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Oscillation in odd-order neutral delay differential equations

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Abstract. Consider the odd-order functional differential equation

$$(x(t)-\alpha x(t-\tau))^{(n)}+p(t)f(x(t-\sigma))=0, \tag{*}$$

where $0\leqslant\alpha<1$, τ, $\sigma\in(0,\infty)$, $p\in C([0,\infty),(0,\infty))$, $f\in C^1(R,R)$ such that f is increasing, $xf(x)>0$ for $x\neq 0$ and f satisfies a generalized linear condition

$$\liminf_{x\to 0}\left|\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\right|=1.$$

Here we prove that every solution of (*) oscillates if

$$\liminf_{t\to\infty}\int_{t-\sigma/n}^{t}\sigma^{n-1}p(s)\,\mathrm{d}s>\frac{1}{e}(1-\alpha)(n-1)!\left(\frac{n}{n-1}\right)^{n-1}.$$

This result generalizes a recent result of Gopalsamy *et al.* [6].

Keywords. Functional differential equations; oscillation of all solutions.

1. Introduction

In a remarkable result Ladas [4] proved that every solution of the first-order delay differential equation

$$x'(t)+px(t-\sigma)=0, \tag{1}$$

where $p,\sigma\in(0,\infty)$ oscillates (i.e., every solution has an unbounded set of zeros in $(0,\infty)$) if and only if

$$p\sigma>\frac{1}{e}. \tag{2}$$

The result was extended by authors in [5] for general *odd*-order differential equation

$$x^{(n)}(t)+px(t-\sigma)=0, \tag{3}$$

replacing (2) by

$$p^{1/n}\left(\frac{\sigma}{n}\right)>\frac{1}{e}. \tag{4}$$

The first result was further improved (see [7]) for equations with variable coefficients with the statement that

$$\liminf_{t\to\infty}\int_{t-\sigma}^{t} p(s)\,ds > \frac{1}{e},$$

and

$$\limsup_{t\to\infty}\int_{t-\sigma}^{t} p(s)\,ds > \frac{1}{e}$$

are respectively sufficient and necessary conditions for every solution of

$$x'(t) + p(t)x(t-\sigma) = 0,$$

where $p\in C([0,\infty),(0,\infty))$, to be oscillatory. But a similar extension for

$$x^{(n)}(t) + p(t)x(t-\sigma) = 0 \tag{5}$$

has not been proved yet.

Recently, Gopalsamy *et al* [6] proved that

$$\liminf_{t\to\infty}\int_{t-\sigma}^{t} (t-s)^{n-1}p(s)\,ds > (1-\alpha)(n-1)! \tag{6}$$

implies that every solution of the *odd*-order differential equation

$$(x(t) - \alpha x(t-\tau))^{(n)} + p(t)x(t-\sigma) = 0 \tag{7}$$

oscillates, where $0 \leqslant \alpha < 1$. Indeed, for $\alpha = 0$ and $p(t) = p\in(0,\infty)$, (6) reduces to

$$p\sigma^n > n!,$$

that is,

$$p^{1/n}\left(\frac{\sigma}{n}\right) > \frac{1}{n}(n!)^{1/n}, \tag{8}$$

which is the sufficient condition for oscillation of (3). In view of the condition given in (4), the lower bound of $p^{1/n}(\sigma/n)$ in (8) is comparatively larger than that of $(1/e)$.

In this paper we prove a result, a particular case of which shows that all solutions of (7) are oscillatory if

$$\liminf_{t\to\infty}\int_{t-\sigma/n}^{t} \sigma^{n-1}p(s)\,ds > \frac{1}{e}(1-\alpha)(n-1)!\left\{\frac{n}{n-1}\right\}^{n-1}. \tag{9}$$

When $p(t) = p\in(0,\infty)$ and $\alpha = 0$, the above condition reduces to

$$p^{1/n}\left(\frac{\sigma}{n}\right) > \frac{1}{n}\left(\frac{1}{e}\left(\frac{n}{n-1}\right)^{n-1}\right)^{1/n}(n!)^{1/n}. \tag{10}$$

In view of the known inequality

$$\begin{aligned}\left\{\frac{1}{e}\left(\frac{n}{n-1}\right)^{n-1}\right\}^{1/n} &= \left\{\frac{1}{e}\left(1+\frac{1}{n-1}\right)^{n-1}\right\}^{1/n}\\ &= \left\{\frac{1}{e}\left(\sum_{r=0}^{n-1} C(n-1,r)\left(\frac{1}{n-1}\right)^{r}\right)\right\}^{1/n}\\ &\leqslant \left\{\frac{1}{e}\left(\sum_{r=0}^{n-1}\frac{1}{r!}\right)\right\}^{1/n}\end{aligned}$$

$$\leqslant \left\{ \frac{1}{e} \left(\sum_{r=0}^{\infty} \frac{1}{r!} \right) \right\}^{1/n} = 1, \tag{11}$$

where $C(n-1,r)$ is the $(r+1)$th binomial coefficient in the expansion of $(1+1/(n-1))^{n-1}$, our condition is weaker than that of (8). We give examples to support our claim.

