Proc. Indian Acad. Sci. (Math. Sci.), Vol. 105, No. 2, May 1995, pp. 201–205.

Absolute summability of infinite series

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MS received 4 November 1993; revised 22 April 1994

Abstract. It is shown in [4] that if a normal matrix A satisfies some conditions then $|C,1|_k$ summability implies $|A|_k$ summability where $k \geqslant 1$. In the present paper, we consider the converse implication.

Keywords. Normal matrix; $|C,1|_k$ summability; $|A|_k$ summability.

1. Introduction

By u_n^α and t_n^α we denote, respectively, the Cesàro means of order $\alpha(\alpha > -1)$ of the sequences (s_n) and (r_n), where (s_n) is the partial sums of the series Σx_n and $r_n = nx_n$. The series Σx_n is then called absolutely summable (C,α) with index k, or simply summable $|C,\alpha|_k, k \geqslant 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|u_n^\alpha - u_{n-1}^\alpha|^k < \infty, [1]. \tag{1}$$

Since $t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha)$, [3], condition (1) can be written in the form

$$\sum_{n=1}^{\infty} n^{-1}|t_n^\alpha|^k < \infty. \tag{2}$$

Let $A = (a_{nv})$ be a normal matrix, i.e., lower-semi matrix with non-zero diagonal entries. By (T_n) we denote the A-transform of the sequence (s_n), i.e.,

$$T_n = \sum_{v=0}^{n} a_{nv}s_v; \quad n = 0,1,\ldots$$

We say that the series Σx_n is summable $|A|_k, k \geqslant 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|T_n - T_{n-1}|^k < \infty. \tag{3}$$

Given a normal matrix $A = (a_{nv})$, we associate two lower-semi matrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}; \quad n,v = 0,1,\ldots, \quad \hat{a}_{00} = \bar{a}_{00} = a_{00},$$

$$\hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v} \quad \text{for } n = 1, 2, \ldots$$

If A is a normal matrix, then $A' = (a'_{nv})$ will denote the inverse of A. Clearly, if A is normal then $\hat{A} = (\hat{a}_{nv})$ is normal and it has two-sided inverse $\hat{A}' = (\hat{a}'_{nv})$, which is also normal (see [2]).

Note that, if A is normal then

$$T_n = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \sum_{i=v}^{n} a_{ni} x_v = \sum_{v=0}^{n} \bar{a}_{nv} x_v$$

and

$$\Delta T_{n-1} = T_n - T_{n-1} = \sum_{v=0}^{n} (\bar{a}_{nv} - \bar{a}_{n-1,v}) x_v = \sum_{v=0}^{n} \hat{a}_{nv} x_v \quad (a_{n-1,n} = 0),$$

which implies

$$x_n = \sum_{v=0}^{n} \hat{a}'_{nv} \Delta T_{v-1} \quad (T_{-1} = 0). \tag{4}$$

In connection with the absolute summability we have the following theorem.

Theorem A. *Suppose that, for $k \geqslant 1$,*

$$\sum_{n=1}^{\infty} n^{k-1} |\hat{a}_{n0}|^k < \infty, \quad \sum_{v=1}^{n} |\Delta \hat{a}_{nv}| = O(1/n), \quad \sum_{n=v}^{\infty} |\Delta \hat{a}_{nv}| = O(1/v),$$

$$\sum_{v=1}^{n} (1/v) |\hat{a}_{nv}| = O(1/n) \quad \text{and} \quad \sum_{n=v}^{\infty} |\hat{a}_{nv}| = O(1),$$

then if Σx_n is summable $|C, 1|_k$, it is also summable $|A|_k$, where $\Delta \hat{a}_{nv} = \hat{a}_{nv} - \hat{a}_{n,v+1}$ [4].

Furthermore it is shown in [4] that the conditions of Theorem A are satisfied whenever A is (C, α), $\alpha \geqslant 1$. This deduces that $|C, 1|_k$ summability implies $|C, \alpha|_k$, $k \geqslant 1, \alpha \geqslant 1$, summability which is a well-known result.

We may now ask what conditions should be imposed on $A = (a_{nv})$ so that the converse implication holds in Theorem A. It is the object of this paper to answer this question.

