or singular for large t. The latter is a pre-requisite for the applicability of the method of stationary phase [Stoker, [12], §6.8]. Then

$$\zeta \simeq -(k_0t/g^{1/2}\rho r)(\tanh k_0h/|P''(k_0)|)^{1/2}$$
$$\times\,\mathrm{Im}[T_n(k_0,t,t_1)\exp i\{rP(k_0)-\pi/4\}] + O(r'^{-3/2}),$$
$$\text{as } r' \equiv (r/h)\to\infty \tag{34}$$

where

$$P(k) = (t+t_1)r^{-1}(gk\tanh kh)^{1/2} - k + \pi/4r,$$
$$P''(k) \equiv P_j''(k) \quad \text{as obtained from (18), and } k = k_0$$

is the non-negative real root of $P'(k) = 0$.

This approximation holds under the conditions

$$(t+t_1)\sqrt{gh}\,r^{-1} > 1, \quad r \gg r_0(t), \quad k_0 r \gg 1. \tag{35}$$

For large t, $T_n(k_0, t, t_1)$ approximates to

$$T_n(k_0, t, t_1) \simeq (i/t)(gk_0 \tanh k_0 h)^{-1/2} r_0^2(t)(t+t_1)^{-n} \times \bar{F}(k_0 r_0(t)) e^{-i}(t+t_1)\sqrt{gk_0 \tanh(k_0 h)}. \tag{36}$$

Therefore

$$\zeta \simeq -(g\rho r)^{-1}[k_0/|P''(k_0)|]^{+1/2} r_0^2(t)(t+t_1)^{-n} \bar{F}(k_0 r_0(t)) \cos(k_0 r). \tag{37}$$

This result is used below to determine the wave height caused by a Taylor point blast above the fluid surface.

5.1 *Pressure inside a blast wave: Taylor's formula*

For an intense explosion of constant total energy E occurring at a point O' at a height H above the ground, the pressure $p(r, z, t)$ inside the expanding spherical blast wave and the radius $R(t)$ of the shock wave at time t from the moment of the explosion are given by the following formulae due to Taylor [13]:

$$p = 0{\cdot}133 R^{-3} E f_1(\eta), \tag{38}$$

$$f_1(\eta) = \frac{2\gamma}{\gamma+1}\left[\frac{\gamma+1}{\gamma} - \frac{\eta^{n-1}}{\gamma}\right]^{-(2\gamma^2+7\gamma-3/7-\gamma)}, \tag{39}$$

$$\eta = (z^2 + r^2)^{1/2}/R, \tag{40}$$

$$t = 0{\cdot}926 R^{5/2} \rho_0^{1/2} E^{-1/2}. \tag{41}$$

Here $n = (7\gamma - 1)/(\gamma^2 - 1)$, γ = ratio of specific heats of air $\simeq 1.4$.
z = depth of a point vertically downwards from O'.
r = distance of a point P from the perpendicular $O'O$ on the surface.

The surface pressure distribution in the present problem may therefore be taken as

$$p_0(r, t) = \begin{cases} 0{\cdot}133 R^{-3} E f_1(\sqrt{H^2 + r^2}/\sqrt{H^2 + r_0^2}), & r < r_0(t), \\ 0, & r > r_0(t), \end{cases} \tag{42}$$

where

$$r_0^2(t) = R^2(t+t_1) - R^2(t_1) \tag{43}$$

and t_1 = time taken by the shock to reach the surface.

5.2 *Adjustment of Taylor's formula to the wave problem*

The pressure model (10) without the one for $r_0(t)$ results from the above when $H = 0$. The same model may be retained when H is small compared with $r_0(t)$ or R. To this purpose, a Lagrange expansion (MacRobert [7], § 54) of $p_0(r, t)$ is useful.