2. Main results

Consider the *odd*-order nonlinear functional differential equation

$$(x(t) - \alpha x(t-\tau))^{(n)} + p(t) f(x(t-\sigma)) = 0, \tag{E}$$

with the assumptions that

$$\begin{aligned} &p \in C(R^+, R^+),\ f \in C(R, R) \text{ such that } f \text{ is increasing},\\ &x f(x) > 0 \text{ for } x \neq 0,\ |f(x)| \to \infty \text{ as } |x| \to \infty,\ 0 \leqslant \alpha < 1,\\ &n > 1 \text{ is an odd integer and } \tau, \sigma \in (0, \infty). \end{aligned} \tag{H}$$

Let $\delta = \max\{\tau, \sigma\}$ and $\phi \in C([T-\delta, T], R)$. By a solution of (E) in $[T, \infty)$, we mean a function $x \in C([T, \infty), R)$ such that $x(t) = \phi(t)$, $T - \delta \leqslant t \leqslant T$, $(x(t) - \alpha x(t-\tau)) \in C^{(n)}([T, \infty), R)$ and $x(t)$ satisfies (E) for $t \geqslant T$.

As usual, a solution $x(t)$ of (E) is called oscillatory if it has zeros for arbitrarily large t and nonoscillatory, otherwise.

We say (E) is *generalized sublinear* if f satisfies

$$\liminf_{x \to 0} \left| \left(\frac{\mathrm{d}f}{\mathrm{d}x} \right) \right| > 1,$$

superlinear if

$$\liminf_{x \to 0} \left| \left(\frac{\mathrm{d}f}{\mathrm{d}x} \right) \right| < 1 \tag{12}$$

and *linear* if

$$\liminf_{x \to 0} \left| \left(\frac{\mathrm{d}f}{\mathrm{d}x} \right) \right| = 1, \tag{13}$$

which includes the cases $f(x) = x^\alpha$, $0 < \alpha < 1$, $1 < \alpha < \infty$ and $\alpha = 1$, respectively.

In what follows, we list the following two results for our use in sequel.

Theorem 1 ([3], *Lemma 1*). *Suppose that* $g \in C^{(n)}([T, \infty), (0, \infty))$ *such that* $g^{(i)}(t)$ *has no zeros in* $[T, \infty)$ $(i = 1, 2, \ldots (n-1))$ *and* $g^{(n)}(t) \leqslant 0$ *for* $t \geqslant T$. *If* $\beta \in (0, \infty)$ *then*

$$g(t-\beta) \geqslant \frac{\beta^{n-1}}{(n-1)!} g^{(n-1)}(t), \quad t \geqslant T + 2\beta.$$

Theorem 2 ([7], *Theorem 2.1.1*). *If* $\beta \in (0, \infty)$, $Q \in C([T, \infty), (0, \infty))$, $T > 0$ *and*

$$\liminf_{t \to \infty} \int_{t-\beta}^{t} Q(s)\, \mathrm{d}s > \frac{1}{e},$$

then the first-order differential inequality

$$y'(t) + Q(t)y(t-\beta) \leqslant 0$$

has no eventually positive solutions.

Our main theorem is as follows.

Theorem 3. *Suppose that (H) holds and f satisfies* (13). *Then* (9) *implies that every solution of* (E) *oscillates.*

Proof. Since (9) holds, there exists $0 < \varepsilon < 1$ such that

$$(1-\varepsilon)^2 \liminf_{t\to\infty} \int_{t-\sigma/n}^{t} \sigma^{n-1} p(s)\mathrm{d}s > \frac{1}{e}(1-\alpha)(n-1)!\left(\frac{n}{n-1}\right)^{n-1}. \tag{14}$$

To the contrary, assume that $x(t)$ is a nonoscillatory solution of (E). Let $x(t) > 0$ for $t \geqslant t_0$. (The case for $x(t) < 0$, $t \geqslant t_0$ may be treated similarly.) Setting

$$z(t) = x(t) - \alpha x(t-\tau), \tag{15}$$

from (E) it may be observed that $z^{(n)}(t) \leqslant 0$ for $t \geqslant t_0 + \sigma$. Consequently, there exists $T \geqslant t_0 + \sigma$ such that $z^{(i)}(t)$ $(i = 0, 1, 2, 3 \ldots (n-1))$, has no zeros in $[T, \infty)$.

