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On L^1-convergence of modified complex trigonometric sums

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Abstract. We study here L^1-convergence of a complex trigonometric sum and obtain a new necessary and sufficient condition for the L^1-convergence of Fourier series.

Keywords. L^1-convergence of modified complex trigonometric sums; L^1-convergence of Fourier series; Dirichlet kernel; Fejér kernel.

1. Introduction

It is well known that if a trigonometric series converges in L^1 to a function $f \in L^1$, then it is the Fourier series of the function f. Riesz [1, Vol. II, Ch. VIII §22] gave a counter example showing that in a metric space L we cannot expect the converse of the abovesaid result to hold true. This motivated the various authors to study L^1-convergence of trigonometric series. During their investigations some authors introduced modified trigonometric sums as these sums approximate their limits better than the classical trigonometric series in the sense that they converge in L^1-metric to the sum of the trigonometric series whereas the classical series itself may not.

Let the partial sums of the complex trigonometric series

$$\sum_{|n| \leqslant \infty} c_n \mathrm{e}^{int}$$

be denoted by

$$S_n(C,t) = \sum_{|k| \leqslant n} c_k \mathrm{e}^{ikt}, t \in T = \mathbb{R}/2\pi Z.$$

If the trigonometric series is the Fourier series of some $f \in L^1$, we shall write $c_n = \hat{f}(n)$ for all n and $S_n(C,t) = S_n(f,t) = S_n(f)$.

If $a_k = \mathrm{o}(1)$ as $k \to \infty$, and $\sum_{k=1}^{\infty} k^2 |\Delta^2(a_k/k)| < \infty$, then we say that the series $\sum_{k=1}^{\infty} a_k \Phi_k(x)$, where $\Phi_k(x)$ is $\cos kx$ or $\sin kx$, belongs to the class $\mathbb{R}$. Kano [2] proved that if $\sum_{k=1}^{\infty} a_k \Phi_k(x)$ belongs to the class $\mathbb{R}$, then it is a Fourier series or equivalently, it represents an integrable function. Ram and Kumari [3] introduced modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

and

$$g_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \sin kx$$

and studied their L^1-convergence. The aim of this paper is to study the L^1-convergence of the complex form of the above sums.

Let

$$D_n(t) = \frac{1}{2} + \sum_{m=1}^{n} \cos mt = \frac{\sin\left(n + \frac{1}{2}\right)t}{2\sin\frac{t}{2}},$$

$$\tilde{D}_n(t) = \sum_{m=1}^{n} \sin mt = \frac{\cos\frac{t}{2} - \cos\left(n + \frac{1}{2}\right)t}{2\sin\frac{t}{2}},$$

and

$$\tilde{K}_n(t) = \frac{1}{n+1}\sum_{m=0}^{n} \tilde{D}_m(t) = \frac{1}{4\sin^2\frac{t}{2}}\left[\sin t - \frac{\sin(n+1)t}{n+1}\right]$$

denote the Dirichlet's kernel, the conjugate Dirichlet's kernel, and the conjugate Fejér's kernel respectively. Let $E_n(t) = \sum_{k=0}^{n} e^{ikt}$. Then the first differentials $D'_n(t)$ and $\tilde{D}'_n(t)$ of $D_n(t)$ and $\tilde{D}_n(t)$ can be written as

$$D'_n(t) = E'_n(t) + E'_{-n}(t)$$

$$2i\tilde{D}'_n(t) = E'_n(t) - E'_{-n}(t),$$

where $E'_n(t)$ denotes the first differential of $E_n(t)$. The complex form of the above modified sums is

$$g_n(C,t) = S_n(C,t) + \frac{i}{n+1}[c_{n+1}E'_n(t) - c_{-(n+1)}E'_{-n}(t)].$$

We introduce here a new class R^* of sequence as follows:

Definition. A null sequence $\langle c_n \rangle$ of complex numbers belongs to the class R^* if

$$\sum_{k=1}^{\infty} \left|\Delta\left(\frac{c_{-k} - c_k}{k}\right)\right| k \log k < \infty, \tag{1.1}$$

$$\sum_{k=1}^{\infty} k^2 \left|\Delta^2\left(\frac{c_k}{k}\right)\right| < \infty. \tag{1.2}$$

2. Lemmas. The proof of our result is based upon the following lemmas, of which the first three are due to Sheng [4]:

Lemma 1. $\|D'_n(t)\|_1 = 4/\pi(n \log n) + o(n)$

Lemma 2. $\|\tilde{D}'_n(t)\|_1 = o(n \log n)$.