Writing

$$\mu = (H^2 + r^2)/(H^2 + r_0^2),$$
$$\mu_0 = (r/r_0)^2, \tag{44}$$
$$f_1(\sqrt{\mu}) = f_2(\mu),$$

we get

$$\mu = \mu_0 + (H/r_0)^2(1-\mu),$$

and

$$f_2(\mu) = f_2(\mu_0) + \sum_1^\infty \frac{1}{m!}\left(\frac{H}{r_0}\right)^{2m} \frac{\mathrm{d}^{m-1}}{\mathrm{d}\mu_0^{m-1}}[(1-\mu_0)^m f_2'(\mu_0)]. \tag{45}$$

Also

$$\left(\frac{H}{r_0}\right)^{2m} = \left(\frac{R^2 - H^2}{H^2}\right)^{-m} = (H/R)^{2m}\left(1 - \frac{H^2}{R^2}\right)^{-m}$$
$$= (H/R)^{2m}\left[1 + m\left(\frac{H}{R}\right)^2 + \frac{m(m+1)}{2!}\left(\frac{H}{R}\right)^4 + \cdots\right]. \tag{46}$$

Consequently,

$$p_0(r,t) = 0{\cdot}121(E^2\rho_0^3)^{1/5}(t+t_1)^{-6/5}\left[f_2(\mu_0) + \sum_{m=1}^\infty \sum_{l=0}^\infty \frac{m(m+1)\cdots(m+l-1)}{m!\,l!}\right.$$
$$\left.\times \{t_1/(t+t_1)\}^{4(m+l)/5} F_m(\mu_0)\right], \quad \text{when } \mu_0 < 1;$$

$$p_0(r,t) = 0, \qquad \text{when } \mu_0 > 1, \tag{47}$$

since

$$R(t+t_1) = (0{\cdot}926)^{-2/5}(E/\rho_0)^{1/5}(t+t_1)^{2/5}$$
$$= 1{\cdot}031(E/\rho_0)^{1/5}(t+t_1)^{2/5}. \tag{48}$$

Here

$$f_2(\mu_0) = \frac{2\gamma}{\gamma+1}\left[\frac{\gamma+1}{\gamma} - \frac{\mu_0^{(n-1)/2}}{\gamma}\right]^{-(2\gamma^2+7\gamma-3/7-\gamma)} \tag{49}$$

$$F_m(\mu_0) = \frac{d^{m-1}}{d\mu_0^{m-1}}[(1-\mu_0)^m f_2'(\mu_0)]. \tag{50}$$

5.3 *Asymptotic wave height*

Subject to the validity of the linearised wave theory, the asymptotic expression for ζ under the conditions (35) is

$$\zeta \simeq -0{\cdot}129(E^4\rho_0^{1/5}/g\rho r)\{k_0/|P''(k_0)|\}^{1/2}(t+t_1)^{-6/5}$$
$$\times \{(t+t_1)^{4/5} - t_1^{4/5}\}\cos(k_0 r)$$
$$\times \left[\tilde{f}_2(k_0 r_0(t)) + \sum_{m=1}^\infty \sum_{l=0}^\infty \frac{m(m+1)\cdots(m+l-1)}{m!\,l!}\right.$$
$$\left.\times \{t_1/(t+t_1)\}^{4(m+l)/5}\tilde{F}_m(k_0 r_0(t))\right], \tag{51}$$

where

$$\tilde{f}_2(k) = \int_0^1 \alpha f_2(\alpha^2) J_0(k\alpha) \mathrm{d}\alpha. \tag{52}$$

The last result shows that a good approximation to ζ for small H is obtained if only the first term $\tilde{f}_2(k_0 r_0(t))$ in the square bracket is retained. Further evaluation of ζ can be accomplished, it seems, only by numerical methods.

6. Some characteristics of the motion

In both the expressions (29) and (51) for the wave elevation ζ, $k_0 r \gg 1$. The factor $\cos k_0 r$ in both therefore changes its sign rapidly so that we may regard $k_0 r$ as the phase and the co-factor of $\cos k_0 r$ as the amplitude of ζ in either expression. The phase is not directly affected by the velocity parameter u in (29) or by $r_0(t)$ in (29) and (51).