2. The main result

Theorem B. *Let $A = (a_{nv})$ be a normal matrix such that*

(i) $1 = O(v a_{vv})$,

(ii) $(a_{vv} - a_{v+1,v}) = O(a_{vv} a_{v+1,v+1})$

(iii) $$\sum_{v=i}^{\infty} (v+2) |\hat{a}'_{v+2,i}| = O(i+1). \tag{5}$$

If Σx_n is summable $|A|_k$, then it is also summable $|C, 1|_k, k \geqslant 1$.

Proof. By T_n and t_n we denote the A-transform and $(C, 1)$-mean of the series Σx_n and the sequence (nx_n), respectively. Then it follows from (3) that

$$
\begin{aligned}
t_n &= (n+1)^{-1} \sum_{v=1}^{n} v x_v = (n+1)^{-1} \sum_{v=1}^{n} v \left\{ \sum_{r=0}^{v} \hat{a}'_{vr} \Delta T_{r-1} \right\} \\
&= (n+1)^{-1} \sum_{v=1}^{n} v \left\{ \hat{a}'_{v,v-1} \Delta T_{v-2} + \hat{a}'_{vv} \Delta T_{v-1} + \sum_{r=0}^{v-2} \hat{a}'_{vr} \Delta T_{r-1} \right\} \\
&= (n+1)^{-1} \left\{ \sum_{v=1}^{n-1} v \hat{a}'_{vv} \Delta T_{v-1} + n \hat{a}'_{nn} \Delta T_{n-1} + \sum_{v=2}^{n} v \hat{a}'_{v,v-1} \Delta T_{v-2} + \hat{a}'_{10} \Delta T_{-1} \right. \\
&\quad \left. + \sum_{v=2}^{n} v \sum_{r=0}^{v-2} \hat{a}'_{vr} \Delta T_{r-1} \right\} \\
&= (n+1)^{-1} \sum_{v=1}^{n-1} \{ v \hat{a}'_{vv} + (v+1) \hat{a}'_{v+1,v} \} \Delta T_{v-1} + n(n+1)^{-1} \hat{a}'_{nn} \Delta T_{n-1} \\
&\quad + (n+1)^{-1} \hat{a}'_{10} \Delta T_{-1} + (n+1)^{-1} \sum_{v=0}^{n-2} (v+2) \sum_{r=0}^{v} \hat{a}'_{v+2,r} \Delta T_{r-1}.
\end{aligned}
$$

By considering the equality

$$
\sum_{k=v}^{n} \hat{a}'_{nk} \hat{a}_{kv} = \delta_{nv},
$$

where δ_{nv} is the Kronecker delta, we have

$$
\begin{aligned}
v \hat{a}'_{vv} + (v+1) \hat{a}'_{v+1,v} &= v / a_{vv} + (v+1)(-\hat{a}_{v+1,v} / a_{vv} a_{v+1,v+1}) \\
&= v / a_{vv} + (v+1)[-(a_{v+1,v} + a_{v+1,v+1} - a_{vv}) / a_{vv} a_{v+1,v+1}] \\
&= (v+1)[1 / a_{v+1,v+1} - a_{v+1,v} / a_{vv} a_{v+1,v+1}] - 1 / a_{vv}
\end{aligned}
$$

and so

$$
\begin{aligned}
t_n &= (n+1)^{-1} \sum_{v=1}^{n-1} \{ (v+1)[1 / a_{v+1,v+1} - a_{v+1,v} / a_{vv} a_{v+1,v+1}] - 1 / a_{vv} \} \Delta T_{v-1} \\
&\quad + n(n+1)^{-1} (1 / a_{nn}) \Delta T_{n-1} + \hat{a}'_{10} \Delta T_{-1} (n+1)^{-1} \\
&\quad + (n+1)^{-1} \sum_{r=0}^{n-2} \Delta T_{r-1} \sum_{v=r}^{n-2} (v+2) \hat{a}'_{v+2,r}
\end{aligned}
$$

which implies, by virtue of (5i), (5ii) and (5iii), that

$$
\begin{aligned}
t_n &= O\left\{ (n+1)^{-1} \sum_{v=1}^{n-1} v |\Delta T_{v-1}| + n |\Delta T_{n-1}| + (n+1)^{-1} \right\} \\
&= w_{n,1} + w_{n,2} + w_{n,3}, \text{ say.}
\end{aligned}
$$