Suppose that $z(t) < 0$, $t \geqslant T$. Since n is odd, $z^{(n)}(t) \leqslant 0$, $t \geqslant T$ implies that $z'(t) < 0, t \geqslant T$. On the other hand, let

$$\limsup_{t\to\infty} x(t) = \mu.$$

If $\mu = \infty$, there exists a sequence of real numbers $\langle t_n \rangle_{n=1}^{\infty}$ such that $t_n \to \infty$, $x(t_n) \to \infty$ as $n \to \infty$ and $x(s) < x(t_n)$ for $s < t_n$. From (15) we see that

$$z(t_n) = x(t_n) - \alpha x(t_n - \tau) \geqslant (1-\alpha)x(t_n),$$

which further gives

$$\lim_{n\to\infty} z(t_n) = \infty,$$

a contradiction to the fact that $z(t) < 0, t \geqslant T$. In case μ is finite, there exists a sequence $\langle t_n \rangle_{n=1}^{\infty}$ such that $t_n \to \infty$, $x(t_n) \to \mu$ as $n \to \infty$. Since $\langle x(t_n - \tau) \rangle_{n=1}^{\infty}$ is a bounded sequence of real numbers, it admits a convergent subsequence. Let $\langle s_n \rangle_{n=1}^{\infty}$ be the subsequence for which $\langle x(s_n - \tau) \rangle_{n=1}^{\infty}$ converges to a real number λ. Clearly $\lambda \leqslant \mu$. Again $x(s_n) \to \mu$ as $n \to \infty$. Now

$$\lim_{n\to\infty} z(s_n) = \lim_{n\to\infty} (x(s_n) - \alpha x(s_n - \tau)) \geqslant (1-\alpha)\lambda,$$

that is,

$$\lim_{n\to\infty} z(s_n) \geqslant 0,$$

which is a contradiction to the fact that $z(t)$ is negative and decreasing function. Hence $z(t) < 0$, $t \geqslant T$ is impossible.

Let $z(t) > 0, t \geqslant T$. Clearly, it follows that $z'(t) < 0, t \geqslant T$. Indeed, otherwise, $z'(t) > 0$, $t \geqslant T$ implies that $\liminf_{t\to\infty} z(t) > 0$ and consequently

$$\liminf_{t\to\infty} x(t) = \liminf_{t\to\infty} (z(t) + \alpha x(t-\tau)) > 0.$$

Integrating (E) from T to t and using the above observation along with the fact that (9) implies

$$\int^{\infty} p(s)\,\mathrm{d}s = \infty,$$

we see that $z^{(n-1)}(t) \to -\infty$ as $t \to \infty$. Consequently, $z(t) \to -\infty$ as $t \to \infty$, a contradiction. Further $z^{(n)}(t) \leqslant 0$ implies that

$$z^{(j)}(t) z^{(j+1)}(t) \leqslant 0, \quad 0 \leqslant j \leqslant (n-1).$$

Consequently,

$$\lim_{t\to\infty} z^{(j)}(t) = 0, \quad 1 \leqslant j \leqslant (n-1),$$

and

$$\lim_{t\to\infty} z(t) = k \geqslant 0.$$

If $k > 0$, then repeating the argument applied earlier we lead to a contradiction. Hence $k = 0$. From (13) it follows that

$$\liminf_{y\to 0} \left(\frac{\mathrm{d}f}{\mathrm{d}y}\right) = 1.$$

Taking $y(t) = z^{(n-1)}(t)$, and from the definition of limit infimum it follows that for every $\varepsilon > 0$ there exists a large positive number M such that

$$\left(\frac{\mathrm{d}f}{\mathrm{d}y}\right) > (1-\varepsilon) \quad \text{for } t \geqslant M. \tag{16}$$

From (15) we see that

$$x(t) = z(t) + \alpha x(t-\tau), \quad t \geqslant T. \tag{17}$$

The repeated application of (17) on it, as per the idea in the paper of Gopalsamy *et al* [6] results in

$$x(t) \geqslant z(t)\left(\sum_{n=0}^{N} \alpha^n\right), \quad t \geqslant T + N\tau.$$

From the above inequality it follows that there exists $M_1 \geqslant T + N\tau$ such that

$$x(t) > z(t)\frac{(1-\varepsilon)^2}{(1-\alpha)}, \quad t \geqslant M_1. \tag{18}$$