Since $\mathrm{d}k_0/\mathrm{d}t$ is positive as per (18), the degree of oscillation of level at any point becomes more rapid with time. Since $\frac{\mathrm{d}}{\mathrm{d}t}(\tau^2 - k_0 h)$ at first diminishes with τ (up to the value given by the equation $-2\tau^2 P_j''(k_0) = h$) and then increases with it, the oscillation at any point in shallow water is somewhat more rapid at first and less rapid thereafter than what would happen if the sea were deep.

Denoting $t\sqrt{gh}/(2r)$ by τ_0, we have for

$$k_0 \sim \kappa(\tau_0), \quad P_j''(k_0)[\text{or } P''(k_0)] \sim (\tau_0/\tau)[P_j''(\kappa) \text{ or } P''(\kappa)] \equiv P_0''(\kappa),$$

(say) and equation $(P_j'(k))_{u=0} = 0)$ may be written as $t/\psi(\kappa) = 1$, where

$$[\psi(k)]^{-1} = (\sqrt{gh}/2r)[\{\tanh kh/kh\}^{1/2} + \{kh/\tanh kh\}^{1/2}\operatorname{sech}^2 kh]. \tag{53}$$

The amplitude of ζ in (29) then varies as

$$r^{-1} t^{2-n} \kappa^{1/2} |P_0''(\kappa)|^{-1/2} \bar{F}(\kappa u t).$$

From (53), it appears that $kh = O(\tau_0^2)$ and $P''(\kappa) = O(\kappa^{-1})$ when $\kappa h(\text{or } \tau) \gg 1$. As n is usually >1, one finds that the amplitude $\to 0$ as $t \to \infty$ when $\bar{F}(x)$ is $O(x^{-1})$ or of a higher order of smallness as $x \to \infty$.

The times of maximum amplitude at any point are given by

$$t_n = \frac{2r}{\sqrt{gh}} \frac{(a_n \tanh a_n)^{1/2}}{\tanh a_n + a_n \operatorname{sech}^2 a_n}, \quad n = 1, 2, 3, \ldots,$$

where

$$\kappa = a_n, \quad n = 1, 2, 3, \ldots$$

satisfy the equation

$$\left[2x\left(\frac{1}{\kappa P_0''(\kappa)} - 1\right)\frac{\bar{F}'(x)}{\bar{F}(x)}\right]_{x = \kappa u \psi(\kappa)} = 3 - 2n + \left\{\frac{P_0'''(\kappa)}{P_0''^2(\kappa)} - \frac{1}{\kappa P_0''(\kappa)}\right\}.$$

Therefore, the points of maximum amplitude at a distance r travel outwards with the corresponding constant velocities

$$\frac{1}{2}\sqrt{gh}\frac{\tanh a_n + a_n \operatorname{sech}^2 a_n}{(a_n \tanh a_n)^{1/2}}.$$

The amplitude at any point becomes nearly zero at times

$$\tau_n = \frac{2r}{\sqrt{gh}}\frac{(b_n \tanh b_n)^{1/2}}{\tanh b_n + b_n \operatorname{sech}^2 b_n}$$

where

$$\kappa = b_n, \quad n = 1, 2, 3, \ldots$$

satisfy the equation $\bar{F}(\kappa u\psi(\kappa)) = 0$. These points of minimum amplitude travel outwards with the corresponding constant velocities

$$\frac{1}{2}\sqrt{gh}\frac{\tanh b_n + b_n \operatorname{sech}^2 b_n}{(b_n \tanh b_n)^{1/2}}.$$

The values of a_n and b_n increase with n. Hence, the outer rings spread out faster than the inner ones. A similar discussion may be given for (51).