Lemma 3. For each non-negative integer n, there holds

$$\| c_n E_n'(t) + c_{-n} E_{-n}'(t) \|_1 = o(1), \quad n \to \infty$$

if and only if $nc_n \log|n| = o(1)$, $|n| \to \infty$, *where* $\langle c_n \rangle$ *is a complex sequence.*

Lemma 4. (i) There exist positive constants α *and* β *such that*

$$\alpha(\log n) \leqslant \| \tilde{K}_n(t) \|_1 \leqslant \beta(\log n)$$

(ii) $\| \tilde{K}_n'(t) \|_1 = o(n)$.

Proof. The existence of β follows from the fact that $\| \tilde{D}_n(t) \|_1 = o(\log n)$. Further, we have

$$\begin{aligned} 2\pi \| \tilde{K}_n(t) \|_1 &\geqslant \int_0^\pi \tilde{K}_n(t)\,dt \\ &= \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \int_0^\pi \sin kt\,dt \\ &= \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)(1 - \cos k\pi)/k \\ &= \frac{1}{n+1} \sum_{k=0}^{n} \left[\sum_{j=0}^{k} (1 - \cos j\pi)/j \right] \\ &\geqslant M(\log n!)/(n+1) \end{aligned}$$

for some constant M, the last step being the consequence of the relation $\sum_{v=1}^{n} \log v = \log n!$. Using Sterling's asymptotic formula $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$, we then have

$$\| \tilde{K}_n(t) \|_1 \geqslant \alpha \log n.$$

This completes the proof of (i). To prove (ii) we have,

$$|\tilde{D}_n'(t)| = \left| \sum_{k=0}^{n} k \cos kt \right| \leqslant n(n+1)/2$$

and so

$$|\tilde{K}_n'(t)| \leqslant (n+1)^{-1} \sum_{k=0}^{n} |\tilde{D}_k'(t)| = o(n^2).$$

This implies that

$$\int_{|t| \leqslant \pi/n} |\tilde{K}_n'(t)|\,dt = o(n).$$

Differentiating $\tilde{K}_n(t)$ we get

$$\tilde{K}_n'(t) = \Sigma_{1n}(t) - \Sigma_{2n}(t) + \Sigma_{3n}(t),$$

where

$$\Sigma_{1n}(t) = \{\cos t - \cos(n+1)t\} \Big/ \left(4 \sin^2 \frac{t}{2}\right),$$

$$\Sigma_{2n}(t) = (2\sin^2 t)\Big/\left(2\sin\frac{t}{2}\right)^4,$$

$$\Sigma_{3n}(t) = \{2\sin t\sin(n+1)t\}/(n+1)\left(2\sin\frac{t}{2}\right)^4.$$

Obviously, $|\Sigma_{jn}(t)| = o(|t|^{-2})$ for $j = 1, 2$, and $(n+1)|\Sigma_{3n}(t)| = o(|t|^{-3})$. Using these estimates, we get

$$\int_{\pi/n \leqslant |t| \leqslant \pi} |\tilde{K}_n'(t)|\,dt = o\left(\int_{\pi/n \leqslant |t| \leqslant \pi} t^{-2}\,dt\right) + o\left(\frac{1}{n+1}\int_{\pi/n \leqslant |t| \leqslant \pi} t^{-3}\,dt\right)$$
$$= o(n).$$

Combining the above estimates, we infer that $\|\tilde{K}_n'(t)\|_1 = o(n)$.

Lemma 5. Let $n \geqslant 1$ and $0 < \varepsilon < \pi$. Then there exists $A_\varepsilon > 0$ such that for all $\varepsilon \leqslant |t| \leqslant \pi$

(i) $|E_n'(t)| \leqslant A_\varepsilon n/|t|$,
(ii) $|E_{-n}'(t)| \leqslant A_\varepsilon n/|t|$,
(iii) $|D_n'(t)| \leqslant 2A_\varepsilon n/|t|$, *and*
(iv) $|\tilde{D}_n'(t)| \leqslant A_\varepsilon n/|t|$.