To prove the theorem, it is enough to show that

$$
\sum_{n=1}^{\infty} n^{-1} |w_{n,i}|^k < \infty, \quad \text{for } i = 1, 2, 3.
$$

Now it follows from Hölder's inequality that

$$\sum_{n=2}^{m+1} n^{-1}|w_{n,1}|^k = O\left\{\sum_{n=2}^{m+1} n^{-k-1}\left\{\sum_{v=1}^{n-1} v|\Delta T_{v-1}|\right\}^k\right\}$$

$$= O\left\{\left(\sum_{n=2}^{m+1} n^{-2}\sum_{v=1}^{n-1} v^k|\Delta T_{v-1}|^k\right)\left(n^{-1}\sum_{v=1}^{n-1} 1\right)^{k-1}\right\}$$

$$= O\left\{\sum_{v=1}^{m} v^k|\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} n^{-2}\right\}$$

$$= O\left\{\sum_{v=1}^{\infty} v^{k-1}|\Delta T_{v-1}|^k\right\} < \infty,$$

and

$$\sum_{n=1}^{\infty} n^{-1}|w_{n,2}|^k = O\left\{\sum_{n=1}^{\infty} n^{-1}|n\Delta T_{n-1}|^k\right\} < \infty.$$

Finally,

$$\sum_{n=1}^{\infty} n^{-1}|w_{n,3}|^k = O\left\{\sum_{n=1}^{\infty} n^{-k-1}\right\} < \infty, \quad k \geqslant 1.$$

Hence the proof of the theorem is completed.

3. Applications

Let (p_n) be a sequence of positive real numbers such that $P_n = p_0 + p_1 + \cdots + p_n$, $P_{-1} = p_{-1} = 0$. The Riesz (weighted mean) matrix is defined by $a_{nv} = p_v/P_n$ for $0 \leqslant v \leqslant n$ and $a_{nv} = 0$ for $v > n$. From now on, we suppose that $A = (a_{nv})$ is a weighted mean matrix with $P_n \to \infty$ and $n \to \infty$. Hence if no confusion is likely to arise, we say that Σx_n is summable $|R, p_n|_k, k \geqslant 1$, if (3) holds.

With this notation we have

COROLLARY 1

Let (p_n) be a sequence of positive real numbers such that $P_n = O(np_n)$. Then if Σx_n is summable $|R, p_n|_k$, it is also summable $|C, 1|_k, k \geqslant 1$.

Proof. Applying Theorem B with $A = (a_{nv})$, a weighted mean matrix, we see that (5ii) clearly holds and (5i) is reduced to the condition $P_n = O(np_n)$. On the other hand, a small calculation reveals that

$$\bar{a}_{nv} = (P_n - P_{v-1})/P_n, \quad \hat{a}_{nv} = p_n P_{v-1}/P_n P_{n-1}$$

and

$$\hat{a}'_{nv} = \begin{cases} -P_{n-2}/p_{n-1} & \text{if } v = n-1 \\ P_n/p_n & \text{if } v = n \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get

$$\sum_{v=i}^{\infty} (v+2)|\hat{a}'_{v+2,i}| = 0 \text{ for all } i,$$

and so the proof is completed.

COROLLARY 2

Let (p_n) *be a sequence of positive real numbers with* $np_n = O(P_n)$. *Then if* Σx_n *is summable* $|C, 1|_k$, *it is also summable* $|R, p_n|_k$, $(k \geqslant 1)$.

Proof. Apply Theorem A.

Now the next result which appears in [5] is a consequence of Corollaries 1 and 2.

COROLLARY 3

Suppose that (p_n) *is a sequence of positive real numbers such that*

$$np_n = O(P_n) \quad \text{and} \quad P_n = (nP_n).$$

Then the summability $|C, 1|_k$ *is equivalent to the summability* $|R, p_n|_k, k \geqslant 1$.

Acknowledgement

This paper was supported by the Scientific and Technical Research Council of Turkey (TBAG–ÇG2).

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