In Theorem 1, replacing $g(t)$ by $z(t-\sigma/n)$ and β by $\left(\frac{n-1}{n}\right)\sigma$ we get

$$z(t-\sigma) > \frac{1}{(n-1)!}\left(\frac{n-1}{n}\sigma\right)^{n-1} z^{(n-1)}(t-\sigma/n), \quad t \geqslant T + 3\sigma. \tag{19}$$

Using (19) in the inequality obtained by replacing t by $t-\sigma$ in (18) we get

$$x(t-\sigma)\geqslant Kz^{(n-1)}(t-\sigma/n),\quad t\geqslant \max\{M_1+\sigma, T+3\sigma\}=T_0,$$

where

$$K=\frac{(1-\varepsilon)^2}{(1-\alpha)}\frac{1}{(n-1)!}\left(\frac{n-1}{n}\sigma\right)^{n-1}. \tag{20}$$

Since f is increasing,

$$f(x(t-\sigma))\geqslant f(Kz^{(n-1)}(t-\sigma/n)),\quad t\geqslant T_0. \tag{21}$$

From (E) and (21) it follows that

$$z^{(n)}(t)+p(t)f(Kz^{(n-1)}(t-\sigma/n))\leqslant 0,\quad t\geqslant T_0. \tag{22}$$

Multiplying both sides of (22) by

$$\frac{\mathrm{d}}{\mathrm{d}y}(f(y)),\quad \text{where } y=z^{(n-1)}(t),$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(y(t)))+\left(p(t)\frac{\mathrm{d}f}{\mathrm{d}y}\right)f(Ky(t-\sigma/n))\leqslant 0,\quad t\geqslant T_0. \tag{23}$$

Set

$$H(t)=f(Ky(t)),\quad t\geqslant T_0.$$

Now $z^{(n-1)}(t)>0, t\geqslant T_0$ implies that $H(t)>0, t\geqslant T_0$. From (23) and (16) it follows that

$$\frac{\mathrm{d}H}{\mathrm{d}t}+(1-\varepsilon)Kp(t)H(t-\sigma/n)\leqslant 0,\quad t\geqslant T_1=\max\{T_0, M\}.$$

Hence $H(t)$ is an eventually positive solution of the differential inequality given in Theorem 2, where

$$Q(t)=(1-\varepsilon)Kp(t).$$

and

$$\beta=\sigma/n.$$

But, by (14),

$$\liminf_{t\to\infty}\int_{t-\sigma/n}^{t}Q(s)\,\mathrm{d}s=\liminf_{t\to\infty}\int_{t-\sigma/n}^{t}K(1-\varepsilon)p(s)\,\mathrm{d}s>\frac{1}{e},$$

a contradiction to Theorem 2. Hence (E) cannot have a nonoscillatory solution.

This completes the proof of this theorem.

Example. Consider the equation

$$(x(t)-\tfrac{1}{2}x(t-1))^{(3)}+3(\tfrac{19}{20}+e^{-t})x(t-1)=0,\quad t\geqslant 1.$$

Since (6) fails to hold, Theorem 4.1 of Gopalsamy *et al* [6] is not applicable, but (9) holds and hence Theorem 3 shows that every solution of it oscillates.

Remark. In view of the inequality

$$\frac{1}{n}\left(\frac{1}{e}\left(\frac{n}{n-1}\right)^{n-1}\right)^{1/n}(n!)^{1/n} < \frac{1}{2}\left(1+\frac{1}{n}\right), \tag{24}$$

it follows from (10) that

$$p^{1/n}\left(\frac{\sigma}{n}\right) > \frac{1}{2}\left(1+\frac{1}{n}\right),$$

or in particular,

$$p^{1/n}\left(\frac{\sigma}{n}\right) > \frac{2}{3} \tag{25}$$

implies that every solution of eq. (3) oscillates. Indeed, $n > 1$ and odd gives that

$$n! = 1(n-1)2(n-2)\cdots\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}\right).n.$$

Since the arithmetic mean exceeds the geometric mean $r(n-r) < \left(\frac{n}{2}\right)^2$ for every r and hence

$$n! \leqslant \left(\frac{n}{2}\right)^{n-1} n = \left(1-\frac{1}{2}\right)^{n-1} n^n.$$

Consequently, using Binomial theorem we get

$$\frac{1}{n}(n!)^{1/n} \leqslant \left(1-\frac{1}{2}\right)^{(1-(1/n))} \leqslant 1-\frac{1}{2}\left(1-\frac{1}{n}\right) = \frac{1}{2}\left(1+\frac{1}{n}\right). \tag{26}$$

Now (24) follows from (11) and (26). Since $n \geqslant 3$, (25) follows from (26).

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