7. A particular case

Let

$$f(r,t) = D(t+t_1)^{-n} = F(r)(t+t_1)^{-n}, \quad r < r_0(t) = ut.$$

Therefore

$$\begin{aligned}\bar{F}(k_0 ut) &= \int_0^1 \alpha F(\alpha) J_0(\alpha k_0 ut)\mathrm{d}\alpha \\ &= \frac{D}{k_0 ut}\int_0^1 (k_0 \alpha ut) J_0(\alpha k_0 ut)\mathrm{d}\alpha \\ &= \frac{D}{k_0 ut} J_1(k_0 ut).\end{aligned}$$

Then (29) gives

$$\frac{g\rho t_1}{D}\zeta = -\frac{(ut)t_1 J_1(k_0 ut)}{k_0^{1/2} r\,(t+t_1)^n}|p_j''(k_0)|^{-1/2}\cos(k_0 r), \quad r \gg r_0(t) = ut, \; k_0 r \gg 1. \tag{54}$$

By (18), we have

$$P_j''(k) = \frac{g^{1/2}(t+t_1)}{4r(k\tanh kh)^{3/2}}[4kh(\tanh kh)(\operatorname{sech}^2 kh)(1 + k\tanh kh) - (\tanh kh + kh \operatorname{sech}^2 kh)^2]. \tag{55}$$

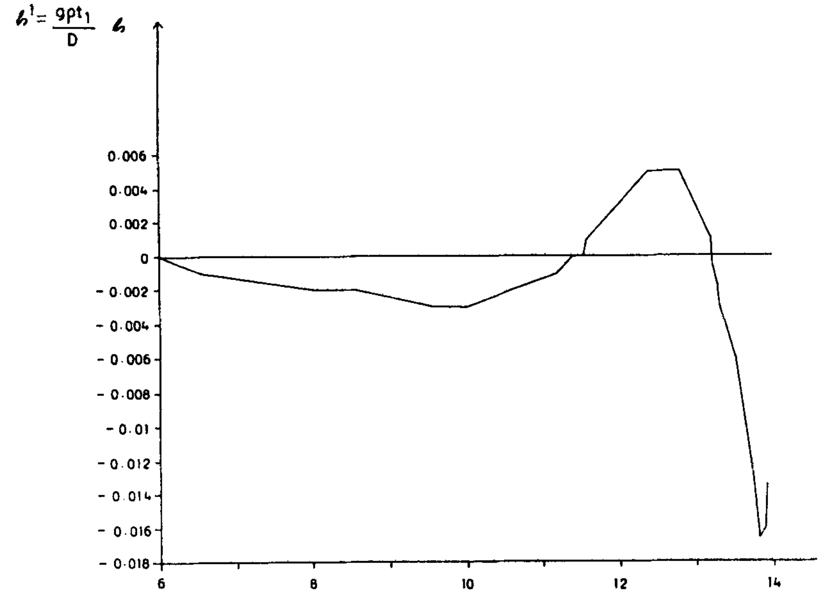

Figure 1. Variation of ζ^1 with r. $u = 0{\cdot}05$, $n = 1$, $g = 32$, $t_1 = 0{\cdot}5$, $t = 2$, $h = 1$.

Now let us take

$$J_1(k_0 ut) \simeq \left(\frac{2}{\pi k_0 ut}\right)^{1/2} \cos\left(k_0 ut - \frac{3}{4}\pi\right). \tag{56}$$

Using (55) and (56) in (54), we get

$$\begin{aligned}\frac{\sqrt{2\pi} g\rho t_1}{D}\zeta = &\left[\frac{(ut)t_1}{rk_0^{1/2}}\right]\left[\frac{(k_0 ut)^{-1/2}}{(t+t_1)^n}\right]\left[\frac{g^{1/2}(t+t_1)}{r(k_0 \tanh k_0 h)^{3/2}}\right]^{-1/2} \\ &\times [(4k_0 h \tanh k_0 h \operatorname{sech}^2 k_0 h)(1 + k_0 \tanh k_0 h) \\ &\quad - (\tanh k_0 h + k_0 h \operatorname{sech}^2 k_0 h)^2]^{-1/2} \\ &\times \cos(k_0 r)\cos(k_0 ut - 3\pi/4), \quad r > ut. \end{aligned} \tag{57}$$

The variation of

$$\zeta' = \frac{\sqrt{2\pi} g\rho t_1}{D}\zeta$$

with r as shown in figure 1.

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