Proof. We have

$$i^{-1}E_n'(t) = \sum_{k=0}^{n} k\,e^{ikt} = \sum_{k=0}^{n} k(E_k(t) - E_{k-1}(t))$$
$$= \sum_{k=0}^{n} (\Delta k)E_k(t) + (n+1)E_n(t).$$

Since $|E_n(t)| \leqslant A_\varepsilon/|t|$ for some constant A_ε, we have

$$|E_n'(t)| \leqslant \frac{A_\varepsilon}{|t|}\left[\sum_{k=0}^{n} 1 + (n+1)\right] \leqslant A_\varepsilon n/|t|.$$

Since

$$E_{-n}'(t) = (-1)E_n'(-t),$$

we obtain $|E_{-n}'(t)| \leqslant A_\varepsilon n/|t|$. The other two inequalities follow from $D_n'(t) = E_n'(t) + E_{-n}'(t)$ and $2i\tilde{D}_n'(t) = E_n'(t) - E_{-n}'(t)$.

3. Main theorem

We prove the following result.

Theorem. *Let $c_n \in R^*$. Then there exists $f(t)$ such that*

$$\lim_{n\to\infty} g_n(C,t) = f(t) \text{ for all } 0 < |t| \leqslant \pi, \tag{3.1}$$

$$f(t) \in L^1(T) \text{ and } \|g_n(C,t) - f(t)\|_1 = o(1) \text{ as } n \to \infty, \tag{3.2}$$

$$\|S_n(f,t) - f(t)\|_1 = o(1) \text{ as } n\to\infty \text{ if and only if } \hat{f}(n)\log|n| = o(1) \text{ as } |n| \to \infty. \tag{3.3}$$

Proof. We have, by using Abel's transformation,

$$g_n(C,t)=S_n(C,t)+\frac{\mathrm{i}}{n+1}[c_{n+1}E'_n(t)-c_{-(n+1)}E'_{-n}(t)]$$

$$=2\sum_{k=1}^{n}\Delta\left(\frac{c_k}{k}\right)\tilde{D}'_k(t)+\sum_{k=1}^{n}\Delta\left(\frac{c_{-k}-c_k}{k}\right)\mathrm{i}E'_{-k}(t).$$

By Lemma 5, we get

$$\sum_{k=1}^{\infty}\left|\Delta\left(\frac{c_k}{k}\right)\tilde{D}'_k(t)\right|\leqslant(A_1/|t|)\sum_{k=1}^{\infty}\left\{k\left|\Delta\left(\frac{c_k}{k}\right)\right|\right\}$$

$$\leqslant(A_1/|t|)\left\{\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}k\left|\Delta^2\left(\frac{c_j}{j}\right)\right|\right\}$$

$$=(A_1/|t|)\left\{\sum_{j=1}^{\infty}\left(\sum_{k=1}^{j}k\right)\left|\Delta^2\left(\frac{c_j}{j}\right)\right|\right\}$$

$$=O\left\{(A_1/|t|)\left(\sum_{j=1}^{\infty}j^2\right)\left|\Delta^2\left(\frac{c_j}{j}\right)\right|\right\}<\infty,$$

and

$$\sum_{k=3}^{\infty}\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)E'_{-k}(t)\right|\leqslant(A_1/|t|)\left\{\sum_{k=3}^{\infty}k\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)\right|\right\}$$

$$\leqslant(A_1/|t|)\left\{\sum_{k=3}^{\infty}k(\log k)\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)\right|\right\}<\infty,$$

where A_1 is a suitable constant. These imply that

$$f(t)=2\left\{\sum_{k=1}^{\infty}\Delta\left(\frac{c_k}{k}\right)\tilde{D}'_k(t)\right\}+\mathrm{i}\left\{\sum_{k=1}^{\infty}\Delta\left(\frac{c_{-k}-c_k}{k}\right)E'_{-k}(t)\right\}$$

exists and thus (3.1) follows.

Further, for $t\neq 0$, we have

$$f(t)-g_n(C,t)=2\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_k}{k}\right)\tilde{D}'_k(t)+\mathrm{i}\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_{-k}-c_k}{k}\right)E'_{-k}(t)$$

$$=2\sum_{k=n+1}^{\infty}(k+1)\Delta^2\left(\frac{c_k}{k}\right)\tilde{K}'_k(t)-2(n+1)\Delta\left(\frac{c_{n+1}}{n+1}\right)\tilde{K}'_n(t)$$

$$+\mathrm{i}\sum_{k=n+1}^{\infty}\Delta\left(\frac{c_{-k}-c_k}{k}\right)E'_{-k}(t).$$

Thus

$$\|f(t)-g_n(C,t)\|_1\leqslant 2\sum_{k=n+1}^{\infty}(k+1)\left|\Delta^2\left(\frac{c_k}{k}\right)\right|\int_{-\pi}^{\pi}|\tilde{K}'_k(t)|\,\mathrm{d}t$$

$$+2(n+1)\left|\Delta\left(\frac{c_{n+1}}{n+1}\right)\right|\int_{-\pi}^{\pi}|\tilde{K}'_{n+1}(t)|\,\mathrm{d}t$$

$$+\sum_{k=n+1}^{\infty}\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)\right|\int_{-\pi}^{\pi}|E'_{-k}(t)|\,dt.$$

But, by Lemma 4,

$$\int_{-\pi}^{\pi}|\tilde{K}'_k(t)|\,dt=o(k).$$

Also

$$\left|\Delta\left(\frac{c_{n+1}}{n+1}\right)\right|=\left|\sum_{k=n+1}^{\infty}\Delta^2\left(\frac{c_k}{k}\right)\right|$$

$$\leqslant\sum_{k=n+1}^{\infty}\frac{k^2}{k^2}\left|\Delta^2\left(\frac{c_k}{k}\right)\right|\leqslant(n+1)^{-2}\sum_{k=n+1}^{\infty}k^2\left|\Delta^2\left(\frac{c_k}{k}\right)\right|$$

$$=o((n+1)^{-2}),$$

by the hypothesis of the theorem. Lemma 1 and Lemma 2 imply that

$$\int_{-\pi}^{\pi}|E'_{-k}(t)|\,dt=o(k\log k).$$

Therefore,

$$\|f(t)-g_n(C,t)\|_1=o\left(\sum_{k=n+1}^{\infty}(k+1)^2\left|\Delta^2\left(\frac{c_k}{k}\right)\right|\right)+o(1)$$

$$+o\left(\sum_{k=n+1}^{\infty}\left|\Delta\left(\frac{c_{-k}-c_k}{k}\right)\right|k\log k\right)$$

$=o(1)$, by the hypothesis of the theorem.

Since $g_n(C,t)$ is a polynomial, it follows that $f\in L^1(T)$, which proves the assertion (3.2).
We notice further that

$$\|f-S_n(f)\|_1=\|f-g_n(C,t)+g_n(C,t)-S_n(f)\|_1$$

$$\leqslant\|f-g_n(C,t)\|_1+\|g_n(C,t)-S_n(f)\|_1$$

$$=\|f-g_n(C,t)\|_1+\left\|\frac{i}{n+1}(\hat{f}(n+1))E'_n(t)-\hat{f}(-(n+1))E'_{-n}(t))\right\|_1$$

and

$$\left\|\frac{i}{n+1}(\hat{f}(n+1)E'_n(t)-\hat{f}(-(n+1))E'_{-n}(t))\right\|_1=\|g_n(C,t)-S_n(f)\|_1$$

$$\leqslant\|f-S_n(f)\|_1+\|f-g_n(C,t)\|_1.$$

Since $\|f-g_n(C,t)\|=o(1)$, $n\to\infty$, by (3.2), and by Lemma 3, $\|\hat{f}(n)E'_n(t)-\hat{f}(-n)E'_{-n}(t)\|_1=o(n)$, $n\to\infty$ if and only if $\hat{f}(n)\log|n|=o(1)$, $|n|\to\infty$, the assertion (3.3) follows.

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